On Meager Additive and Null Additive Sets
in the Cantor Space \(2^\omega\) and in \(\mathbb{R}\)

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Summary. Let \(T\) be the standard Cantor–Lebesgue function that maps the Cantor space \(2^\omega\) onto the unit interval \((0, 1)\). We prove within ZFC that for every \(X \subseteq 2^\omega\), \(X\) is meager additive in \(2^\omega\) iff \(T(X)\) is meager additive in \((0, 1)\). As a consequence, we deduce that the cartesian product of meager additive sets in \(\mathbb{R}\) remains meager additive in \(\mathbb{R} \times \mathbb{R}\). In this note, we also study the relationship between null additive sets in \(2^\omega\) and \(\mathbb{R}\).

1. Introduction. Assume that \((2^\omega, \oplus)\) denotes the Cantor space \(2^\omega\) with modulo 2 coordinatewise addition, and \(\mathbb{R}\) is the additive group of real numbers, both with the standard topology and measure. We will say that \(X \subseteq 2^\omega\) is \textit{meager additive} (respectively, \textit{null additive}) iff for every meager (respectively, null) set \(A\), \(X \oplus A = \{x \oplus a : x \in X, a \in A\}\) is meager (respectively, null) in \(2^\omega\). Analogously we define \textit{meager additive} and \textit{null additive} sets in \(\mathbb{R}\). The following question was asked by T. Bartoszyński (personal communication): Suppose that there exists an uncountable meager (respectively, null) additive set in \((2^\omega, \oplus)\). Is it true that there is an uncountable meager (respectively, null) additive set in \(\mathbb{R}\)? And how about the converse implication?

We give a complete answer (in ZFC) to the category part of this question, and we partially answer the measure version.

To start, let us notice that \(2^\omega\) can be related to the interval \((0, 1)\) through the Cantor–Lebesgue continuous function \(T : 2^\omega \to (0, 1)\) given by

\[
T(x) = \sum_{i \in \omega} \frac{x(i)}{2^{i+1}}.
\]
It is well known that $T$ is category and measure preserving, and one-to-one except on the countable set of sequences that are eventually zero (respectively, one). Thus depending on the context, a subset $X$ of $2^{\omega}$ is often identified with $T(X) = \{T(x) : x \in X\}$, and $Y \subseteq (0,1)$ is identified with $T^{-1}(Y)$. Let us also notice that instead of using meager (respectively, null) additive sets in $\mathbb{R}$, we may consider meager (respectively, null) additive sets in $((0,1), +_1)$, where $+_1$ denotes modulo 1 addition. Clearly, the latter sets are meager (respectively, null) additive in $((0,1), +_1)$, where $x +_1 y = x + y$ if $x + y \leq 1$, and $x +_1 y = x + y - 1$ if $x + y > 1$. Conversely, if $X$ is a meager (respectively, null) additive set in $((0,1), +_1)$, then $X \setminus \{1\}$ is meager (respectively, null) additive in $((0,1), +_1)$.

We use standard terminology and notation. $\omega^\uparrow$ stands for the set of all increasing functions $f : \omega \to \omega$. For $n \in \omega$ and $f \in \omega^\uparrow$, 

$$[f(n), f(n + 1)) = \{k \in \omega : f(n) \leq k < f(n + 1)\},$$

and if $s \in 2^{|f(n), f(n + 1)|}$, then 

$$[s] = \{x \in 2^{\omega} : x|\langle f(n), f(n + 1) \rangle = s\}.$$ 

The quantifiers $\exists^{\infty} n$, $\forall^{\infty} n$ denote “for infinitely many $n$” and “for all but finitely many $n$”, respectively.

