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# On Meager Additive and Null Additive Sets in the Cantor Space $2^\omega$ and in $\mathbb{R}$

by

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**Summary.** Let $T$ be the standard Cantor–Lebesgue function that maps the Cantor space $2^\omega$ onto the unit interval $\langle 0,1\rangle$. We prove within ZFC that for every $X\subseteq 2^\omega$, $X$ is meager additive in $2^\omega$ iff $T(X)$ is meager additive in $\langle 0,1\rangle$. As a consequence, we deduce that the cartesian product of meager additive sets in $\mathbb{R}$ remains meager additive in $\mathbb{R}\times\mathbb{R}$. In this note, we also study the relationship between null additive sets in $2^\omega$ and $\mathbb{R}$.

**1. Introduction.** Assume that $(2^\omega,\oplus)$ denotes the Cantor space $2^\omega$ with modulo 2 coordinatewise addition, and $\mathbb{R}$ is the additive group of real numbers, both with the standard topology and measure. We will say that $X\subseteq 2^\omega$ is *meager additive* (respectively, *null additive*) iff for every meager (respectively, null) set $A$, $X\oplus A=\{x\oplus a: x\in X,\ a\in A\}$ is meager (respectively, null) in $2^\omega$. Analogously we define *meager additive* and *null additive* sets in $\mathbb{R}$. The following question was asked by T. Bartoszyński (personal communication): Suppose that there exists an uncountable meager (respectively, null) additive set in $(2^\omega,\oplus)$. Is it true that there is an uncountable meager (respectively, null) additive set in $\mathbb{R}$? And how about the converse implication?

We give a complete answer (in ZFC) to the category part of this question, and we partially answer the measure version.

To start, let us notice that $2^\omega$ can be related to the interval $\langle 0,1\rangle$ through the Cantor–Lebesgue continuous function $T:2^\omega\to\langle 0,1\rangle$ given by

$$T(x)=\sum_{i\in\omega}\frac{x(i)}{2^{i+1}}.$$

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It is well known that $T$ is category and measure preserving, and one-to-one except on the countable set of sequences that are eventually zero (respectively, one). Thus depending on the context, a subset $X$ of $2^\omega$ is often identified with $T(X) = \{T(x) : x \in X\}$, and $Y \subseteq \langle 0,1\rangle$ is identified with $T^{-1}(Y)$. Let us also notice that instead of using meager (respectively, null) additive sets in $\mathbb{R}$, we may consider meager (respectively, null) additive sets in $(\langle 0,1),+_1)$, where $+_1$ denotes modulo 1 addition. Clearly, the latter sets are meager (respectively, null) additive in $(\langle 0,1\rangle,+_1)$, where $x+_1 y = x+y$ if $x+y\le 1$, and $x+_1 y = x+y-1$ if $x+y>1$. Conversely, if $X$ is a meager (respectively, null) additive set in $(\langle 0,1\rangle,+_1)$, then $X\setminus\{1\}$ is meager (respectively, null) additive in $(\langle 0,1),+_1)$.

We use standard terminology and notation. $\omega^{\omega\uparrow}$ stands for the set of all increasing functions $f:\omega\to\omega$. For $n\in\omega$ and $f\in\omega^{\omega\uparrow}$,

$$[f(n),f(n+1)) = \{k\in\omega : f(n)\le k< f(n+1)\},$$

and if $s\in 2^{[f(n),f(n+1))}$, then

$$[s] = \{x\in 2^\omega : x\restriction[f(n),f(n+1)) = s\}.$$

The quantifiers $\exists^\infty n$, $\forall^\infty n$ denote "for infinitely many $n$" and "for all but finitely many $n$", respectively.

