On Stochastic Differential Equations with Reflecting Boundary Condition in Convex Domains

by

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Summary. Let \( D \) be an open convex set in \( \mathbb{R}^d \) and let \( F \) be a Lipschitz operator defined on the space of adapted càdlàg processes. We show that for any adapted process \( H \) and any semimartingale \( Z \) there exists a unique strong solution of the following stochastic differential equation (SDE) with reflection on the boundary of \( D \):

\[
X_t = H_t + \int_0^t F(X_s^{-}, dZ_s) + K_t, \quad t \in \mathbb{R}^+.
\]

Our proofs are based on new \textit{a priori} estimates for solutions of the deterministic Skorokhod problem.

1. Introduction. In the present paper we consider the following SDE with reflection on the boundary \( \partial D \) of an open convex set \( D \subset \mathbb{R}^d \):

(1.1) \[
X_t = H_t + \int_0^t \langle F(X_s^{-}, dZ_s) + K_t, \quad t \in \mathbb{R}^+.
\]

Here \( Z = (Z_t) \) is an \( (\mathcal{F}_t) \) adapted semimartingale with \( Z_0 = 0 \), \( H = (H_t) \) is an \( (\mathcal{F}_t) \) adapted process with \( H_0 \in \bar{D} = D \cup \partial D \) and \( F \) is a Lipschitz operator on the space of adapted càdlàg processes (for a precise definition see Section 3).

The problem of existence and uniqueness of solutions of (1.1) was discussed for the first time by Skorokhod [7] in the case where \( d = 1 \), \( D = \)

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(0, \infty), H = X_0 \in [0, \infty) and Z is a standard Wiener process \(W\). Next, many attempts have been made to generalize Skorokhod’s results to a larger class of domains or a larger class of driving processes \(Z\). In particular, existence and uniqueness of solutions of (1.1) with \(Z = W\) for an arbitrary convex set \(D\) was proved by Tanaka [11]. SDEs driven by general semimartingales were considered in detail by Śomiński [8, 9]. Let us stress, however, that in the above mentioned papers [8, 9, 11] it is assumed that \(H = X_0\) and that \(F(X)_{s-} = f(X_{s-})\), where \(f\) is a Lipschitz continuous function.

In the present paper we show existence and uniqueness of solutions of the SDE (1.1) for an arbitrary \((\mathcal{F}_t)\) adapted process \(H = (H_t)\) with \(H_0 \in \bar{D}\) and an arbitrary Lipschitz operator \(F\), and thus we generalize the results of [8, 9, 11] considerably. The proof of our main result is based on new a priori estimates for the solution of the deterministic Skorokhod problem

\[(1.2)\]

\[x_t = y_t + k_t, \quad t \in \mathbb{R}^+,\]

associated with a \(y \in \mathbb{D}([0, \infty); \mathbb{R}^d)\) such that \(y_0 \in \bar{D}\). These estimates say that for any \(T \in \mathbb{R}^+\) and any \(a \in D\), \(\sup_{t \leq T} |x_t - a|\) and \(|k|_T\) are bounded by constants depending only on \(y_0\), \(\sup_{t \leq T} |y_t - a|\) and the modulus of continuity \(\omega'_y\). As a consequence, we prove existence and uniqueness of solutions of the Skorokhod problem (1.2) in an arbitrary open convex set \(D\). In this way we solve Tanaka’s problem (see [11, Remark 2.3]) concerning existence of a solution of the Skorokhod problem associated with \(y \in \mathbb{D}([0, \infty); \mathbb{R}^d)\) (in the case where \(y\) is continuous, Tanaka’s problem was solved earlier by Cépa [2]).