Suppose that $f \in \omega^\uparrow$ and $x \in 2^{\omega}$. We define a meager set in $2^{\omega}$ by 

$$B_{f,x} = \{y \in 2^{\omega} : \forall^{\infty} n \; x|\langle f(n), f(n + 1) \rangle \neq y|\langle f(n), f(n + 1) \rangle\}.$$ 

It is well known (see [BJ, Theorem 2.2.4, p. 10]) that every meager set in $2^{\omega}$ is a subset of a set of the above form, for some $f \in \omega^\uparrow$ and $x \in 2^{\omega}$. For $n \in \omega$, we let 

$$B_{\langle f(n), f(n + 1) \rangle,x} = \{y \in 2^{\omega} : x|\langle f(n), f(n + 1) \rangle \neq y|\langle f(n), f(n + 1) \rangle\},$$

and we denote by $\emptyset$ (respectively, $\mathbb{1}$) the constantly zero (respectively, one) function in $2^{\omega}$. One can easily check that for every $n \in \omega$ and $x \in 2^{\omega}$,

\[ x \oplus B_{\langle f(n), f(n + 1) \rangle, \emptyset} = B_{\langle f(n), f(n + 1) \rangle, x} = \sum_{i=0}^{f(n+1)-1} \frac{x(i)}{2^{i+1}} +_1 B_{\langle f(n), f(n + 1) \rangle, \emptyset}. \]

In the identity ($\ast$), which plays an important role in the second part of this paper, $+_1$ is addition in $\langle 0, 1 \rangle$, the sets $B_{\langle f(n), f(n + 1) \rangle,x}$ and $B_{\langle f(n), f(n + 1) \rangle, \emptyset}$ are identified with their images under $T$, and the sequence of ones $(1, 1, \ldots)$ is excluded from the left-hand side of the equation.

2. Main theorems

**Theorem 1.** For every meager additive set $X$ in $(2^{\omega}, \oplus)$, $T(X)$ is meager additive in $((0,1), +_1)$. 

Proof. Let \( X \) be a meager additive set in \( 2^\omega \). By the Bartoszyński–Judah–Shelah characterization (see [BJ, Theorem 2.7.17, p. 95]), for every \( f \in \omega^{\omega^1} \), there are \( g \in \omega^{\omega^1} \) and \( y \in 2^\omega \) such that

\[
\forall x \in X \forall \infty n \exists k \ g(n) \leq f(k) < f(k + 1) < g(n + 1) \text{ and } x \upharpoonright [f(k), f(k + 1)) = y \upharpoonright [f(k), f(k + 1)).
\]

For \( k \in \omega \) and \( f \in \omega^{\omega^1} \), let

\[
G^f_k = \{ [s] : s \in 2^{[f(k), f(k + 1))} \}.
\]

Notice that each \([s]\) from \( G^f_k \), treated as a subset of \( \langle 0, 1 \rangle \), is a union of \( 2^{f(k)} \) intervals of diameter \( 1/2^{f(k + 1)} \) each. Define \( h_k = y \upharpoonright [f(k), f(k + 1)) \) for \( k \in \omega \) and \( y \in 2^\omega \). Thus, clearly,

\[
\forall f \in \omega^{\omega^1} \exists g \in \omega^{\omega^1} \exists \{ [h_k] \}_{k \in \omega} \forall k : [h_k] \in G^f_k,
\]

so that any \( x \in X \) belongs to all but finitely many sets of the form

\[
X^f_n = \bigcup_{k : g(n) \leq f(k) < f(k + 1) < g(n + 1)} [h_k].
\]

Claim 2. Let \( P \) be a closed nowhere dense subset of \( \langle 0, 1 \rangle \). Given \( d > 0 \) and \( k \in \omega \), there are \( \varepsilon, \delta > 0 \) such that for any interval \( I \) with \( \text{diam}(I) \geq d \), and each set \( J \) that consists of \( k \) intervals of diameter at most \( \varepsilon \), we have

\[
(J + 1) P \cap I' = \emptyset,
\]

where \( I' \) is some interval included in \( I \) with \( \text{diam}(I') \geq \delta \).