Suppose that $f\in\omega^{\omega\uparrow}$ and $x\in 2^\omega$. We define a meager set in $2^\omega$ by

$$B_{f,x} = \{y\in 2^\omega : \forall^\infty n\ x\restriction[f(n),f(n+1)) \ne y\restriction[f(n),f(n+1))\}.$$

It is well known (see [BJ, Theorem 2.2.4, p. 10]) that every meager set in $2^\omega$ is a subset of a set of the above form, for some $f\in\omega^{\omega\uparrow}$ and $x\in 2^\omega$. For $n\in\omega$, we let

$$B_{[f(n),f(n+1)),x} = \{y\in 2^\omega : x\restriction[f(n),f(n+1)) \ne y\restriction[f(n),f(n+1))\},$$

and we denote by $\mathbb{O}$ (respectively, $\mathbb{1}$) the constantly zero (respectively, one) function in $2^\omega$. One can easily check that for every $n\in\omega$ and $x\in 2^\omega$,

$$(*)\quad x\oplus B_{[f(n),f(n+1)),\mathbb{O}} = B_{[f(n),f(n+1)),x} = \sum_{i=0}^{f(n+1)-1}\frac{x(i)}{2^{i+1}} +_1 B_{[f(n),f(n+1)),\mathbb{O}}.$$

In the identity $(*)$, which plays an important role in the second part of this paper, $+_1$ is addition in $\langle 0,1)$, the sets $B_{[f(n),f(n+1)),x}$ and $B_{[f(n),f(n+1)),\mathbb{O}}$ are identified with their images under $T$, and the sequence of ones $(1,1,\ldots)$ is excluded from the left-hand side of the equation.

## 2. Main theorems

THEOREM 1. *For every meager additive set $X$ in $(2^\omega,\oplus)$, $T(X)$ is meager additive in $(\langle 0,1\rangle,+_1)$.*

*Proof.* Let $X$ be a meager additive set in $2^\omega$. By the Bartoszyński–Judah–Shelah characterization (see [BJ, Theorem 2.7.17, p. 95]), for every $f \in \omega^{\omega\uparrow}$, there are $g \in \omega^{\omega\uparrow}$ and $y \in 2^\omega$ such that

$$\forall x \in X\ \forall^\infty n\ \exists k\ g(n) \le f(k) < f(k+1) < g(n+1) \text{ and } x\restriction[f(k), f(k+1)) = y\restriction[f(k), f(k+1)).$$

For $k \in \omega$ and $f \in \omega^{\omega\uparrow}$, let

$$G^f_k = \{[s] : s \in 2^{[f(k),f(k+1))}\}.$$

Notice that each $[s]$ from $G^f_k$, treated as a subset of $\langle 0, 1\rangle$, is a union of $2^{f(k)}$ intervals of diameter $1/2^{f(k+1)}$ each. Define $h_k = y\restriction[f(k), f(k+1))$ for $k \in \omega$ and $y \in 2^\omega$. Thus, clearly,

$$\forall f \in \omega^{\omega\uparrow}\ \exists g \in \omega^{\omega\uparrow}\ \exists\{[h_k]\}_{k\in\omega}\ \forall k\quad [h_k] \in G^f_k,$$

so that any $x \in X$ belongs to all but finitely many sets of the form

$$X^f_n = \bigcup_{k\,:\, g(n)\le f(k)<f(k+1)<g(n+1)} [h_k].$$

Claim 2. *Let $P$ be a closed nowhere dense subset of $\langle 0, 1\rangle$. Given $d > 0$ and $k \in \omega$, there are $\varepsilon, \delta > 0$ such that for any interval $I$ with* $\mathrm{diam}(I) \ge d$, *and each set $J$ that consists of $k$ intervals of diameter at most $\varepsilon$, we have*

$$(J +_1 P) \cap I' = \emptyset,$$

*where $I'$ is some interval included in $I$ with* $\mathrm{diam}(I') \ge \delta$.

*Proof.* It suffices to prove the assertion for a fixed open interval $I$ with $\mathrm{diam}(I) = d > 0$, since any interval of diameter no smaller than $d$ can be translated modulo 1 so as to cover $I$. For every $\langle x_0, \ldots, x_{k-1}\rangle \in \langle 0, 1\rangle^k$, find open intervals $A_{x_0}, \ldots, A_{x_{k-1}}$ of equal diameters with middle points $x_0, \ldots, x_{k-1}$ such that $(A_{x_0} \cup \cdots \cup A_{x_{k-1}}) +_1 P$ is disjoint from some open interval $I_{\langle x_0,\ldots,x_{k-1}\rangle} \subseteq I$. Let $\{A_{\langle x_0,\ldots,x_{k-1}\rangle}\}_{\langle x_0,\ldots,x_{k-1}\rangle\in\langle 0,1\rangle^k}$ be the open cover of $\langle 0, 1\rangle^k$ that consists of the cartesian products of the middle thirds of the sets $A_{x_0}, \ldots, A_{x_{k-1}}$. Use compactness of $\langle 0, 1\rangle^k$ to choose $\varepsilon$ and $\delta$. ■

