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On Stochastic Differential Equations with Reflecting Boundary Condition in Convex Domains

by

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Summary. Let D be an open convex set in $\mathbb{R}^d$ and let F be a Lipschitz operator defined on the space of adapted càdlàg processes. We show that for any adapted process H and any semimartingale Z there exists a unique strong solution of the following stochastic differential equation (SDE) with reflection on the boundary of D:

$$X_t = H_t + \int_0^t \langle F(X)_{s-}, dZ_s\rangle + K_t, \qquad t \in \mathbb{R}^+.$$

Our proofs are based on new *a priori* estimates for solutions of the deterministic Skorokhod problem.

1. Introduction. In the present paper we consider the following SDE with reflection on the boundary ∂D of an open convex set $D \subset \mathbb{R}^d$:

$$X_t = H_t + \int_0^t \langle F(X)_{s-}, dZ_s\rangle + K_t, \qquad t \in \mathbb{R}^+. \tag{1.1}$$

Here $Z = (Z_t)$ is an $(\mathcal{F}_t)$ adapted semimartingale with $Z_0 = 0$, $H = (H_t)$ is an $(\mathcal{F}_t)$ adapted process with $H_0 \in \overline{D} = D \cup \partial D$ and F is a Lipschitz operator on the space of adapted càdlàg processes (for a precise definition see Section 3).

The problem of existence and uniqueness of solutions of (1.1) was discussed for the first time by Skorokhod [7] in the case where $d = 1$, $D =$

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$(0,\infty)$, $H = X_0 \in [0,\infty)$ and Z is a standard Wiener process W. Next, many attempts have been made to generalize Skorokhod's results to a larger class of domains or a larger class of driving processes Z. In particular, existence and uniquness of solutions of (1.1) with $Z = W$ for an arbitrary convex set D was proved by Tanaka [11]. SDEs driven by general semimartingales were considered in detail by Słomiński [8, 9]. Let us stress, however, that in the above mentioned papers [8, 9, 11] it is assumed that $H = X_0$ and that $F(X)_{s-} = f(X_{s-})$, where f is a Lipschitz continuous function.

In the present paper we show existence and uniqueness of solutions of the SDE (1.1) for an arbitrary $(\mathcal{F}_t)$ adapted process $H = (H_t)$ with $H_0 \in \bar{D}$ and an arbitrary Lipschitz operator F, and thus we generalize the results of [8, 9, 11] considerably. The proof of our main result is based on new *a priori* estimates for the solution of the deterministic Skorokhod problem

$$x_t = y_t + k_t, \qquad t \in \mathbb{R}^+, \tag{1.2}$$

associated with a $y \in \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ such that $y_0 \in \bar{D}$. These estimates say that for any $T \in \mathbb{R}^+$ and any $a \in D$, $\sup_{t\le T}|x_t - a|$ and $|k|_T$ are bounded by constants depending only on y_0, $\sup_{t\le T}|y_t - a|$ and the modulus of continuity ω'_y. As a consequence, we prove existence and uniqueness of solutions of the Skorokhod problem (1.2) in an arbitrary open convex set D. In this way we solve Tanaka's problem (see [11, Remark 2.3]) concerning existence of a solution of the Skorokhod problem associated with $y \in \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ (in the case where y is continuous, Tanaka's problem was solved earlier by Cépa [2]).

Notation. $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ is the space of all mappings $y : \mathbb{R}^+ \to \mathbb{R}^d$ which are right-continuous and admit left-hand limits. Every process appearing in what follows is assumed to have its trajectories in the space $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ endowed with the Skorokhod topology J_1. If $X = (X^1,\ldots,X^d)$ is a semimartingale then $[X]_t = \sum_{i=1}^d [X^i]_t$, where $[X^i]$ stands for the quadratic variation process of X^i, $i = 1,\ldots,d$. Similarly, $\langle X\rangle_t = \sum_{i=1}^d \langle X^i\rangle_t$, where $\langle X^i\rangle_t$ stands for the predictable compensator of $[X^i]$, $i = 1,\ldots,d$. If $K = (K^1,\ldots,K^d)$ is a process with locally finite variation, then $|K|_t = \sum_{i=1}^d |K^i|_t$, where $|K^i|_t$ is the total variation of K^i on $[0,t]$. For $x \in \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$, $\delta > 0$, $T \in \mathbb{R}^+$ we denote by $\omega'_x(\delta,T)$ the modulus of continuity of x on $[0,T]$, i.e.