**Notation.** \(\mathbb{D}([0, \infty); \mathbb{R}^d)\) is the space of all mappings \(y : \mathbb{R}^+ \to \mathbb{R}^d\) which are right-continuous and admit left-hand limits. Every process appearing in what follows is assumed to have its trajectories in the space \(\mathbb{D}([0, \infty); \mathbb{R}^d)\) endowed with the Skorokhod topology \(J_1\). If \(X = (X^1, \ldots, X^d)\) is a semimartingale then \([X]_t = \sum_{i=1}^d [X^i]_t\), where \([X^i]_t\) stands for the quadratic variation process of \(X^i\), \(i = 1, \ldots, d\). Similarly, \((X)_t = \sum_{i=1}^d (X^i)_t\), where \((X^i)_t\) stands for the predictable compensator of \([X^i], i = 1, \ldots, d\). If \(K = (K^1, \ldots, K^d)\) is a process with locally finite variation, then \(|K|_t = \sum_{i=1}^d |K^i|_t\), where \(|K^i|_t\) is the total variation of \(K^i\) on \([0, t]\). For \(x \in \mathbb{D}([0, \infty); \mathbb{R}^d), \delta > 0, T \in \mathbb{R}^+\) we denote by \(\omega_x'(\delta, T)\) the modulus of continuity of \(x\) on \([0, T]\), i.e.

\[\omega_x'(\delta, T) = \inf \{\max_{i \leq r} \omega_x([t_{i-1}, t_i]); 0 = t_0 < \cdots < t_r = T, \inf_{i < r} (t_i - t_{i-1}) \geq \delta\},\]

where \(\omega_x(I) = \sup_{s, t \in I} |x_s - x_t|\).

2. The Skorokhod problem. Let \(D\) be an open convex domain in \(\mathbb{R}^d\) and let \(\mathcal{N}_x\) denote the set of inward normal unit vectors at \(x \in \partial D\).

The following remark can be found in Menaldi [4] and Storm [10].
Remark 2.1. (i) \( n \in \mathcal{N}_x \) if and only if \( \langle y - x, n \rangle \geq 0 \) for every \( y \in \overline{D} \), where \( \langle \cdot, \cdot \rangle \) denotes the usual inner product in \( \mathbb{R}^d \).

(ii) If \( \text{dist}(x, \overline{D}) > 0 \), then there exists a unique \( \Pi(x) \in \partial D \) such that \( |\Pi(x) - x| = \text{dist}(x, \overline{D}) \). Moreover, \( (\Pi(x) - x)/|\Pi(x) - x| \in \mathcal{N}_{\Pi(x)} \).

(iii) For every \( a \in D \) and \( n \in \mathcal{N}_x \),

\[
\langle x - a, n \rangle \leq -\text{dist}(a, \partial D).
\]

Let \( y \in \mathcal{D}(\mathbb{R}^+, \mathbb{R}^d) \) with \( y_0 \in \overline{D} \). We say that a pair \((x, k)\) is a solution of the Skorokhod problem associated with \( y \) if

\[
\begin{align*}
(2.1) \quad x_t &= y_t + k_t, \quad t \in \mathbb{R}^+, \\
(2.2) \quad x_t &\in \overline{D}, \quad t \in \mathbb{R}^+, \\
(2.3) \quad k &\text{ is a function with locally bounded variation, } k_0 = 0, \text{ and }
\end{align*}
\]

\[
k_t = \int_0^t n_s \, d|k|_s, \quad |k|_t = \int_0^t 1_{\{x_s \in \partial D\}} \, d|k|_s, \quad t \in \mathbb{R}^+,
\]

where \( n_s \in \mathcal{N}_{x_s} \) if \( x_s \in \partial D \).

The following estimates on the solution of the Skorokhod problem will prove extremely useful in the proofs of our main results.

Theorem 2.2. Let \((x, k)\) be a solution of the Skorokhod problem associated with \( y \), and let \( y_0 \in \overline{D} \). Then for any \( T > 0, \eta > 0 \) and \( a \in D \) such that

\[
\omega'_y(\eta, T) < \frac{\text{dist}(a, \partial D)}{2}
\]

we have

\[
\begin{align*}
(\text{i}) \quad &\sup_{t \leq T} |x_t - a| \leq 6([T/\eta] + 1) \sup_{t \leq T} |y_t - a|, \\
(\text{ii}) \quad &|k|_T \leq \frac{71([T/\eta] + 1)^3}{\text{dist}(a, \partial D)} \sup_{t \leq T} |y_t - a|^2
\end{align*}
\]

([T/\eta] denotes the largest integer less than or equal to \( T/\eta \)).

