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DIFFERENTIAL GEOMETRY

Isotropic Immersions of Complex Space Forms into Real Space Forms and Mean Curvatures

by

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Summary. Using an inequality related to the mean curvature, we give a sufficient condition for an isotropic immersion of a complex space form into a real space form to be parallel.

1. Introduction. Parallel submanifolds of real space forms are ones of the most interesting objects in differential geometry (cf. [F-1, T]). Here, we have in mind the following fact: "all parallel immersions of Riemannian symmetric spaces of rank one into a real space form are isotropic but there exist many isotropic immersions of these spaces into a real space form, which are not parallel." For example, the fourth standard minimal immersion $f : S^3(1/8) \to S^{24}(1)$ is isotropic, but not parallel. So, sufficient conditions for these isotropic immersions to be parallel are worth considering. In this paper, we are interested in parallel immersions of a complex space form into a real space form.

We have proved the following theorem in the previous paper [B]:

THEOREM A. *Let f be a λ-isotropic immersion of an $n(\geq 2)$-dimensional compact oriented real space form $M^n(c;\mathbb{R})$ of constant sectional curvature c into an m-dimensional real space form $\widetilde{M}^m(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. Suppose that the mean curvature vector field $\mathfrak{h}$ and the mean curvature $H = \|\mathfrak{h}\|$ satisfy the following two inequalities*:

(i) $H^2 \leq \frac{2(n+1)}{n}c - \tilde{c}$,
(ii) $0 \leq (1-n)\Delta H^2 + n\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle$,

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where Δ denotes the Laplacian on $M^n(c;\mathbb{R})$. Then f is a parallel immersion. Moreover f is locally equivalent to one of the following:

(I) *f is a totally umbilic immersion of $S^n(c)$ into $\widetilde{M}^m(\tilde{c};\mathbb{R})$, where $c \geq \tilde{c}$ and $H^2 \equiv c - \tilde{c}$.*

(II) *$f = f_2 \circ f_1 : S^n(c) \xrightarrow{f_1} S^{n+n(n+1)/2-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R})$, where f_1 is the second standard minimal immersion, f_2 is a totally umbilic immersion, $2(n+1)c/n \geq \tilde{c}$ and $H^2 \equiv 2(n+1)c/n - \tilde{c}$.*

Remark 1. We remark that Theorem A is no longer true if we omit condition (ii).

In this paper, we consider the case where the submanifold is a complex space form, and get the following theorem:

Theorem 1. *Let f be a λ-isotropic immersion of a complex $n(\geq 2)$-dimensional complex space form $M^n(4c;\mathbb{C})$ of constant holomorphic sectional curvature $4c$ into an m-dimensional real space form $\widetilde{M}^m(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. Suppose that the mean curvature H satisfies*

$$H^2 \leq \frac{2(n+1)}{n}c - \tilde{c}.$$

Then f is a parallel immersion. Moreover f is locally equivalent to one of the following:

(I) *f is a totally geodesic immersion of $\mathbb{C}^n$ $(= \mathbb{R}^{2n})$ into $\mathbb{R}^m$, where $H \equiv 0$.*

(II) *f is a totally umbilic immersion of $\mathbb{C}^n$ into $\mathbb{R}H^m(\tilde{c})$, where $H^2 \equiv -\tilde{c}$.*

(III) *$f = f_2 \circ f_1 : \mathbb{C}P^n(4c) \xrightarrow{f_1} S^{n^2+2n-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R})$, where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion, $2(n+1)c/n \geq \tilde{c}$ and $H^2 \equiv 2(n+1)c/n - \tilde{c}$.*

The main purpose of this paper is to prove Theorem 1.

Remark 2. It is interesting to compare Theorem A with Theorem 1: the submanifold in Theorem 1 need not be compact, and Theorem 1 does not require condition (ii) of Theorem A.

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2. Preliminaries. Let $f : M \to \widetilde{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\widetilde{M}$ with metric $\langle\ ,\ \rangle$, and σ the second fundamental form. We recall the notion of isotropic immersion (cf. [O]): the immersion f is said to be *isotropic* if $\|\sigma(X,X)\|/\|X\|^2$ is constant for all $x \in M$ and all tangent vectors X $(\neq 0)$ to M at x. If we define the function λ on M by $x(\in M) \mapsto \|\sigma(X,X)\|/\|X\|^2$, then the immersion f is also said to be λ-*isotropic*. Note that totally umbilic immersions are isotropic, but not vice versa.