$$\omega'_x(\delta,T) = \inf\{\max_{i\le r}\omega_x([t_{i-1},t_i))\,;0 = t_0 < \cdots < t_r = T,\ \inf_{i<r}(t_i - t_{i-1}) \ge \delta\},$$

where $\omega_x(I) = \sup_{s,t\in I}|x_s - x_t|$.

2. The Skorokhod problem. Let D be an open convex domain in $\mathbb{R}^d$ and let $\mathcal{N}_x$ denote the set of inward normal unit vectors at $x \in \partial D$.

The following remark can be found in Menaldi [4] and Storm [10].

REMARK 2.1. (i) $\mathbf{n} \in \mathcal{N}_x$ if and only if $\langle y - x, \mathbf{n} \rangle \geq 0$ for every $y \in \bar{D}$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^d$.

(ii) If $\operatorname{dist}(x, \bar{D}) > 0$, then there exists a unique $\Pi(x) \in \partial D$ such that $|\Pi(x) - x| = \operatorname{dist}(x, \bar{D})$. Moreover, $(\Pi(x) - x)/|\Pi(x) - x| \in \mathcal{N}_{\Pi(x)}$.

(iii) For every $a \in D$ and $\mathbf{n} \in \mathcal{N}_x$,

$$\langle x - a, \mathbf{n} \rangle \leq -\operatorname{dist}(a, \partial D).$$

Let $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ with $y_0 \in \bar{D}$. We say that a pair (x, k) is a *solution of the Skorokhod problem* associated with y if

(2.1) $x_t = y_t + k_t$, $t \in \mathbb{R}^+$,

(2.2) $x_t \in \bar{D}$, $t \in \mathbb{R}^+$,

(2.3) k is a function with locally bounded variation, $k_0 = 0$, and

$$k_t = \int_0^t \mathbf{n}_s \, d|k|_s, \quad |k|_t = \int_0^t \mathbf{1}_{\{x_s \in \partial D\}} \, d|k|_s, \qquad t \in \mathbb{R}^+,$$

where $\mathbf{n}_s \in \mathcal{N}_{x_s}$ if $x_s \in \partial D$.

The following estimates on the solution of the Skorokhod problem will prove extremely useful in the proofs of our main results.

THEOREM 2.2. *Let (x, k) be a solution of the Skorokhod problem associated with y, and let $y_0 \in \bar{D}$. Then for any $T > 0$, $\eta > 0$ and $a \in D$ such that*

$$\omega'_y(\eta, T) < \frac{\operatorname{dist}(a, \partial D)}{2} \tag{2.4}$$

we have

(i) $\sup_{t \leq T} |x_t - a| \leq 6([T/\eta] + 1) \sup_{t \leq T} |y_t - a|$,

(ii) $|k|_T \leq \dfrac{71([T/\eta] + 1)^3}{\operatorname{dist}(a, \partial D)} \sup_{t \leq T} |y_t - a|^2$

(*$[T/\eta]$ denotes the largest integer less than or equal to T/η*).

Proof. (i) We proceed along the lines of the proof of Theorem 3.2 in [2]. Let $0 \leq t \leq T$. It is easily seen that

$$\begin{aligned}
|x_t - a|^2 &= |y_t - a|^2 + \langle k_t, k_t \rangle + 2\int_0^t \langle y_t - a, dk_u \rangle \\
&= |y_t - a|^2 + 2\int_0^t \langle k_u, dk_u \rangle - \sum_{u \leq t} |\Delta k_u|^2 + 2\int_0^t \langle y_t - a, dk_u \rangle \\
&= |y_t - a|^2 + 2\int_0^t \langle x_u - a, dk_u \rangle + 2\int_0^t \langle y_t - y_u, dk_u \rangle - \sum_{u \leq t} |\Delta k_u|^2.
\end{aligned}$$