Proof. (i) We proceed along the lines of the proof of Theorem 3.2 in [2]. Let \( 0 \leq t \leq T \). It is easily seen that

\[
|x_t - a|^2 = |y_t - a|^2 + \langle k_t, k_t \rangle + 2 \int_0^t \langle y_t - a, dk_u \rangle
\]

\[
= |y_t - a|^2 + 2 \int_0^t \langle k_u, dk_u \rangle - \sum_{u \leq t} |\Delta k_u|^2 + 2 \int_0^t \langle y_t - a, dk_u \rangle
\]

\[
= |y_t - a|^2 + 2 \int_0^t \langle x_u - a, dk_u \rangle + 2 \int_0^t \langle y_t - y_u, dk_u \rangle - \sum_{u \leq t} |\Delta k_u|^2.
\]
Therefore, for any $0 \leq s \leq t \leq T$,
\[
|x_t - a|^2 - |x_s - a|^2 = |y_t - a|^2 - |y_s - a|^2 + 2\int_s^t \langle x_u - a, dk_u \rangle
\]
\[
- 2\int_s^t \langle y_u - y_s, dk_u \rangle + 2\langle k_t, y_t - y_s \rangle - \sum_{s<u\leq t} |\Delta k_u|^2.
\]
By Remark 2.1(iii), $2\int_s^t \langle x_u - a, dk_u \rangle \leq -2\text{dist}(a, \partial D)|k|^t_s$, where $|k|^t_s = |k_t - |k||_s$. Hence
\[
|x_t - a|^2 - |x_s - a|^2 \leq |y_t - a|^2 - |y_s - a|^2 - 2\text{dist}(a, \partial D)|k|^t_s
\]
\[
- 2\int_s^t \langle y_u - y_s, dk_u \rangle + 2\langle y_t - a, y_t - y_s \rangle
\]
\[
- 2\langle a - x_t, y_t - y_s \rangle - \sum_{s<u\leq t} |\Delta k_u|^2
\]
\[
\leq 5\sup_{t \leq T} |y_t - a|^2 + 4\sup_{t \leq T} |y_t - a| \cdot \sup_{t \leq T} |x_t - a|
\]
\[
- 2\text{dist}(a, \partial D)|k|^t_s - 2\int_s^t \langle y_u - y_s, dk_u \rangle - \sum_{s<u\leq t} |\Delta k_u|^2.
\]
On the other hand, since $y \in D(\mathbb{R}^+, \mathbb{R}^d)$ it follows that there exist $\eta > 0$ and a subdivision $(s_k)$ of $[0, T]$ such that $0 = s_0 < s_1 < \cdots < s_r = T$, $\eta \leq s_k - s_{k-1}$, $k = 1, \ldots, r-1$, where $r = \lceil T/\eta \rceil + 1$, and
\[
(2.5) \quad \omega_y([s_{k-1}, s_k]) < \frac{\text{dist}(a, \partial D)}{2}.
\]
Using (2.5) we obtain
\[
- \int_{s_{k-1}}^{s_k} \langle y_u - y_{s_{k-1}}, dk_u \rangle \leq \Bigg| \int_{(s_{k-1}, s_k)} \langle y_u - y_{s_{k-1}}, dk_u \rangle \Bigg| - \langle \Delta y_{s_k}, \Delta k_{s_k} \rangle
\]
\[
\leq \frac{\text{dist}(a, \partial D)}{2} |k_{s_{k-1}}^{s_k} - \langle \Delta y_{s_k}, \Delta k_{s_k} \rangle|.
\]
Therefore
\[
2\left( - \int_{s_{k-1}}^{s_k} \langle y_u - y_{s_{k-1}}, dk_u \rangle - \text{dist}(a, \partial D)|k_{s_{k-1}}^{s_k}| \right)
\]
\[
\leq 2\left( \frac{\text{dist}(a, \partial D)}{2} |k_{s_{k-1}}^{s_k} - \text{dist}(a, \partial D)|k_{s_{k-1}}^{s_k} - \langle \Delta y_{s_k}, \Delta k_{s_k} \rangle \right)
\]
\[
= - \text{dist}(a, \partial D)|k_{s_{k-1}}^{s_k} - 2\langle \Delta y_{s_k}, \Delta k_{s_k} \rangle,
\]