Let $P$ be a closed nowhere dense subset of $\langle 0, 1\rangle$. We construct $f \in \omega^{\omega\uparrow}$ and a sequence $\{d_n\}_{n\in\omega}$ of positive real numbers iteratively using Claim 2. Set $d_0 = 1/2$, $f(0) = 0$, and assume that $f(n)$, $d_n$ have already been constructed. Define $f(n + 1)$ so that for any interval $I$ with $\mathrm{diam}(I) \ge d_n/2^n$, and any finite set $J$ consisting of at most $2^{f(n)}$ intervals of diameter at most $1/2^{f(n+1)}$ each, $J +_1 P$ is disjoint from a certain interval $I' \subseteq I$ with $\mathrm{diam}(I') \ge d_{n+1}$.

Now let $\{I_n\}_{n\in\omega}$ be a fixed bijective enumeration of all rational open intervals in $\langle 0, 1\rangle$, and suppose that $g \in \omega^{\omega\uparrow}$ and $\{[h_k]\}_{k\in\omega}$ are chosen for $f$ as in the Bartoszyński–Judah–Shelah characterization. Set $\overline{d}_0 = \mathrm{diam}(I_0)$.

Since $d_n/2^n \to 0$, we can pick $k_0$ so large that $d_{k_0}/2^{k_0} \le \overline{d}_0$. Let $n_0 \ge 0$ be the maximum $n$ satisfying $g(n) \le f(k_0)$. Using the properties of the function $f$ defined above, we choose a sequence of open intervals $\{I_0^k\}_{k\,:\,g(n_0)\le f(k)<f(k+1)<g(n_0+1)}$ so that

$$\begin{aligned} ([h_{k_0}] +_1 P) \cap I_0^{k_0} &= \emptyset, \quad I_0^{k_0} \subseteq I_0, \\ ([h_{k_0+1}] +_1 P) \cap I_0^{k_0+1} &= \emptyset, \quad I_0^{k_0+1} \subseteq I_0^{k_0}, \quad \text{etc.} \end{aligned}$$

We let $I_0' = \bigcap_{j\,:\,f(k_0+j+1)<g(n_0+1)} I_0^{k_0+j}$. Then

$$\Big(\Big(\bigcup_{k\,:\,g(n_0)\le f(k)<f(k+1)<g(n_0+1)} [h_k]\Big) +_1 P\Big) \cap I_0' = \emptyset.$$

Now set $\overline{d}_1 = \operatorname{diam}(I_1)$. As before, we choose $k_1$ with $d_{k_1}/2^{k_1} \le \overline{d}_1$, and we pick $n_1$ satisfying $g(n_1) > g(n_0+1)$, and a sequence of open intervals $\{I_1^k\}_{k\,:\,g(n_1)\le f(k)<f(k+1)<g(n_1+1)}$. Let $I_1' = \bigcap_{j\,:\,f(k_1+j+1)<g(n_1+1)} I_1^{k_1+j}$. Then

$$\Big(\Big(\bigcup_{k\,:\,g(n_1)\le f(k)<f(k+1)<g(n_1+1)} [h_k]\Big) +_1 P\Big) \cap I_1' = \emptyset.$$

We follow this scenario for $I_2, I_3, \ldots$ etc.