Therefore, for any $0 \le s \le t \le T$,

$$|x_t - a|^2 - |x_s - a|^2 = |y_t - a|^2 - |y_s - a|^2 + 2\int_s^t \langle x_u - a, dk_u\rangle - 2\int_s^t \langle y_u - y_s, dk_u\rangle + 2\langle k_t, y_t - y_s\rangle - \sum_{s<u\le t} |\Delta k_u|^2.$$

By Remark 2.1(iii), $2\int_s^t \langle x_u - a, dk_u\rangle \le -2\,\mathrm{dist}(a,\partial D)|k|_s^t$, where $|k|_s^t = |k|_t - |k|_s$. Hence

$$\begin{aligned}|x_t - a|^2 - |x_s - a|^2 &\le |y_t - a|^2 - |y_s - a|^2 - 2\,\mathrm{dist}(a,\partial D)|k|_s^t \\ &\quad - 2\int_s^t \langle y_u - y_s, dk_u\rangle - 2\langle y_t - a, y_t - y_s\rangle \\ &\quad - 2\langle a - x_t, y_t - y_s\rangle - \sum_{s<u\le t} |\Delta k_u|^2 \\ &\le 5\sup_{t\le T}|y_t - a|^2 + 4\sup_{t\le T}|y_t - a|\cdot\sup_{t\le T}|x_t - a| \\ &\quad - 2\,\mathrm{dist}(a,\partial D)|k|_s^t - 2\int_s^t \langle y_u - y_s, dk_u\rangle - \sum_{s<u\le t}|\Delta k_u|^2.\end{aligned}$$

On the other hand, since $y \in \mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ it follows that there exist $\eta > 0$ and a subdivision (s_k) of $[0,T]$ such that $0 = s_0 < s_1 < \cdots < s_r = T$, $\eta \le s_k - s_{k-1}$, $k = 1,\ldots,r-1$, where $r = [T/\eta] + 1$, and

$$\omega_y([s_{k-1}, s_k)) < \frac{\mathrm{dist}(a,\partial D)}{2}. \tag{2.5}$$

Using (2.5) we obtain

$$\begin{aligned}-\int_{s_{k-1}}^{s_k} \langle y_u - y_{s_{k-1}}, dk_u\rangle &\le \Big|\int_{(s_{k-1},s_k)} \langle y_u - y_{s_{k-1}}, dk_u\rangle\Big| - \langle \Delta y_{s_k}, \Delta k_{s_k}\rangle \\ &\le \frac{\mathrm{dist}(a,\partial D)}{2}\,|k|_{s_{k-1}}^{s_k} - \langle \Delta y_{s_k}, \Delta k_{s_k}\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}&2\Big(-\int_{s_{k-1}}^{s_k} \langle y_u - y_{s_{k-1}}, dk_u\rangle - \mathrm{dist}(a,\partial D)|k|_{s_{k-1}}^{s_k}\Big) \\ &\quad\le 2\Big(\frac{\mathrm{dist}(a,\partial D)}{2}|k|_{s_{k-1}}^{s_k} - \mathrm{dist}(a,\partial D)|k|_{s_{k-1}}^{s_k} - \langle \Delta y_{s_k}, \Delta k_{s_k}\rangle\Big) \\ &\quad= -\,\mathrm{dist}(a,\partial D)|k|_{s_{k-1}}^{s_k} - 2\langle \Delta y_{s_k}, \Delta k_{s_k}\rangle,\end{aligned}$$