Proof. It suffices to prove the assertion for a fixed open interval \( I \) with \( \text{diam}(I) = d > 0 \), since any interval of diameter no smaller than \( d \) can be translated modulo 1 so as to cover \( I \). For every \( \langle x_0, \ldots, x_{k-1} \rangle \in \langle 0, 1 \rangle^k \), find open intervals \( A_{x_0}, \ldots, A_{x_{k-1}} \) of equal diameters with middle points \( x_0, \ldots, x_{k-1} \) such that \( (A_{x_0} \cup \cdots \cup A_{x_{k-1}}) + 1 P \) is disjoint from some open interval \( I_{\langle x_0, \ldots, x_{k-1} \rangle} \subseteq I \). Let \( \{ A_{\langle x_0, \ldots, x_{k-1} \rangle} \}_{\langle x_0, \ldots, x_{k-1} \rangle \in \langle 0, 1 \rangle^k} \) be the open cover of \( \langle 0, 1 \rangle^k \) that consists of the cartesian products of the middle thirds of the sets \( A_{x_0}, \ldots, A_{x_{k-1}} \). Use compactness of \( \langle 0, 1 \rangle^k \) to choose \( \varepsilon \) and \( \delta \). \( \blacksquare \)

Let \( P \) be a closed nowhere dense subset of \( \langle 0, 1 \rangle \). We construct \( f \in \omega^{\omega^1} \) and a sequence \( \{ d_n \}_{n \in \omega} \) of positive real numbers iteratively using Claim 2. Set \( d_0 = 1/2, f(0) = 0, \) and assume that \( f(n), d_n \) have already been constructed. Define \( f(n + 1) \) so that for any interval \( I \) with \( \text{diam}(I) \geq d_n/2^n \), and any finite set \( J \) consisting of at most \( 2^{f(n)} \) intervals of diameter at most \( 1/2^{f(n + 1)} \) each, \( J + 1 P \) is disjoint from a certain interval \( I' \subseteq I \) with \( \text{diam}(I') \geq d_{n + 1} \).

Now let \( \{ I_n \}_{n \in \omega} \) be a fixed bijective enumeration of all rational open intervals in \( \langle 0, 1 \rangle \), and suppose that \( g \in \omega^{\omega^1} \) and \( \{ [h_k] \}_{k \in \omega} \) are chosen for \( f \) as in the Bartoszyński–Judah–Shelah characterization. Set \( d_0 = \text{diam}(I_0) \).
Since \( d_n/2^n \to 0 \), we can pick \( k_0 \) so large that \( d_{k_0}/2^{k_0} \leq \tilde{d}_0 \). Let \( n_0 \geq 0 \) be the maximum \( n \) satisfying \( g(n) \leq f(k_0) \). Using the properties of the function \( f \) defined above, we choose a sequence of open intervals \( \{I_k^0\}_{k : g(n_0) \leq f(k) < f(k+1) < g(n_0+1)} \) so that

\[
([h_{k_0}] + 1 \ P) \cap I_0^{k_0} = \emptyset, \quad I_0^{k_0} \subseteq I_0,
([h_{k_0}+1] + 1 \ P) \cap I_0^{k_0+1} = \emptyset, \quad I_0^{k_0+1} \subseteq I_0^{k_0}, \quad \text{etc.}
\]

We let \( I'_0 = \bigcap_j f(k_0+j+1) < g(n_0+1) \ I_0^{k_0+j} \). Then

\[
\left( \bigcup_{k : g(n_0) \leq f(k) < f(k+1) < g(n_0+1)} [h_k] + 1 \ P \right) \cap I'_0 = \emptyset.
\]

Now set \( \tilde{d}_1 = \text{diam}(I_1) \). As before, we choose \( k_1 \) with \( d_{k_1}/2^{k_1} \leq \tilde{d}_1 \), and we pick \( n_1 \) satisfying \( g(n_1) > g(n_0 + 1) \), and a sequence of open intervals \( \{I_1^k\}_{k : g(n_1) \leq f(k) < f(k+1) < g(n_1+1)} \). Let \( I'_1 = \bigcap_j f(k_1+j+1) < g(n_1+1) \ I_1^{k_1+j} \). Then

\[
\left( \bigcup_{k : g(n_1) \leq f(k) < f(k+1) < g(n_1+1)} [h_k] + 1 \ P \right) \cap I'_1 = \emptyset.
\]

We follow this scenario for \( I_2, I_3, \ldots \) etc.