and, as a consequence,

\begin{equation}
(2.6) \quad |x_{s_k} - a|^2 - |x_{s_{k-1}} - a|^2 \\
\leq 5 \sup_{t \leq T} |y_t - a|^2 + 4 \sup_{t \leq T} |y_t - a| \cdot \sup_{t \leq T} |x_t - a| \\
- \text{dist}(a, \partial D)||k_{s_k}^{s_k} - 2(\Delta y_{s_k}, \Delta k_{s_k}) - \sum_{s_{k-1} < u \leq s_k} |\Delta k_u|^2 \\
\leq 5 \sup_{t \leq T} |y_t - a|^2 + 4 \sup_{t \leq T} |y_t - a| \cdot \sup_{t \leq T} |x_t - a| - \text{dist}(a, \partial D)||k_{s_k}^{s_k} + |\Delta y_{s_k}|^2 \\
\leq 9 \sup_{t \leq T} |y_t - a|^2 + 4 \sup_{t \leq T} |y_t - a| \cdot \sup_{t \leq T} |x_t - a| - \text{dist}(a, \partial D)||k_{s_k}^{s_k}.
\end{equation}

From (2.6) it follows immediately that

\[ |x_{s_k} - a|^2 - |x_{s_{k-1}} - a|^2 \leq 9 \sup_{t \leq T} |y_t - a|^2 + 4 \sup_{t \leq T} |y_t - a| \cdot \sup_{t \leq T} |x_t - a|. \]

For given \( t \in [0, T] \) set \( k_0 = \max\{k : s_k \leq t\} \). Then

\[ |x_t - a|^2 = \sum_{k=1}^{k_0} (|x_{s_k} - a|^2 - |x_{s_{k-1}} - a|^2) + |x_t - a|^2 - |x_{s_{k_0}} - a|^2 + |x_0 - a|^2 \\
\leq r (9 \sup_{t \leq T} |y_t - a|^2 + 4 \sup_{t \leq T} |y_t - a| \cdot \sup_{t \leq T} |x_t - a|) + \sup_{t \leq T} |y_t - a|^2, \]

which implies that

\[ \sup_{t \leq T} |x_t - a|^2 \leq 18 r^2 \sup_{t \leq T} |y_t - a|^2 + \sup_{t \leq T} |x_t - a|^2 / 2. \]

Hence

\begin{equation}
(2.7) \quad \sup_{t \leq T} |x_t - a|^2 \leq 36 r^2 \sup_{t \leq T} |y_t - a|^2,
\end{equation}

and the proof of (i) is complete.

(ii) Using (2.6) and (2.7) gives

\[ \text{dist}(a, \partial D)||k_{s_k}^{s_k} \leq 9 \sup_{t \leq T} |y_t - a|^2 + 4 \sup_{t \leq T} |y_t - a| \cdot \sup_{t \leq T} |x_t - a| \\
+ |x_{s_{k-1}} - a|^2 - |x_{s_k} - a|^2 \\
\leq 17 \sup_{t \leq T} |y_t - a|^2 + \frac{3}{2} \sup_{t \leq T} |x_t - a|^2 \leq 71 r^2 \sup_{t \leq T} |y_t - a|^2 \]

for \( k = 1, \ldots, r \). Since \( |k_T| \leq \sum_{k=1}^{r} |k_{s_k}^{s_k}|, \) this proves (ii). \( \blacksquare \)

Corollary 2.3. If \( \{y^n\} \) is relatively compact in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d) \) then

(i) \( \sup_{n} \sup_{t \leq T} |x^n_t| < \infty \) and \( \sup_{n} |k^n|_T < \infty \) for every \( T > 0 \),

(ii) \( \{(x^n, k^n)\} \) is relatively compact in \( \mathbb{D}(\mathbb{R}^+, \mathbb{R}^{2d}) \).
Proof. (i) Clearly, sup\(n x^n_0 = \sup_n |y^n_0| < \infty\). Since \(\{y^n\}\) is relatively compact, for any \(a \in D, T > 0\) there exists \(\eta > 0\) such that sup\(n \omega_{y^n}(\eta, T) < \text{dist}(a, \partial D)/2\). Moreover, relative compactness of \(\{y^n\}\) implies that sup\(n \sup_{t \leq T} |y^n| < \infty\). Therefore (i) follows from Theorem 2.2.