and, as a consequence,

$$
\begin{aligned}
(2.6)\qquad & |x_{s_k}-a|^2-|x_{s_{k-1}}-a|^2 \\
&\le 5\sup_{t\le T}|y_t-a|^2+4\sup_{t\le T}|y_t-a|\cdot\sup_{t\le T}|x_t-a| \\
&\quad -\operatorname{dist}(a,\partial D)|k|_{s_{k-1}}^{s_k}-2\langle \Delta y_{s_k},\Delta k_{s_k}\rangle-\sum_{s_{k-1}<u\le s_k}|\Delta k_u|^2 \\
&\le 5\sup_{t\le T}|y_t-a|^2+4\sup_{t\le T}|y_t-a|\cdot\sup_{t\le T}|x_t-a|-\operatorname{dist}(a,\partial D)|k|_{s_{k-1}}^{s_k}+|\Delta y_{s_k}|^2 \\
&\le 9\sup_{t\le T}|y_t-a|^2+4\sup_{t\le T}|y_t-a|\cdot\sup_{t\le T}|x_t-a|-\operatorname{dist}(a,\partial D)|k|_{s_{k-1}}^{s_k}.
\end{aligned}
$$

From (2.6) it follows immediately that

$$
|x_{s_k}-a|^2-|x_{s_{k-1}}-a|^2\le 9\sup_{t\le T}|y_t-a|^2+4\sup_{t\le T}|y_t-a|\cdot\sup_{t\le T}|x_t-a|.
$$

For given $t\in[0,T]$ set $k_0=\max\{k\,;s_k\le t\}$. Then

$$
\begin{aligned}
|x_t-a|^2 &= \sum_{k=1}^{k_0}(|x_{s_k}-a|^2-|x_{s_{k-1}}-a|^2)+|x_t-a|^2-|x_{s_{k_0}}-a|^2+|x_0-a|^2 \\
&\le r(9\sup_{t\le T}|y_t-a|^2+4\sup_{t\le T}|y_t-a|\cdot\sup_{t\le T}|x_t-a|)+\sup_{t\le T}|y_t-a|^2,
\end{aligned}
$$

which implies that

$$
\sup_{t\le T}|x_t-a|^2\le 18r^2\sup_{t\le T}|y_t-a|^2+\sup_{t\le T}|x_t-a|^2/2.
$$

Hence

$$
(2.7)\qquad \sup_{t\le T}|x_t-a|^2\le 36r^2\sup_{t\le T}|y_t-a|^2,
$$

and the proof of (i) is complete.

(ii) Using (2.6) and (2.7) gives

$$
\begin{aligned}
\operatorname{dist}(a,\partial D)|k|_{s_{k-1}}^{s_k} &\le 9\sup_{t\le T}|y_t-a|^2+4\sup_{t\le T}|y_t-a|\cdot\sup_{t\le T}|x_t-a| \\
&\quad +|x_{s_{k-1}}-a|^2-|x_{s_k}-a|^2 \\
&\le 17\sup_{t\le T}|y_t-a|^2+\frac{3}{2}\sup_{t\le T}|x_t-a|^2\le 71r^2\sup_{t\le T}|y_t-a|^2
\end{aligned}
$$

for $k=1,\dots,r$. Since $|k|_T\le\sum_{k=1}^r|k|_{s_{k-1}}^{s_k}$, this proves (ii). ■

COROLLARY 2.3. *If* $\{y^n\}$ *is relatively compact in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^d)$ *then*

(i) $\sup_n\sup_{t\le T}|x_t^n|<\infty$ *and* $\sup_n|k^n|_T<\infty$ *for every* $T>0$,

(ii) $\{(x^n,k^n)\}$ *is relatively compact in* $\mathbb{D}(\mathbb{R}^+,\mathbb{R}^{2d})$.

Proof. (i) Clearly, $\sup_n |x_0^n| = \sup_n |y_0^n| < \infty$. Since $\{y^n\}$ is relatively compact, for any $a \in D$, $T > 0$ there exists $\eta > 0$ such that $\sup_n \omega'_{y^n}(\eta, T) < \operatorname{dist}(a, \partial D)/2$. Moreover, relative compactness of $\{y^n\}$ implies that $\sup_n \sup_{t\le T} |y_t^n| < \infty$. Therefore (i) follows from Theorem 2.2.