Finally, we conclude that

\[
\bigcap_{j \in \omega} \left( \bigcup_{k : g(n_j) \leq f(k) < f(k+1) < g(n_j+1)} [h_k] \right) + 1 \ P
\]

is meager. Thus \( \bigcap_{n \geq 0} X_n^f + 1 \ P \) is meager. In a similar way we show that \( \bigcap_{n \geq m} X_n^f + 1 \ P \) is meager for all \( m \geq 1 \). This finishes the proof of Theorem 1. \( \blacksquare \)

**Corollary 3.** Suppose that \( X, Y \) are meager additive sets in \( 2^\omega \). Then their cartesian product is meager additive in \( \langle (0,1), +1 \rangle \times \langle (0,1), +1 \rangle \).

**Proof.** First notice that one can apply the same argument as in the proof of Theorem 1 to show that \( X \times \{0\} \) and \( \{0\} \times Y \) are meager additive in the product \( \langle (0,1), +1 \rangle \times \langle (0,1), +1 \rangle \) with modulo 1 coordinatewise addition (denoted by \(+\)). Then use the fact that for any meager set \( A \) in the product, \( X \times Y + A = X \times \{0\} + (\{0\} \times Y + A) \) is meager. \( \blacksquare \)

**Theorem 4.** Let \( X \) be meager additive in \( \langle (0,1), +1 \rangle \). Then there exists \( t \in \langle 0,1 \rangle \) such that \( T^{-1}(X +_1 t) \) is meager additive in \( (2^\omega, \oplus) \).

**Proof.** Our goal is to show that for any meager additive set \( X \) in \( \langle (0,1), +1 \rangle \), there exists \( t \in \langle 0,1 \rangle \) such that for every meager set \( A \) in \( 2^\omega \), one can find a meager set \( B \) in \( \langle 0,1 \rangle \) satisfying

\[
T(T^{-1}(X +_1 t) \oplus A) \subseteq (X +_1 t) +_1 B.
\]
Claim 5. Let $F$ be an $F_\sigma$ meager subset of $\langle 0, 1 \rangle$. Then there is $t \in \langle 0, 1 \rangle$ such that $(F + 1 \cdot t) \cap Q = \emptyset$, where $Q$ is the set of all rational numbers in $\langle 0, 1 \rangle$.

Proof. Let $t$ be such that $-1 \cdot t \not\in F + 1 \cdot Q$. ■

Claim 6. Suppose that $X$ is a meager additive subset of $\langle 0, 1 \rangle$. Then there is a $\sigma$-compact set $F$ disjoint from $Q$ such that $X + 1 \cdot t \subseteq F$ for some $t \in \langle 0, 1 \rangle$.

Proof. Since $X$ is meager, there exists a meager $F_\sigma$ set $\tilde{F}$ such that $X \subseteq \tilde{F}$. Apply Claim 5 to find $t$ with $(\tilde{F} + 1 \cdot t) \cap Q = \emptyset$, and then put $F = \tilde{F} + 1 \cdot t$. Notice that we can assume that $F$ is a $\sigma$-compact subset of $\langle 0, 1 \rangle$. ■

Claim 7. Suppose that $F$ is a $\sigma$-compact subset of $\langle 0, 1 \rangle$ disjoint from $Q$, and $h \in \omega^{\omega^\uparrow}$. Then there is an increasing sequence $\{n_k\}_{k \in \omega}$ of natural numbers such that

$$F \subseteq \{ x \in 2^\omega : \forall^\infty k \, x|_{[h(n_k), h(n_{k+1})]} \neq 1 \}.$$ 

Proof. We define a continuous function $g : F \rightarrow \omega^\omega$ as follows. For $x \in F$, we let $g(x)(0) = h(0)$, $g(x)(n) = \min \{ m : m > \max \{ h(n), n \}, m = h(k) \}$ for some $k \in \omega$, and $x|_n[g(x)(n-1), g(x)(m)) \neq 1 \}$. As $g$ is continuous, its range is a $\sigma$-compact subset of $\omega^\omega$, thus there exists $G \in \omega^{\omega^\uparrow}$, with $\text{range}(G) \subseteq \text{range}(h)$, such that for every $x \in F$,

$$\exists n_0 \forall k \geq n_0 \ g(x)(k) \leq G(k).$$

For $k \in \omega$, let $G^k$ be the $k$-fold iteration of $G$. We define $n_k$ to be the $n$ such that $h(n) = G^{2k}(0)$, for $k \in \omega$. ■