(ii) Since \(\{y^n\}\) is relatively compact, for any \(T > 0\) and \(\varepsilon > 0\) there exist \(\delta > 0\) and \(0 = n_0 < n_1 < \cdots < n_r = T\) such that \(\delta \leq n_{k-1} - n_k\) and \(\omega_{y^n}([-n_{k-1}, n_k]) \leq \varepsilon\) for \(k = 1, \ldots, r-1\). By [11, Lemma 2.2], for \(n \in \mathbb{N}\) and \(t, s \in \mathbb{R}^+\) we have

\[
|x^n_t - x^n_s|^2 \leq |y^n_t - y^n_s|^2 + \sum_{s \leq t} |y^n_u - y^n_n, dk_u^n|.
\]

Therefore

\[
\sup_{s_{k-1} \leq t < s_k} |x^n_t - x^n_{s_{k-1}}|^2 \leq \omega_{y^n}([s_{k-1}, s_k]) + \omega_{y^n}([s_{k-1}, s_k]) |k^n| T \leq \varepsilon(\varepsilon + |k^n| T),
\]

and hence, max\(k \leq r\) \(\omega_{x^n}([s_{k-1}, s_k]) \leq (\varepsilon(\varepsilon + |k^n| T))^{1/2}\). As a consequence,

\[
\lim_{t \downarrow 0} \sup_{n} \omega_{(x^n, y^n)}(\delta, T) = 0,
\]

which together with (i) shows that \(\{(x^n, y^n)\}\) is relatively compact in \(D(\mathbb{R}^+, \mathbb{R}^{2d})\). Since \(k^n = x^n - y^n\), (ii) follows. ■

**Corollary 2.4.** Let \((x, k)\) be a solution of the Skorokhod problem associated with \(y\) such that \(y_0 \in \overline{D}\), and let \(a \in D\). Set \(\tau = \inf\{t > 0; |y_t - y_0| \geq \text{dist}(a, \partial D)/2\}\). Then

\[
\sup_{t < \tau} |x_t - a| \leq 6 \sup_{t < \tau} |y_t - a|, \quad |k|_{\tau} \leq \frac{71}{\text{dist}(a, \partial D)} \sup_{t < \tau} |y_t - a|^2.
\]

**Proof.** It suffices to put \(r = 1, s_0 = 0, s_1 = \tau\) in the proof of Theorem 2.2. ■

**Theorem 2.5.** Assume \(\{y^n\} \subset D(\mathbb{R}^+, \mathbb{R}^d)\), \(y^n_0 \in \overline{D}\) and let \((x^n, k^n)\) denote the solution of the Skorokhod problem associated with \(y^n\), \(n \in \mathbb{N}\). If \(y^n \to y\) in \(D(\mathbb{R}^+, \mathbb{R}^d)\) and \(y_0 \in \overline{D}\) then there exists a unique solution \((x, k)\) of the Skorokhod problem associated with \(y\) and

\[
(x^n, k^n) \to (x, k) \quad \text{in} \ D(\mathbb{R}^+, \mathbb{R}^{2d}).
\]

**Proof.** By Corollary 2.3(ii), the sequence \(\{(x^n, y^n, k^n)\}\) is relatively compact in \(D(\mathbb{R}^+, \mathbb{R}^{3d})\). Therefore, there exists a subsequence \((n') \subset (n)\) and a pair \((x', k')\) such that

\[
(x'^n, y^n, k'^n) \to (x', y, k') \quad \text{in} \ D(\mathbb{R}^+, \mathbb{R}^{3d}).
\]

By [11, Lemma 2.2] the solution of the Skorokhod problem associated with \(y\) is unique. Therefore, the proof is completed by showing that \((x', k')\) is a solution of the Skorokhod problem. Obviously \(x' = y + k'\). Moreover, by Corollary 2.3(i), sup\(n |k^n|_T < \infty\), \(T \in \mathbb{R}^+\), which implies that \(|k'|_T < \infty\),

To check (2.3), we first note that it is equivalent to the following two conditions: for any bounded continuous $f : \overline{D} \to \mathbb{R}^d$ such that $f(x) = 0$ for $x \in \partial D$ we have

$$\int_0^t \langle f(x_s), dk_s \rangle = 0, \quad t \in \mathbb{R}^+,$$

(2.9)

and for any continuous $\widehat{x} : \mathbb{R}^+ \to \overline{D}$ the function

$$t \mapsto \int_0^t \langle \widehat{x}_s - x_s, dk_s \rangle, \quad t \in \mathbb{R}^+,$$

(2.10)

is non-decreasing (see e.g. [1]).