(ii) Since $\{y^n\}$ is relatively compact, for any $T > 0$ and $\varepsilon > 0$ there exist $\delta > 0$ and $0 = s_0 < s_1 < \dots < s_r = T$ such that $\delta \le s_k - s_{k-1}$ and $\omega_{y^n}([s_{k-1}, s_k)) \le \varepsilon$ for $k = 1, \dots, r-1$. By [11, Lemma 2.2], for $n \in \mathbb{N}$ and $t, s \in \mathbb{R}^+$ we have

$$|x_t^n - x_s^n|^2 \le |y_t^n - y_s^n|^2 + \int_s^t \langle y_t^n - y_u^n, dk_u^n\rangle.$$

Therefore

$$\sup_{s_{k-1}\le t<s_k} |x_t^n - x_{s_{k-1}}^n|^2 \le \omega_{y^n}^2([s_{k-1}, s_k)) + \omega_{y^n}([s_{k-1}, s_k))|k^n|_T \le \varepsilon(\varepsilon + |k^n|_T),$$

and hence, $\max_{k\le r} \omega_{x^n}([s_{k-1}, s_k)) \le (\varepsilon(\varepsilon + |k^n|_T))^{1/2}$. As a consequence,

$$\lim_{\delta\downarrow 0} \sup_n \omega'_{(x^n, y^n)}(\delta, T) = 0,$$

which together with (i) shows that $\{(x^n, y^n)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d})$. Since $k^n = x^n - y^n$, (ii) follows. ∎

COROLLARY 2.4. *Let (x, k) be a solution of the Skorokhod problem associated with y such that $y_0 \in \overline{D}$, and let $a \in D$. Set $\tau = \inf\{t > 0\,; |y_t - y_0| \ge \operatorname{dist}(a, \partial D)/2\}$. Then*

$$\sup_{t<\tau} |x_t - a| \le 6 \sup_{t<\tau} |y_t - a|, \qquad |k|_{\tau-} \le \frac{71}{\operatorname{dist}(a, \partial D)} \sup_{t<\tau} |y_t - a|^2.$$

Proof. It suffices to put $r = 1$, $s_0 = 0$, $s_1 = \tau$ in the proof of Theorem 2.2. ∎

THEOREM 2.5. *Assume $\{y^n\} \subset \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, $y_0^n \in \overline{D}$ and let (x^n, k^n) denote the solution of the Skorokhod problem associated with y^n, $n \in \mathbb{N}$. If $y^n \to y$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ and $y_0 \in \overline{D}$ then there exists a unique solution (x, k) of the Skorokhod problem associated with y and*

$$(x^n, k^n) \to (x, k) \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}).$$

Proof. By Corollary 2.3(ii), the sequence $\{(x^n, y^n, k^n)\}$ is relatively compact in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$. Therefore, there exists a subsequence $(n') \subset (n)$ and a pair (x', k') such that

$$(x^{n'}, y^{n'}, k^{n'}) \to (x', y, k') \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d}). \tag{2.8}$$

By [11, Lemma 2.2] the solution of the Skorokhod problem associated with y is unique. Therefore, the proof is completed by showing that (x', k') is a solution of the Skorokhod problem. Obviously $x' = y + k'$. Moreover, by Corollary 2.3(i), $\sup_n |k^n|_T < \infty$, $T \in \mathbb{R}^+$, which implies that $|k'|_T < \infty$,

$T \in \mathbb{R}^+$. To check (2.3), we first note that it is equivalent to the following two conditions: for any bounded continuous $f : \bar{D} \to \mathbb{R}^d$ such that $f(x) = 0$ for $x \in \partial D$ we have

$$\int_0^t \langle f(x_s), dk_s \rangle = 0, \quad t \in \mathbb{R}^+, \tag{2.9}$$

and for any continuous $\widehat{x} : \mathbb{R}^+ \to \bar{D}$ the function

$$t \mapsto \int_0^t \langle \widehat{x}_s - x_s, dk_s \rangle,\ t \in \mathbb{R}^+, \text{ is non-decreasing} \tag{2.10}$$

(see e.g. [1]).