Claim 8. Assume that $k, n \in \omega$, $n \geq 2$ and $f \in \omega^{\omega^\uparrow}$. If $X \subseteq 2^\omega$ and $x|_n[f(k+1), f(k+n)) \neq 1$ for every $x \in X$, then

$$X \oplus B_{\langle f(k), f(k+1) \rangle, 0} \subseteq X + 1 \cdot B_{\langle f(k), f(k+n) \rangle, 0}.$$ 

Proof. Fix $s \in 2^{f(k)}$ and identify $B_{\langle f(k), f(k+1) \rangle, 0}$ and $B_{\langle f(k), f(k+n) \rangle, 0}$ with their images under $T$. Then the distance between the left endpoint of the interval $\{ y \in 2^\omega : y|_{f(k)} = s, y \in B_{\langle f(k), f(k+n) \rangle, 0} \}$ and the left endpoint of the interval $\{ y \in 2^\omega : y|_{f(k)} = s, y \in B_{\langle f(k), f(k+1) \rangle, 0} \}$ is equal to $1/2^{f(k+1)} - 1/2^{f(k+n)}$. Since $x|_n[f(k+1), f(k+n)) \neq 1$, we have

$$\sum_{i=f(k+1)}^{\infty} \frac{x(i)}{2^{i+1}} \leq \frac{1}{2^{f(k+1)}} - \frac{1}{2^{f(k+n)}}.$$ 

Thus $B_{\langle f(k), f(k+1) \rangle, 0}$ remains included in the modulo 1 (in $\langle 0, 1 \rangle$) translation
of $B_{[f(k),f(k+1)],\emptyset}$ by $\sum_{i=f(k+1)}^{\infty} x(i)/2^{i+1}$, that is,

$$B_{[f(k),f(k+1)],\emptyset} \subseteq \sum_{i=f(k+1)}^{\infty} \frac{x(i)}{2^{i+1}} + 1 B_{[f(k),f(k+1)],\emptyset}.$$  

By the identity (*) from Section 1, we obtain

$$B_{[f(k),f(k+1)],x} = \sum_{i=0}^{f(k+1)-1} \frac{x(i)}{2^{i+1}} + 1 B_{[f(k),f(k+1)],\emptyset} \subseteq \sum_{i=0}^{f(k+1)-1} \frac{x(i)}{2^{i+1}} + 1 \sum_{i=f(k+1)}^{\infty} \frac{x(i)}{2^{i+1}} + 1 B_{[f(k),f(k+1)],\emptyset} = x + 1 B_{[f(k),f(k+1)],\emptyset}.\quad \blacksquare$$

Now assume that $X$ is meager additive in $((0,1),+1)$, and let $t$ be such that $(X + 1 t) \cap Q = \emptyset$. Set $X_m = \{ x \in 2^\omega : x \in X + 1 t \text{ and } \forall k \geq m \}$. Let $m_0 \in \omega$ be fixed. Suppose that $x \in X_{m_0}$. Then for every $m \geq m_0$,

$$x \oplus \bigcap_{n \geq m} B_{[h(n),h(n+1)],\emptyset} = \bigcap_{n \geq m} (x \oplus B_{[h(n),h(n+1)],\emptyset}) \subseteq \bigcap_{k \geq m, k \text{ even}} (x \oplus B_{[h(n_k),h(n_{k+1})],\emptyset}).$$

By Claim 8, for every $n_k \geq m$,

$$x \oplus B_{[h(n_k),h(n_{k+1})],\emptyset} = B_{[h(n_k),h(n_{k+1})],x} \subseteq x + 1 B_{[h(n_k),h(n_{k+2})],\emptyset}.$$  