By (2.8) and [3, Proposition 2.9],

$$\int_0^t \langle f(x^n_{s}), dk^n_{s} \rangle \to \int_0^t \langle f(x'_s), dk'_s \rangle \quad \text{in} \, \mathbb{D}(\mathbb{R}^+, \mathbb{R}).$$

(2.11)

On the other hand, since $(x^n_{s}, k^n_{s})$ is a solution of the Skorokhod problem, for each $n'$ we have $\int_0^t \langle f(x^n_{s}), dk^n_{s} \rangle = 0$. Therefore (2.11) gives (2.9). Furthermore, by (2.8), $(\widehat{x}, x'^n_{s}, k'^n_{s}) \to (\widehat{x}, x'_s, k')$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$. Hence, using once again [3, Proposition 2.9], we obtain

$$\int_0^t \langle \widehat{x}_s - x'^n_{s}, dk'^n_{s} \rangle \to \int_0^t \langle \widehat{x}_s - x'_s, dk'_s \rangle \quad \text{in} \, \mathbb{D}(\mathbb{R}^+, \mathbb{R}),$$

(2.12)

which implies (2.10), because the functions $t \mapsto \int_0^t \langle \widehat{x}_s - x'^n_{s}, dk'^n_{s} \rangle$ are non-decreasing. ■

**Theorem 2.6.** For every $y \in \mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$ such that $y_0 \in \overline{D}$ there exists a unique solution of the Skorokhod problem associated with $y$.

**Proof.** Let $\{y^n\}$ be the sequence of discretizations of $y$ defined by $y^n_t = y_{k/n}, \ t \in [k/n, (k+1)/n), \ n \in \mathbb{N}$. We check at once that for every $n \in \mathbb{N}$ the pair $(x^n_{s}, k^n_{s})$ defined by

$$\begin{cases}
x^n_0 = y_0, \\
x^n_{(k+1)/n} = \Pi(x^n_{k/n} + y_{(k+1)/n} - y_{k/n})
\end{cases}$$

and $x^n_{s} = x^n_{k/n}, \ k^n_{s} = x^n_{s} - y^n_{s}, \ s \in [k/n, (k+1)/n), \ n \in \mathbb{N}$, solves the Skorokhod problem for $y^n$. Since $y^n \to y$ in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}^d)$, the result follows from Theorem 2.5. ■

**3. SDEs in convex domains.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered probability space and let $Y$ be an $(\mathcal{F}_t)$ adapted process with $Y_0 \in \overline{D}$. 
We say that a pair \((X, K)\) of \((\mathcal{F}_t)\) adapted processes solves the Skorokhod problem associated with \(Y\) if for almost every \(\omega \in \Omega\) the pair \((X(\omega), K(\omega))\) is a solution of the Skorokhod problem associated with \(Y(\omega)\).

From Theorem 2.6 it follows that for any process \(Y\) with \(Y_0 \in \overline{D}\) there exists a unique solution of the Skorokhod problem associated with \(Y\). The following remark is due to Slomiński (see [8, Corollary 1]).

**Remark 3.1.** Let \(Y, \tilde{Y}\) be \((\mathcal{F}_t)\) adapted processes of the form \(Y = H + M + V, \tilde{Y} = H + \tilde{M} + \tilde{V}\), where \(M, \tilde{M}\) are local martingales, \(V, \tilde{V}\) are processes with locally bounded variation and \(M_0 = \tilde{M}_0 = V_0 = \tilde{V}_0 = 0\). If \((X, K), (\tilde{X}, \tilde{K})\) are solutions of the Skorokhod problem associated with \(Y, \tilde{Y}\), respectively, then for every \(p \in \mathbb{N}\) there exists \(C_p\) such that

\[
E \sup_{t < \tau} |X_t - \tilde{X}_t|^{2p} \leq C_p E([M - \tilde{M}]_{\tau-}^p + |V - \tilde{V}|_{\tau-}^{2p} + (M - \tilde{M})_{\tau-}^p)
\]

for every \((\mathcal{F}_t)\) stopping time \(\tau\).

Let us denote by \(\mathbb{F}^d\) the class of \(d\)-dimensional \((\mathcal{F}_t)\) adapted processes and by \(\mathbb{M}^d\) the class of \((\mathcal{F}_t)\) adapted processes with values in the set \(\mathbb{R}^d \otimes \mathbb{R}^d\) of \(d\)-dimensional matrices.