Finally, we conclude that

$$\bigcap_{j\in\omega}\Big(\bigcup_{k\,:\,g(n_j)\le f(k)<f(k+1)<g(n_j+1)} [h_k]\Big) +_1 P$$

is meager. Thus $\bigcap_{n\ge 0} X_n^f +_1 P$ is meager. In a similar way we show that $\bigcap_{n\ge m} X_n^f +_1 P$ is meager for all $m \ge 1$. This finishes the proof of Theorem 1. ∎

Corollary 3. *Suppose that $X, Y$ are meager additive sets in $2^\omega$. Then their cartesian product is meager additive in $(\langle 0,1\rangle, +_1) \times (\langle 0,1\rangle, +_1)$.*

*Proof.* First notice that one can apply the same argument as in the proof of Theorem 1 to show that $X \times \{0\}$ and $\{0\} \times Y$ are meager additive in the product $(\langle 0,1\rangle, +_1) \times (\langle 0,1\rangle, +_1)$ with modulo 1 coordinatewise addition (denoted by $+$). Then use the fact that for any meager set $A$ in the product, $X \times Y + A = X \times \{0\} + (\{0\} \times Y + A)$ is meager. ∎

Theorem 4. *Let $X$ be meager additive in $(\langle 0,1), +_1)$. Then there exists $t \in \langle 0,1)$ such that $T^{-1}(X +_1 t)$ is meager additive in $(2^\omega, \oplus)$.*

*Proof.* Our goal is to show that for any meager additive set $X$ in $(\langle 0,1), +_1)$, there exists $t \in \langle 0,1)$ such that for every meager set $A$ in $2^\omega$, one can find a meager set $B$ in $\langle 0,1)$ satisfying

$$T(T^{-1}(X +_1 t) \oplus A) \subseteq (X +_1 t) +_1 B.$$

Claim 5. *Let $F$ be an $F_\sigma$ meager subset of $\langle 0,1)$. Then there is $t \in \langle 0,1)$ such that $(F +_1 t) \cap Q = \emptyset$, where $Q$ is the set of all rational numbers in $\langle 0,1)$.*

*Proof.* Let $t$ be such that $-_1 t \notin F +_1 Q$. ■

Claim 6. *Suppose that $X$ is a meager additive subset of $\langle 0,1)$. Then there is a $\sigma$-compact set $F$ disjoint from $Q$ such that $X +_1 t \subseteq F$ for some $t \in \langle 0,1)$.*

*Proof.* Since $X$ is meager, there exists a meager $F_\sigma$ set $\widetilde{F}$ such that $X \subseteq \widetilde{F}$. Apply Claim 5 to find $t$ with $(\widetilde{F} +_1 t) \cap Q = \emptyset$, and then put $F = \widetilde{F} +_1 t$. Notice that we can assume that $F$ is a $\sigma$-compact subset of $\langle 0,1)$. ■

Claim 7. *Suppose that $F$ is a $\sigma$-compact subset of $\langle 0,1)$ disjoint from $Q$, and $h \in \omega^{\omega\uparrow}$. Then there is an increasing sequence $\{n_k\}_{k\in\omega}$ of natural numbers such that*

$$F \subseteq \{x \in 2^\omega : \forall^\infty k\ x\restriction[h(n_k), h(n_{k+1})) \neq \mathbb{1}\}.$$

*Proof.* We define a continuous function $g : F \to \omega^\omega$ as follows. For $x \in F$, we let $g(x)(0) = h(0)$, $g(x)(n) = \min\{m : m > \max\{h(n), n\},\ m = h(k)$ for some $k \in \omega$, and $x\restriction[g(x)(n-1), g(x)(m)) \neq \mathbb{1}\}$. As $g$ is continuous, its range is a $\sigma$-compact subset of $\omega^\omega$, thus there exists $G \in \omega^{\omega\uparrow}$, with $\mathrm{range}(G) \subset \mathrm{range}(h)$, such that for every $x \in F$,

$$\exists n_0\ \forall k \geq n_0\ g(x)(k) \leq G(k).$$

For $k \in \omega$, let $G^k$ be the $k$-fold iteration of $G$. We define $n_k$ to be the $n$ such that $h(n) = G^{2k}(0)$, for $k \in \omega$. ■

Claim 8. *Assume that $k, n \in \omega$, $n \geq 2$ and $f \in \omega^{\omega\uparrow}$. If $X \subseteq 2^\omega$ and $x\restriction[f(k+1), f(k+n)) \neq \mathbb{1}$ for every $x \in X$, then*

$$X \oplus B_{[f(k),f(k+1)),\mathbb{O}} \subseteq X +_1 B_{[f(k),f(k+n)),\mathbb{O}}.$$