It follows that

$$x \oplus \bigcap_{n \geq m} B_{[h(n),h(n+1)],\emptyset} \subseteq \bigcap_{k \geq m, k \text{ even}} (x + 1 B_{[h(n_k),h(n_{k+2})],\emptyset}) = x + 1 \bigcap_{k \geq m, k \text{ even}} B_{[h(n_k),h(n_{k+2})],\emptyset}.$$  

Consequently,

$$X_{m_0} \oplus \bigcap_{n \geq m} B_{[h(n),h(n+1)],\emptyset} \subseteq X_{m_0} + 1 \bigcap_{k \geq m, k \text{ even}} B_{[h(n_k),h(n_{k+2})],\emptyset}.$$  

As $\bigcap_{k \geq m, k \text{ even}} B_{[h(n_k),h(n_{k+2})],\emptyset}$ is closed nowhere dense, the right hand side above remains meager. Hence the image of $X_{m_0} \oplus \bigcap_{n \geq m} B_{[h(n),h(n+1)],\emptyset}$ under $T$ is meager in $\langle 0,1 \rangle$. Thus $X_{m_0} \oplus \bigcap_{n \geq m} B_{[h(n),h(n+1)],\emptyset}$ is meager in $(2^\omega,\oplus)$. 


Consider now any set of the form $B_{h,z}$, where $h \in \omega^\omega$ and $z \in 2^\omega$. Then

$$X_{m_0} \oplus \bigcap_{n \geq m} B_{[h(n), h(n+1)), z} = X_{m_0} \oplus \bigcap_{n \geq m} B_{[h(n), h(n+1)), 0} \oplus z.$$ 

This proves that $T^{-1}(X +_t t)$ is meager additive in $2^\omega$. ■

**COROLLARY 9.** Suppose that there is a meager additive set in $(\langle 0, 1 \rangle, +_1)$ of cardinality $\kappa$, where $\aleph_0 < \kappa \leq c$. Then there exists a meager additive set in $(2^\omega, \oplus)$ of cardinality $\kappa$.

**Proof.** Apply Theorem 4. ■

From Theorem 4 the following stronger fact follows immediately.

**THEOREM 10.** For every meager additive set $X$ in $(\langle 0, 1 \rangle, +_1)$, $T^{-1}(X)$ is meager additive in $(2^\omega, \oplus)$.

**Proof.** Let $X$ be a meager additive set in $(\langle 0, 1 \rangle, +_1)$. Then, by Theorem 4, $T^{-1}(X +_t t)$ is meager additive in $2^\omega$ for some $t \in \langle 0, 1 \rangle$.

Let $f \in \omega^\omega$. As in the first part of this paper, let $g \in \omega^\omega$ and $\{[h_k]\}_{k \in \omega}$, with each $h_k \in G^f_k$, be such that any $x$ from $X +_t t$ belongs to almost every set of the form

$$\bigcup_{k : g(n) \leq f(k) < f(k+1) < g(n+1)} [h_k].$$

Then there are $\{[h'_k]\}_{k \in \omega}$ and $\{[h''_k]\}_{k \in \omega}$, with $h'_k, h''_k \in G^f_k$ for $k \in \omega$, such that any $x \in X$ belongs to almost every set of the form

$$\bigcup_{k : g(n) \leq f(k) < f(k+1) < g(n+1)} [h'_k] \cup [h''_k].$$

Thus, by applying Claim 2 for subsets of $2^\omega$, we can proceed as in the proof of Theorem 1 to show that $T^{-1}(X)$ is meager additive in $2^\omega$. ■

**REMARK 11.** Notice that by Corollary 3 and Theorem 10, the cartesian product of meager additive sets in $(\langle 0, 1 \rangle, +_1)$ is meager additive in $(\langle 0, 1 \rangle, +_1) \times (\langle 0, 1 \rangle, +_1)$. This can be easily extended to products of meager additive subsets of $\mathbb{R}$ (see Problem 2.4 and Remark 2.5 in [TW]).

Unfortunately, we do not know if one can establish a result analogous to Theorem 4 for null additive sets in $(\langle 0, 1 \rangle, +_1)$. So the following crucial question remains open.