By (2.8) and [3, Proposition 2.9],

$$\int_0^t \langle f(x_s^{n'}), dk_s^{n'} \rangle \to \int_0^t \langle f(x'_s), dk'_s \rangle \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}). \tag{2.11}$$

On the other hand, since $(x^{n'}, k^{n'})$ is a solution of the Skorokhod problem, for each n' we have $\int_0^t \langle f(x_s^{n'}), dk_s^{n'} \rangle = 0$. Therefore (2.11) gives (2.9). Furthermore, by (2.8), $(\widehat{x}, x^{n'}, k^{n'}) \to (\widehat{x}, x', k')$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^{3d})$. Hence, using once again [3, Proposition 2.9], we obtain

$$\int_0^t \langle \widehat{x}_s - x_s^{n'}, dk_s^{n'} \rangle \to \int_0^t \langle \widehat{x}_s - x'_s, dk'_s \rangle \quad \text{in } \mathbb{D}(\mathbb{R}^+, \mathbb{R}),$$

which implies (2.10), because the functions $t \mapsto \int_0^t \langle \widehat{x}_s - x_s^{n'}, dk_s^{n'} \rangle$ are non-decreasing. ∎

THEOREM 2.6. *For every $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $y_0 \in \bar{D}$ there exists a unique solution of the Skorokhod problem associated with y.*

Proof. Let $\{y^n\}$ be the sequence of discretizations of y defined by $y_t^n = y_{k/n}$, $t \in [k/n, (k+1)/n)$, $n \in \mathbb{N}$. We check at once that for every $n \in \mathbb{N}$ the pair (x^n, k^n) defined by

$$\begin{cases} x_0^n = y_0, \\ x_{(k+1)/n}^n = \Pi(x_{k/n}^n + y_{(k+1)/n} - y_{k/n}) \end{cases}$$

and $x_t^n = x_{k/n}^n$, $k_t^n = x_t^n - y_t^n$, $t \in [k/n, (k+1)/n)$, $n \in \mathbb{N}$, solves the Skorokhod problem for y^n. Since $y^n \to y$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, the result follows from Theorem 2.5. ∎

3. SDEs in convex domains. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space and let Y be an $(\mathcal{F}_t)$ adapted process with $Y_0 \in \bar{D}$.

We say that a pair (X, K) of $(\mathcal{F}_t)$ adapted processes *solves the Skorokhod problem associated with* Y if for almost every $\omega \in \Omega$ the pair $(X(\omega), K(\omega))$ is a solution of the Skorokhod problem associated with $Y(\omega)$.

From Theorem 2.6 it follows that for any process Y with $Y_0 \in \overline{D}$ there exists a unique solution of the Skorokhod problem associated with Y. The following remark is due to Słomiński (see [8, Corollary 1]).

REMARK 3.1. Let Y, $\widehat{Y}$ be $(\mathcal{F}_t)$ adapted processes of the form $Y = H + M + V$, $\widehat{Y} = H + \widehat{M} + \widehat{V}$, where $M, \widehat{M}$ are local martingales, $V, \widehat{V}$ are processes with locally bounded variation and $M_0 = \widehat{M_0} = V_0 = \widehat{V_0} = 0$. If (X, K), $(\widehat{X}, \widehat{K})$ are solutions of the Skorokhod problem associated with $Y, \widehat{Y}$, respectively, then for every $p \in \mathbb{N}$ there exists C_p such that

$$E \sup_{t<\tau} |X_t - \widehat{X}_t|^{2p} \le C_p E([M - \widehat{M}]^p_{\tau-} + |V - \widehat{V}|^{2p}_{\tau-} + \langle M - \widehat{M}\rangle^p_{\tau-})$$

for every $(\mathcal{F}_t)$ stopping time τ.

Let us denote by $\mathbb{F}^d$ the class of d-dimensional $(\mathcal{F}_t)$ adapted processes and by $\mathbb{M}^d$ the class of $(\mathcal{F}_t)$ adapted processes with values in the set $\mathbb{R}^d \otimes \mathbb{R}^d$ of d-dimensional matrices.