An *n-dimensional real space form* $M^n(c;\mathbb{R})$ is a Riemannian manifold of constant sectional curvature c, which is locally congruent to a standard sphere $S^n(c)$, a Euclidean space $\mathbb{R}^n$ or a real hyperbolic space $\mathbb{R}H^n(c)$, according as c is positive, zero or negative. An *n-dimensional complex space form* $M^n(c;\mathbb{C})$ is a Kähler manifold of constant holomorphic sectional curvature c, which is locally congruent to a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ $(=\mathbb{R}^{2n})$ or a complex hyperbolic space $\mathbb{C}H^n(c)$, according as c is positive, zero or negative. An *n-dimensional quaternionic space form* $M^n(c;\mathbb{Q})$ is a quaternionic Kähler manifold of constant quaternionic sectional curvature c, which is locally congruent to a quaternionic projective space $\mathbb{Q}P^n(c)$, a quaternionic Euclidean space $\mathbb{Q}^n$ $(=\mathbb{R}^{4n})$ or a quaternionic hyperbolic space $\mathbb{Q}H^n(c)$, according as c is positive, zero or negative.

3. Proof of Theorem 1. Let J be the complex structure on $M^n(4c;\mathbb{C})$. Then the curvature tensor R of $M^n(4c;\mathbb{C})$ is given by

$$\begin{aligned}R(X,Y)Z = c\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX \\ &- \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\}\end{aligned} \tag{3.1}$$

for all $X,Y,Z \in \mathfrak{X}(M^n(4c;\mathbb{C}))$, where we denote by $\mathfrak{X}(M^n(4c;\mathbb{C}))$ the set of all vector fields on $M^n(4c;\mathbb{C})$.

Since f is λ-isotropic, we have $\langle\sigma(X,X),\sigma(X,X)\rangle = \lambda^2\langle X,X\rangle\langle X,X\rangle$ for all $X \in \mathfrak{X}(M^n(4c;\mathbb{C}))$, which is equivalent to

$$\begin{aligned}&\langle\sigma(X,Y),\sigma(Z,W)\rangle + \langle\sigma(X,Z),\sigma(W,Y)\rangle + \langle\sigma(X,W),\sigma(Y,Z)\rangle \\ &\qquad = \lambda^2\{\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle\}\end{aligned} \tag{3.2}$$

for all $X,Y,Z,W \in \mathfrak{X}(M^n(4c;\mathbb{C}))$.

The Gauss equation is written as follows:

$$\begin{aligned}&\langle\sigma(X,Y),\sigma(Z,W)\rangle - \langle\sigma(Z,Y),\sigma(X,W)\rangle \\ &\qquad = \langle R(Z,X)Y,W\rangle - \tilde{c}\{\langle X,Y\rangle\langle Z,W\rangle - \langle Z,Y\rangle\langle X,W\rangle\}\end{aligned} \tag{3.3}$$

for all $X,Y,Z,W \in \mathfrak{X}(M^n(4c;\mathbb{C}))$.

It follows from (3.1)–(3.3) that

$$\begin{aligned}\langle\sigma(X,Y),\sigma(Z,W)\rangle = &\frac{\lambda^2+2(c-\tilde{c})}{3}\langle X,Y\rangle\langle Z,W\rangle \\ &+ \frac{\lambda^2-(c-\tilde{c})}{3}\{\langle X,W\rangle\langle Y,Z\rangle + \langle X,Z\rangle\langle Y,W\rangle\} \\ &+ c\{\langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle\}\end{aligned} \tag{3.4}$$

for all $X, Y, Z, W \in \mathfrak{X}(M^n(4c;\mathbb{C}))$. This yields

$$H^2 = \frac{(n+1)\lambda^2 + 2(n+1)c - (2n-1)\tilde{c}}{3n}, \tag{3.5}$$

$$\|\sigma(X, JX)\|^2 = \frac{\lambda^2 - 4c + \tilde{c}}{3}, \tag{3.6}$$

where $X \in \mathfrak{X}(M^n(4c;\mathbb{C}))$ with $\|X\| = 1$. It follows from (3.5) and (3.6) that

$$H^2 - \frac{2(n+1)}{n}\,c + \tilde{c} = \frac{n+1}{n}\,\|\sigma(X, JX)\|^2 \geq 0,$$

where $X \in \mathfrak{X}(M^n(4c;\mathbb{C}))$ with $\|X\| = 1$.

Therefore, $H^2 \equiv 2(n+1)c/n - \tilde{c}$ by assumption, so that

$$\sigma(X, JX) = 0 \tag{3.7}$$

for all $X \in \mathfrak{X}(M^n(4c;\mathbb{C}))$. From (3.7), we get

$$\sigma(X, Y) = \sigma(JX, JY)$$

for all $X, Y \in \mathfrak{X}(M^n(4c;\mathbb{C}))$. Consequently, the immersion f is parallel (cf. [F-2]). The classification theorem for parallel submanifolds of a real space form completes the proof (cf. [F-1, T]).