*Proof.* Fix $s \in 2^{f(k)}$ and identify $B_{[f(k),f(k+1)),\mathbb{O}}$ and $B_{[f(k),f(k+n)),\mathbb{O}}$ with their images under $T$. Then the distance between the left endpoint of the interval $\{y \in 2^\omega : y\restriction f(k) = s,\ y \in B_{[f(k),f(k+n)),\mathbb{O}}\}$ and the left endpoint of the interval $\{y \in 2^\omega : y\restriction f(k) = s,\ y \in B_{[f(k),f(k+1)),\mathbb{O}}\}$ is equal to $1/2^{f(k+1)} - 1/2^{f(k+n)}$. Since $x\restriction[f(k+1), f(k+n)) \neq \mathbb{1}$, we have

$$\sum_{i=f(k+1)}^{\infty} \frac{x(i)}{2^{i+1}} \leq \frac{1}{2^{f(k+1)}} - \frac{1}{2^{f(k+n)}}.$$

Thus $B_{[f(k),f(k+1)),\mathbb{O}}$ remains included in the modulo 1 (in $\langle 0,1)$) translation of $B_{[f(k),f(k+n)),\mathbb{O}}$ by $\sum_{i=f(k+1)}^{\infty} x(i)/2^{i+1}$, that is,

$$B_{[f(k),f(k+1)),\mathbb{O}} \subseteq \sum_{i=f(k+1)}^{\infty} \frac{x(i)}{2^{i+1}} +_1 B_{[f(k),f(k+n)),\mathbb{O}}.$$

By the identity $(*)$ from Section 1, we obtain

$$\begin{aligned} B_{[f(k),f(k+1)),x} &= \sum_{i=0}^{f(k+1)-1} \frac{x(i)}{2^{i+1}} +_1 B_{[f(k),f(k+1)),\mathbb{O}} \\ &\subseteq \sum_{i=0}^{f(k+1)-1} \frac{x(i)}{2^{i+1}} +_1 \sum_{i=f(k+1)}^{\infty} \frac{x(i)}{2^{i+1}} +_1 B_{[f(k),f(k+n)),\mathbb{O}} \\ &= x +_1 B_{[f(k),f(k+n)),\mathbb{O}}. \ \blacksquare \end{aligned}$$

Now assume that $X$ is meager additive in $(\langle 0,1),+_1)$, and let $t$ be such that $(X +_1 t) \cap Q = \emptyset$. Set $X_m = \{x \in 2^\omega : x \in X +_1 t$ and $\forall k \geq m$ $x\restriction[h(n_k), h(n_{k+1})) \neq \mathbb{1}\}$ for $m \in \omega$, where $\{n_k\}_{k\in\omega}$ is chosen for a given $h \in \omega^{\omega\uparrow}$ as in Claim 7. Obviously, $X +_1 t = \bigcup_{m\in\omega} X_m$. Let $m_0 \in \omega$ be fixed. Suppose that $x \in X_{m_0}$. Then for every $m \geq m_0$,

$$\begin{aligned} x \oplus \bigcap_{n\geq m} B_{[h(n),h(n+1)),\mathbb{O}} &= \bigcap_{n\geq m} (x \oplus B_{[h(n),h(n+1)),\mathbb{O}}) \\ &\subseteq \bigcap_{\substack{k\geq m \\ k \text{ even}}} (x \oplus B_{[h(n_k),h(n_{k+1})),\mathbb{O}}). \end{aligned}$$

By Claim 8, for every $n_k \geq m$,

$$x \oplus B_{[h(n_k),h(n_{k+1})),\mathbb{O}} = B_{[h(n_k),h(n_{k+1})),x} \subseteq x +_1 B_{[h(n_k),h(n_{k+2})),\mathbb{O}}.$$