We say that an operator $F : \mathbb{F}^d \to \mathbb{M}^d$ is *Lipschitz* if

(i) for every $X, Y \in \mathbb{F}^d$ and every stopping time τ, if $X^{\tau-} = Y^{\tau-}$ then $F(X)^{\tau-} = F(Y)^{\tau-}$,

(ii) there exists a one-dimensional, $(\mathcal{F}_t)$ adapted, non-decreasing process $L = (L_t)$ such that $P(\sup_t L_t < \infty) = 1$ and, for every $X, Y \in \mathbb{F}^d$,

$$\|F(X)_t - F(Y)_t\| \le L_t \sup_{s\le t} |X_s - Y_s|, \qquad t \in \mathbb{R}^+.$$

Clearly, if $f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is Lipschitz, that is, there exists $L > 0$ such that $\|f(x) - f(y)\| \le L|x - y|$ for $x, y \in \mathbb{R}^d$, then the operator $F(X)$ defined by $F(X)_t = f(X_t)$ for $X \in \mathbb{F}^d$ is Lipschitz.

Let $F : \mathbb{F}^d \to \mathbb{M}^d$. We say that a pair (X, K) of $(\mathcal{F}_t)$ adapted processes is a *strong solution* of the SDE (1.1) if (1.1) is satisfied and (X, K) is a solution of the Skorokhod problem associated with $Y_t = H_t + \int_0^t \langle F(X)_{s-}, dZ_s\rangle$, $t \in \mathbb{R}^+$.

We can now formulate our main result.

THEOREM 3.2. *Let H be an $(\mathcal{F}_t)$ adapted process such that $H_0 \in \overline{D}$, and let Z be an $(\mathcal{F}_t)$ adapted semimartingale with $Z_0 = 0$. Then for any Lipschitz operator $F : \mathbb{F}^d \to \mathbb{M}^d$ there exists a unique strong solution of the SDE* (1.1).

Proof. By using the arguments from the proof of [6, Chapter V, Theorem 7], we may and do assume that $F(0)_t = 0$ and $L_t < L$ for some constant

$L > 0$. Since Z is a semimartingale, it admits a unique decomposition

$$Z_t = J_t + M_t + B_t, \qquad t \in \mathbb{R}^+,$$

where $J_t = \sum_{s\le t} \Delta Z_u \mathbb{I}\{|\Delta Z_s| > 1]\}$, M is a local square-integrable martingale, $|\Delta M| \le 2$, and B is a predictable process with locally bounded variation, $|\Delta B| \le 1$. Let C_1 be a constant from Remark 3.1. For given $a' \in D$ set $a^2 = (12C_1L^2)^{-1}$ and $\tau' = \inf\{t > 0\,; |H_t - H_0| \ge \mathrm{dist}(a', \partial D)/2\}$.

We first prove existence and uniqueness of a solution of (1.1) on the interval $[0, \tau[$, where $\tau = \inf\{t > 0\,; \max([M]_t, \langle M\rangle_t, |B|_t^2, |J|_t^2) > a^2\} \wedge \tau' \wedge 1$. To this end we set $\tau_k = \inf\{t > 0\,; \sup_{s\le t}|H_s| > k\} \wedge \tau$ for $k \in \mathbb{N}$. For fixed $k \in \mathbb{N}$ we denote by S^2 the class of $(\mathcal{F}_t)$ adapted processes $Y = (Y_t)$ on $[0,1]$ such that $Y_0 \in \bar{D}$, $Y = Y^{\tau_k-}$ and $E\sup_{t\le 1}|Y_t|^2 < \infty$. Then S^2 is a Banach space with the norm $\|Y\|_{S^2} = (E\sup_{t\le 1}|Y_t|^2)^{1/2}$. Define the mapping Φ on S^2 by letting $\Phi(Y)$ be the first component X of the solution (X, K) of the Skorokhod problem associated with $H^{\tau_k-} + \int_0^{\cdot} F(Y)_{s-}\,dZ_s^{\tau_k-}$. We will show that Φ is a contraction mapping on S^2. To see this, we first observe that $\Phi(Y \equiv 0) = H^{\tau_k-} + K^{\tau_k-}$, since $F(0) \equiv 0$. Hence

$$E\sup_{t\le 1}|\Phi(0)_t|^2 \le 2E\sup_{t\le 1}|H_t^{\tau_k-}|^2 + 2E\sup_{t\le 1}|K_t^{\tau_k-}|^2 \le 2k^2 + 2E\sup_{t\le 1}|K_t^{\tau_k-}|^2.$$