We say that an operator \(F : \mathbb{F}^d \rightarrow \mathbb{M}^d\) is Lipschitz if

(i) for every \(X, Y \in \mathbb{F}^d\) and every stopping time \(\tau\), if \(X^{\tau-} = Y^{\tau-}\) then \(F(X)^{\tau-} = F(Y)^{\tau-}\),

(ii) there exists a one-dimensional, \((\mathcal{F}_t)\) adapted, non-decreasing process \(L = (L_t)\) such that \(P(\sup_t L_t < \infty) = 1\) and, for every \(X, Y \in \mathbb{F}^d\),

\[
\|F(X)_t - F(Y)_t\| \leq L_t \sup_{s \leq t} |X_s - Y_s|, \quad t \in \mathbb{R}^+.
\]

Clearly, if \(f : \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d\) is Lipschitz, that is, there exists \(L > 0\) such that \(\|f(x) - f(y)\| \leq L|x - y|\) for \(x, y \in \mathbb{R}^d\), then the operator \(F(X)\) defined by \(F(X)_t = f(X_t)\) for \(X \in \mathbb{F}^d\) is Lipschitz.

Let \(F : \mathbb{F}^d \rightarrow \mathbb{M}^d\). We say that a pair \((X, K)\) of \((\mathcal{F}_t)\) adapted processes is a strong solution of the SDE (1.1) if (1.1) is satisfied and \((X, K)\) is a solution of the Skorokhod problem associated with \(Y_t = H_t + \int_0^t (F(X)_{s-}, dZ_s)\), \(t \in \mathbb{R}^+\).

We can now formulate our main result.

**Theorem 3.2.** Let \(H\) be an \((\mathcal{F}_t)\) adapted process such that \(H_0 \in \overline{D}\), and let \(Z\) be an \((\mathcal{F}_t)\) adapted semimartingale with \(Z_0 = 0\). Then for any Lipschitz operator \(F : \mathbb{F}^d \rightarrow \mathbb{M}^d\) there exists a unique strong solution of the SDE (1.1).

**Proof.** By using the arguments from the proof of [6, Chapter V, Theorem 7], we may and do assume that \(F(0)_t = 0\) and \(L_t < L\) for some constant
$L > 0$. Since $Z$ is a semimartingale, it admits a unique decomposition

$$Z_t = J_t + M_t + B_t, \quad t \in \mathbb{R}^+,$$

where $J_t = \sum_{s \leq t} \Delta Z_u \mathbb{I}(|\Delta Z_s| > 1)$, $M$ is a local square-integrable martingale, $|\Delta M| \leq 2$, and $B$ is a predictable process with locally bounded variation, $|\Delta B| \leq 1$. Let $C_1$ be a constant from Remark 3.1. For given $a' \in D$ set $a^2 = (12C_1 L^2)^{-1}$ and $\tau' = \inf\{t > 0; |H_t - H_0| \geq \text{dist}(a', \partial D)/2\}$.

We first prove existence and uniqueness of a solution of (1.1) on the interval $[0, \tau]$, where $\tau = \inf\{t > 0; \max(|M|_t, \langle M \rangle_t, |B|_t^2, |J|_t^2) > a^2 \} \wedge \tau' \wedge 1$. To this end we set $\tau_k = \inf\{t > 0; \sup_{s \leq t}|H_s| > k\} \wedge \tau$ for $k \in \mathbb{N}$. For fixed $k \in \mathbb{N}$ we denote by $S^2$ the class of $(\mathcal{F}_t)$ adapted processes $Y = (Y_t)$ on $[0, 1]$ such that $Y_0 \in \overline{D}$, $Y = Y^{\tau_k-}$ and $E \sup_{t \leq 1}|Y_t|^2 < \infty$. Then $S^2$ is a Banach space with the norm $\|Y\|_S^2 = (E \sup_{t \leq 1}|Y_t|^2)^{1/2}$. Define the mapping $\Phi$ on $S^2$ by letting $\Phi(Y)$ be the first component $X$ of the solution $(X, K)$ of the Skorokhod problem associated with $H^{\tau_k-} + \int_0^1 F(Y)_{s-} dZ^{\tau_k-}_s$. We will show that $\Phi$ is a contraction mapping on $S^2$. To see this, we first observe that $\Phi(Y \equiv 0) = H^{\tau_k-} + K^{\tau_k-}$, since $F(0) \equiv 0$. Hence

$$E \sup_{t \leq 1}|\Phi(0)_t|^2 \leq 2E \sup_{t \leq 1}|H^{\tau_k-}_t|^2 + 2E \sup_{t \leq 1}|K^{\tau_k-}_t|^2 \leq 2k^2 + 2E \sup_{t \leq 1}|K^{\tau_k-}_t|^2.$$ 