4. Quaternionic case. In this section, we consider the case where the submanifold is a quaternionic space form, and get the following theorem:

THEOREM 2. *Let f be a λ-isotropic immersion of a quaternionic $n(\geq 2)$-dimensional quaternionic space form $M^n(4c;\mathbb{Q})$ of constant quaternionic sectional curvature $4c$ into an m-dimensional real space form $\widetilde{M}^m(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. Suppose that the mean curvature H satisfies*

$$H^2 \leq \frac{2(n+1)}{n}\,c - \tilde{c}.$$

Then f is a parallel immersion. Moreover f is locally equivalent to one of the following:

(I) *f is a totally geodesic immersion of $\mathbb{Q}^n$ $(= \mathbb{R}^{4n})$ into $\mathbb{R}^m$, where $H \equiv 0$.*

(II) *f is a totally umbilic immersion of $\mathbb{Q}^n$ into $\mathbb{R}H^m(\tilde{c})$, where $H^2 \equiv -\tilde{c}$.*

(III) *$f = f_2 \circ f_1 : \mathbb{Q}P^n(4c) \xrightarrow{f_1} S^{2n^2+3n-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R})$, where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion, $2(n+1)c/n \geq \tilde{c}$ and $H^2 \equiv 2(n+1)c/n - \tilde{c}$.*

Proof. Let $\{I, J, K\}$ be the canonical local basis on $M^n(4c;\mathbb{Q})$. Then the curvature tensor R of $M^n(4c;\mathbb{Q})$ is given by

$$\begin{aligned} R(X,Y)Z = c\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle IY,Z\rangle IX - \langle IX,Z\rangle IY \\ &+ \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + \langle KY,Z\rangle KX \\ &- \langle KX,Z\rangle KY + 2\langle X,IY\rangle IZ \\ &+ 2\langle X,JY\rangle JZ + 2\langle X,KY\rangle KZ\} \end{aligned} \tag{4.1}$$

for all $X, Y, Z \in \mathfrak{X}(M^n(4c;\mathbb{Q}))$. It follows from (3.2), (3.3) and (4.1) that

$$\begin{aligned} \langle\sigma(X,Y),\sigma(Z,W)\rangle = &\frac{\lambda^2 + 2(c-\tilde{c})}{3}\langle X,Y\rangle\langle Z,W\rangle \\ &+ \frac{\lambda^2 - (c-\tilde{c})}{3}\{\langle X,W\rangle\langle Y,Z\rangle \\ &+ \langle X,Z\rangle\langle Y,W\rangle\} + c\{\langle IX,W\rangle\langle IY,Z\rangle \\ &+ \langle IX,Z\rangle\langle IY,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle \\ &+ \langle JX,Z\rangle\langle JY,W\rangle + \langle KX,W\rangle\langle KY,Z\rangle \\ &+ \langle KX,Z\rangle\langle KY,W\rangle\} \end{aligned} \tag{4.2}$$

for all $X, Y, Z, W \in \mathfrak{X}(M^n(4c;\mathbb{Q}))$. Equation (4.2) yields

$$H^2 = \frac{(2n+1)\lambda^2 + 4(n+2)c - (4n-1)\tilde{c}}{6n}, \tag{4.3}$$

$$\|\sigma(X,IX)\|^2 = \|\sigma(X,JX)\|^2 = \|\sigma(X,KX)\|^2 = \frac{\lambda^2 - 4c + \tilde{c}}{3}, \tag{4.4}$$

where $X \in \mathfrak{X}(M^n(4c;\mathbb{Q}))$ with $\|X\| = 1$. It follows from (4.3) and (4.4) that

$$\begin{aligned} H^2 - \frac{2(n+1)}{n}c + \tilde{c} &= \frac{2n+1}{2n}\|\sigma(X,IX)\|^2 = \frac{2n+1}{2n}\|\sigma(X,JX)\|^2 \\ &= \frac{2n+1}{2n}\|\sigma(X,KX)\|^2 \geq 0, \end{aligned}$$

where $X \in \mathfrak{X}(M^n(4c;\mathbb{Q}))$ with $\|X\| = 1$.

Therefore, $H^2 \equiv 2(n+1)c/n - \tilde{c}$ by assumption, so that

$$\sigma(X,IX) = \sigma(X,JX) = \sigma(X,KX) = 0 \tag{4.5}$$

for all $X \in \mathfrak{X}(M^n(4c;\mathbb{Q}))$. From (4.5), we get

$$\sigma(X,Y) = \sigma(IX,IY) = \sigma(JX,JY) = \sigma(KX,KY)$$

for all $X, Y \in \mathfrak{X}(M^n(4c;\mathbb{Q}))$. Consequently, f is parallel (cf. [M]). The classification theorem for parallel submanifolds of a real space form completes the proof (cf. [F-1, T]). ∎

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