Therefore, by Corollary 2.4, $\Phi(Y) \in S^2$. Furthermore, by Remark 3.1, for any $Y, \widehat{Y} \in S^2$ we have

$$\begin{aligned}
&E\sup_{t\le 1}|\Phi(Y)_t - \Phi(\widehat{Y})_t|^2 \\
&\qquad \le C_1\Big(E\int_0^{\tau_k-} \|F(Y)_{s-} - F(\widehat{Y})_{s-}\|^2\, d([M]_s + \langle M\rangle_s) \\
&\qquad\quad + E\Big(|B+J|_{\tau_k-} \cdot \int_0^{\tau_k-} \|F(Y)_{s-} - F(\widehat{Y})_{s-}\|^2\, d|B+J|_s\Big)\Big) \\
&\qquad \le C_1(E\sup_{t\le 1}\|F(Y)_{t-} - F(\widehat{Y})_{t-}\|^2([M]_{\tau_k-} + \langle M\rangle_{\tau_k-}) \\
&\qquad\quad + E\sup_{t\le 1}\|F(Y)_{t-} - F(\widehat{Y})_{t-}\|^2|B+J|_{\tau_k-}^2) \\
&\qquad \le 6C_1a^2E\sup_{t\le 1}\|F(Y)_{t-} - F(\widehat{Y})_{t-}\|^2 \\
&\qquad \le 6C_1a^2L^2E\sup_{t\le 1}|Y_t - \widehat{Y}_t|^2 = \frac{1}{2}E\sup_{t\le 1}|Y_t - \widehat{Y}_t|^2.
\end{aligned}$$

From the above we see that $\Phi : S^2 \to S^2$ is a contraction. Hence, by the Banach contraction principle, it has a fixed point X^k, which is a unique solution of (1.1) on $[0,\tau_k[$. Since $P(\tau_k=\tau)\uparrow 1$, putting $X=X^k$ on $[0,\tau_k[$ we obtain a unique solution on $[0,\tau[$. Moreover, putting

$$X_\tau = \Pi(X_{\tau-}+\Delta H_\tau+\langle F(X)_{\tau-},\Delta Z_\tau\rangle)$$

we obtain a solution on $[0,\tau]$, because $F(X^{\tau-})_{\tau-}=F(X)_{\tau-}$ by the definition of F.

Now, we define a sequence of stopping times

$$\tau^0=\tau,\qquad \tau^{k+1}=\tau^k+\inf\{t>0\,;\max([\overline{M}]_t,|\overline{B}|_t^2,|\bar{J}|_t^2)>a^2\}\wedge(\tau'_k\wedge 1),$$

where $\overline{M}.=M_{\tau^k+\cdot}-M_{\tau^k}$, $\overline{B}.=B_{\tau^k+\cdot}-B_{\tau^k}$, $\bar{J}.=J_{\tau^k+\cdot}-J_{\tau^k}$, $\overline{H}.=H_{\tau^k+\cdot}-H_{\tau^k}$, $\tau'_k=\inf\{t>0,|\overline{H}_t|\ge\operatorname{dist}(a',\partial D)/2\}$. By what has been proved there exists a unique solution of (1.1) on $[0,\tau^0]$. By the same method as above, having a unique solution on $[0,\tau^k]$, we can construct a solution of (1.1) on $[0,\tau^{k+1}]$. Since $\tau^k\uparrow\infty$, the theorem follows. ∎

Corollary 3.3. *If $f:\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^d$ is Lipschitz then for every $(\mathcal{F}_t)$ adapted process H with $H_0\in\overline{D}$ and every $(\mathcal{F}_t)$ adapted semimartingale Z with $Z_0=0$ there exists a unique strong solution of the SDE*

$$X_t=H_t+\int_0^t\langle f(X_{s-}),dZ_s\rangle+K_t,\quad t\in\mathbb{R}^+.\ ∎$$

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