Therefore, by Corollary 2.4, $\Phi(Y) \in S^2$. Furthermore, by Remark 3.1, for any $Y, \hat{Y} \in S^2$ we have

$$E \sup_{t \leq 1}|\Phi(Y)_t - \Phi(\hat{Y})_t|^2 \leq C_1 \left( E \int_0^{\tau_k-} \|F(Y)_{s-} - F(\hat{Y})_{s-}\|^2 d(|M|_s + \langle M \rangle_s) \right. $$

$$+ E \left( |B + J|_{\tau_k-} \cdot \int_0^{\tau_k-} \|F(Y)_{s-} - F(\hat{Y})_{s-}\|^2 d|B + J|_s \right) \right)$$

$$\leq C_1 \left( E \sup_{t \leq 1} \|F(Y)_{t-} - F(\hat{Y})_{t-}\|^2 (|M|_{\tau_k-} + \langle M \rangle_{\tau_k-}) \right. $$

$$+ E \sup_{t \leq 1} \|F(Y)_{t-} - F(\hat{Y})_{t-}\|^2 |B + J|^2_{\tau_k-} \right)$$

$$\leq 6C_1 a^2 E \sup_{t \leq 1} \|F(Y)_{t-} - F(\hat{Y})_{t-}\|^2$$

$$\leq 6C_1 a^2 L^2 E \sup_{t \leq 1} |Y_t - \hat{Y}_t|^2 = \frac{1}{2} E \sup_{t \leq 1} |Y_t - \hat{Y}_t|^2.$$

From the above we see that $\Phi : S^2 \rightarrow S^2$ is a contraction. Hence, by the Banach contraction principle, it has a fixed point $X^k$, which is a unique
solution of (1.1) on $[0, \tau_k]$. Since $P(\tau_k = \tau) \uparrow 1$, putting $X = X^k$ on $[0, \tau_k]$ we obtain a unique solution on $[0, \tau]$. Moreover, putting

$$X_\tau = P(X_{\tau-} + \Delta H_\tau + \langle F(X)_{\tau-}, \Delta Z_\tau \rangle)$$

we obtain a solution on $[0, \tau)$, because $F(X^{\tau-})_{\tau-} = F(X)_{\tau-}$ by the definition of $F$.

Now, we define a sequence of stopping times

$$\tau^0 = \tau, \quad \tau^{k+1} = \tau^k + \inf\{t > 0; \max([\bar{M}_t], |\bar{B}_t|^2, |J_t|^2) > a^2\} \wedge (\tau_k \wedge 1),$$

where $\bar{M}_t = M_{\tau^k} - M_{\tau^k}$, $\bar{B}_t = B_{\tau^k} - B_{\tau^k}$, $\bar{J}_t = J_{\tau^k} - J_{\tau^k}$, $\bar{H}_t = H_{\tau^k} - H_{\tau^k}$, $\tau'_k = \inf\{t > 0; |\bar{H}_t| \geq \text{dist}(a', \partial D)/2\}$. By what has been proved there exists a unique solution of (1.1) on $[0, \tau^0]$. By the same method as above, having a unique solution on $[0, \tau^k]$, we can construct a solution of (1.1) on $[0, \tau^{k+1}]$. Since $\tau^k \uparrow \infty$, the theorem follows. \qed

**Corollary 3.3.** If $f : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is Lipschitz then for every $(\mathcal{F}_t)$ adapted process $H$ with $H_0 \in D$ and every $(\mathcal{F}_t)$ adapted semimartingale $Z$ with $Z_0 = 0$ there exists a unique strong solution of the SDE

$$X_t = H_t + \int_0^t \langle f(X_{s-}), dZ_s \rangle + K_t, \quad t \in \mathbb{R}^+.$$ \qed

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