It follows that

$$\begin{aligned} x \oplus \bigcap_{n\geq m} B_{[h(n),h(n+1)),\mathbb{O}} &\subseteq \bigcap_{\substack{k\geq m \\ k \text{ even}}} (x +_1 B_{[h(n_k),h(n_{k+2})),\mathbb{O}}) \\ &= x +_1 \bigcap_{\substack{k\geq m \\ k \text{ even}}} B_{[h(n_k),h(n_{k+2})),\mathbb{O}}. \end{aligned}$$

Consequently,

$$X_{m_0} \oplus \bigcap_{n\geq m} B_{[h(n),h(n+1)),\mathbb{O}} \subseteq X_{m_0} +_1 \bigcap_{\substack{k\geq m \\ k \text{ even}}} B_{[h(n_k),h(n_{k+2})),\mathbb{O}}.$$

As $\bigcap_{k\geq m,\, k \text{ even}} B_{[h(n_k),h(n_{k+2})),\mathbb{O}}$ is closed nowhere dense, the right hand side above remains meager. Hence the image of $X_{m_0} \oplus \bigcap_{n\geq m} B_{[h(n),h(n+1)),\mathbb{O}}$ under $T$ is meager in $\langle 0,1)$. Thus $X_{m_0} \oplus \bigcap_{n\geq m} B_{[h(n),h(n+1)),\mathbb{O}}$ is meager in $(2^\omega, \oplus)$.

Consider now any set of the form $B_{h,z}$, where $h \in \omega^{\omega\uparrow}$ and $z \in 2^\omega$. Then

$$X_{m_0} \oplus \bigcap_{n\geq m} B_{[h(n),h(n+1)),z} = X_{m_0} \oplus \bigcap_{n\geq m} B_{[h(n),h(n+1)),\mathbb{O}} \oplus z.$$

This proves that $T^{-1}(X +_1 t)$ is meager additive in $2^\omega$. ■

Corollary 9. *Suppose that there is a meager additive set in $(\langle 0,1\rangle, +_1)$ of cardinality $\kappa$, where $\aleph_0 < \kappa \leq \mathfrak{c}$. Then there exists a meager additive set in $(2^\omega, \oplus)$ of cardinality $\kappa$.*

*Proof.* Apply Theorem 4. ■

From Theorem 4 the following stronger fact follows immediately.

Theorem 10. *For every meager additive set $X$ in $(\langle 0,1), +_1)$, $T^{-1}(X)$ is meager additive in $(2^\omega, \oplus)$.*

*Proof.* Let $X$ be a meager additive set in $(\langle 0,1), +_1)$. Then, by Theorem 4, $T^{-1}(X +_1 t)$ is meager additive in $2^\omega$ for some $t \in \langle 0,1)$.

Let $f \in \omega^{\omega\uparrow}$. As in the first part of this paper, let $g \in \omega^{\omega\uparrow}$ and $\{[h_k]\}_{k\in\omega}$, with each $h_k \in G_k^f$, be such that any $x$ from $X +_1 t$ belongs to almost every set of the form

$$\bigcup_{k\,:\,g(n)\leq f(k)<f(k+1)<g(n+1)} [h_k].$$

Then there are $\{[h'_k]\}_{k\in\omega}$ and $\{[h''_k]\}_{k\in\omega}$, with $h'_k, h''_k \in G_k^f$ for $k \in \omega$, such that any $x \in X$ belongs to almost every set of the form

$$\bigcup_{k\,:\,g(n)\leq f(k)<f(k+1)<g(n+1)} [h'_k] \cup [h''_k].$$

Thus, by applying Claim 2 for subsets of $2^\omega$, we can proceed as in the proof of Theorem 1 to show that $T^{-1}(X)$ is meager additive in $2^\omega$. ■

Remark 11. Notice that by Corollary 3 and Theorem 10, the cartesian product of meager additive sets in $(\langle 0,1\rangle, +_1)$ is meager additive in $(\langle 0,1\rangle, +_1) \times (\langle 0,1\rangle, +_1)$. This can be easily extended to products of meager additive subsets of $\mathbb{R}$ (see Problem 2.4 and Remark 2.5 in [TW]).

Unfortunately, we do not know if one can establish a result analogous to Theorem 4 for null additive sets in $(\langle 0,1), +_1)$. So the following crucial question remains open.