**QUESTION 12.** Suppose that there is an uncountable null additive set in $(\langle 0, 1 \rangle, +_1)$. Does this imply that its “reasonable” transformation is null additive in $2^\omega$?

Nevertheless, the following theorem holds.
**Theorem 13.** Suppose that $X$ is null additive in $(2^\omega, \oplus)$. Then it is null additive in $(0,1, +1)$.

**Proof.** By Shelah’s characterization (see [BJ, Theorem 2.7.18(3), p. 95]), for every $f \in \omega^{\omega^+}$, there is a sequence $\{I_n\}_{n \in \omega}$, with each $I_n \subseteq 2^{[f(n), f(n+1)]}$ and $|I_n| \leq n$, such that

$$\forall x \in X \forall^\infty n \ x|[f(n), f(n+1)] \in I_n.$$ 

Let $H$ be a null set in $(0,1)$. Then, by Bartoszynski’s theorem, $H$, treated as a subset of $2^\omega$, is contained in a union of two small sets (see [BJ, Theorem 2.5.7, p. 63]). Recall that $A \subseteq 2^\omega$ is small if there are $f \in \omega^{\omega^+}$ and a sequence $\{J_n\}_{n \in \omega}$, with $J_n \subseteq 2^{[f(n), f(n+1)]}$, such that

1. $A \subseteq \{x \in 2^\omega : \exists^\infty n \ x|[f(n), f(n+1)] \in J_n\}$,

2. $$\sum_{n \in \omega} \left|J_n\right| \left\langle 2^{f(n+1)} - f(n) \right\rangle < \infty.$$ 

Thus we may assume for the purpose of this proof that there is $f \in \omega^{\omega^+}$ such that

$$H = \left\{ \sum_{i \in \omega} \frac{x(i)}{2^i+1} : x \in 2^\omega \text{ and } \exists^\infty n \ x|[f(n), f(n+1)] \in J_n \right\},$$

where each $J_n \subseteq 2^{[f(n), f(n+1)]}$ and

$$\forall n \in \omega \frac{|J_n|}{2^{f(n+1)} - f(n)} \leq \frac{1}{2^n}.$$ 

By Shelah’s characterization, $X$, as a subset of $(0,1)$, satisfies

$$X \subseteq \bigcup_{k \in \omega} \bigcap_{n \geq k} X_n,$$

where for $n \in \omega$,

$$X_n = \left\{ \sum_{i \in \omega} \frac{x(i)}{2^i+1} : x \in 2^\omega \text{ and } x|[f(n), f(n+1)] \in \{I^n_1, \ldots, I^n_n\} = I_n \right\}.$$ 

Also,

$$H \subseteq \bigcap_{k \in \omega} \bigcup_{n \geq k} H_n,$$

where for $n \in \omega$,

$$H_n = \left\{ \sum_{i \in \omega} \frac{y(i)}{2^i+1} : y \in 2^\omega \text{ and } y|[f(n), f(n+1)] \in \{J^n_1, \ldots, J^n_{r(n)}\} = J_n \right\},$$

with $r(n) = |J_n|$. Clearly,

$$X +_1 H \subseteq \bigcap_{k \in \omega} \bigcup_{n \geq k} (X_n +_1 H_n).$$
It is easy to see that each $X_{n+1}H_n$ is contained in a union of $n \cdot 2^{f(n)} \cdot |J_n|$ intervals of diameter $2/2^{f(n+1)}$ each. Thus there is a sequence $\{Y_n\}_{n \in \omega}$ of subsets of $\langle 0, 1 \rangle$ satisfying

$$X_{n+1}H \subseteq \bigcap_{k \in \omega} \bigcup_{n \geq k} Y_n, \quad \mu(Y_n) \leq \frac{2n \cdot |J_n|}{2^{f(n+1)} - f(n)} \leq \frac{2n}{2^n}.$$ 

Since $\sum_{n \in \omega} \frac{2n}{2^n}$ is convergent, $X_{n+1}H$ is null. □

Added in proof. O. Zindulka has kindly informed us that he knows how to prove Theorems 1 and 13 by substantially different methods. His results will be published elsewhere.

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