Question 12. *Suppose that there is an uncountable null additive set in $(\langle 0,1), +_1)$. Does this imply that its "reasonable" transformation is null additive in $2^\omega$?*

Nevertheless, the following theorem holds.

THEOREM 13. *Suppose that $X$ is null additive in $(2^\omega, \oplus)$. Then it is null additive in $(\langle 0,1\rangle, +_1)$.*

*Proof.* By Shelah's characterization (see [BJ, Theorem 2.7.18(3), p. 95]), for every $f \in \omega^{\omega\uparrow}$, there is a sequence $\{I_n\}_{n\in\omega}$, with each $I_n \subseteq 2^{[f(n),f(n+1))}$ and $|I_n| \le n$, such that

$$\forall x \in X\ \forall^\infty n\ x{\restriction}[f(n), f(n+1)) \in I_n.$$

Let $H$ be a null set in $\langle 0,1\rangle$. Then, by Bartoszyński's theorem, $H$, treated as a subset of $2^\omega$, is contained in a union of two small sets (see [BJ, Theorem 2.5.7, p. 63]). Recall that $A \subseteq 2^\omega$ is small if there are $f \in \omega^{\omega\uparrow}$ and a sequence $\{J_n\}_{n\in\omega}$, with $J_n \subseteq 2^{[f(n),f(n+1))}$, such that

$$A \subseteq \{x \in 2^\omega : \exists^\infty n\ x{\restriction}[f(n), f(n+1)) \in J_n\}, \tag{1}$$

$$\sum_{n\in\omega} \frac{|J_n|}{2^{f(n+1)-f(n)}} < \infty. \tag{2}$$

Thus we may assume for the purpose of this proof that there is $f \in \omega^{\omega\uparrow}$ such that

$$H = \left\{\sum_{i\in\omega} \frac{x(i)}{2^{i+1}} : x \in 2^\omega \text{ and } \exists^\infty n\ x{\restriction}[f(n), f(n+1)) \in J_n\right\},$$

where each $J_n \subseteq 2^{[f(n),f(n+1))}$ and

$$\forall n \in \omega\ \frac{|J_n|}{2^{f(n+1)-f(n)}} \le \frac{1}{2^n}.$$

By Shelah's characterization, $X$, as a subset of $\langle 0,1\rangle$, satisfies

$$X \subseteq \bigcup_{k\in\omega} \bigcap_{n\ge k} X_n,$$

where for $n \in \omega$,

$$X_n = \left\{\sum_{i\in\omega} \frac{x(i)}{2^{i+1}} : x \in 2^\omega \text{ and } x{\restriction}[f(n), f(n+1)) \in \{I_1^n, \ldots, I_n^n\} = I_n\right\}.$$

Also,

$$H \subseteq \bigcap_{k\in\omega} \bigcup_{n\ge k} H_n,$$

where for $n \in \omega$,

$$H_n = \left\{\sum_{i\in\omega} \frac{y(i)}{2^{i+1}} : y \in 2^\omega \text{ and } y{\restriction}[f(n), f(n+1)) \in \{J_1^n, \ldots, J_{r(n)}^n\} = J_n\right\},$$

with $r(n) = |J_n|$. Clearly,

$$X +_1 H \subseteq \bigcap_{k\in\omega} \bigcup_{n\ge k} (X_n +_1 H_n).$$

It is easy to see that each $X_n +_1 H_n$ is contained in a union of $n \cdot 2^{f(n)} \cdot |J_n|$ intervals of diameter $2/2^{f(n+1)}$ each. Thus there is a sequence $\{Y_n\}_{n\in\omega}$ of subsets of $\langle 0,1\rangle$ satisfying

$$X +_1 H \subseteq \bigcap_{k\in\omega} \bigcup_{n\geq k} Y_n, \qquad \mu(Y_n) \leq \frac{2n \cdot |J_n|}{2^{f(n+1)-f(n)}} \leq \frac{2n}{2^n}.$$

Since $\sum_{n\in\omega} \frac{2n}{2^n}$ is convergent, $X +_1 H$ is null. ∎

**Added in proof.** O. Zindulka has kindly informed us that he knows how to prove Theorems 1 and 13 by substantially different methods. His results will be published elsewhere.

## References

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