BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 57, No. 2, 2009

ALGEBRAIC GEOMETRY

Birational Finite Extensions of Mappings from a Smooth Variety

by

Marek KARAŚ

Presented by Wiesław PLEŚNIAK

Summary. We present an example of finite mappings of algebraic varieties $f : V \to W$, where $V \subset \mathbf{k}^n, W \subset \mathbf{k}^{n+1}$, and $F : \mathbf{k}^n \to \mathbf{k}^{n+1}$ such that $F|_V = f$ and $\operatorname{gdeg} F = 1 < \operatorname{gdeg} f$ ($\operatorname{gdeg} h$ means the number of points in the generic fiber of h). Thus, in some sense, the result of this note improves our result in J. Pure Appl. Algebra 148 (2000) where it was shown that this phenomenon can occur when $V \subset \mathbf{k}^n, W \subset \mathbf{k}^m$ with $m \geq n+2$. In the case $V, W \subset \mathbf{k}^n$ a similar example does not exist.

1. Introduction. Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, and let $f : V \to W$ be a finite mapping of algebraic subsets of $\mathbf{k}^n$ and $\mathbf{k}^m$, respectively. If $n \leq m$, then there exists a finite polynomial mapping $F : \mathbf{k}^n \to \mathbf{k}^m$ such that $F|_V = f$ [10]. Let $\operatorname{gdeg} h$ be the number of points in the generic fiber of a finite mapping h. A natural question is: *what is the relation between* $\operatorname{gdeg} f$ *and* $\min\{\operatorname{gdeg} F \mid F : \mathbf{k}^n \to \mathbf{k}^m$ *finite such that* $F|_V = f\}$? The answer to this question in several different situations is given in [3]–[9].

If $n = m$, then $\operatorname{gdeg} F \geq \operatorname{gdeg} f$ for all finite extensions F of f [5]. The reason is that the image of F is a normal variety (precisely, because $F : \mathbf{k}^n \to \mathbf{k}^n$ is finite, we have $F(\mathbf{k}^n) = \mathbf{k}^n$). But if $n < m$, then there is no obvious obstruction to existence of finite mappings $f : V \to W$ and $F : \mathbf{k}^n \to \mathbf{k}^m$ such that $F|_V = f$ and $\operatorname{gdeg} F < \operatorname{gdeg} f$. For $m \geq n+2$ an example is given in [5]. It is natural to ask whether the similar phenomenon can occur when $m = n+1$. In this short note we give an affirmative answer to this question by proving the following

2000 *Mathematics Subject Classification*: 14Rxx, 14R10.
Key words and phrases: finite mapping, birational mapping, geometric degree.

DOI: 10.4064/bp57-2-4

Theorem 1.1. *Let* $V \subset \mathbf{k}^n$ *and* $W \subset \mathbf{k}^{n+1}$ *be smooth algebraic sets of dimension* k, *and let* $f : V \to W$ *be a finite mapping* (*possibly with* $\operatorname{gdeg} f$ *large*). *If* $2k+1 < n$, *then there exists a finite mapping* $F : \mathbf{k}^n \to \mathbf{k}^{n+1}$ *such that* $F|_V = f$ *and* $\operatorname{gdeg} F = 1$ (*that is*, *birational onto its image*).

One can compare this theorem with the closely related results of M. Artin [1, Theorem (6.1)] and Srinivas [12], which however give the much higher dimension of the ambient space of W.

2. The proof. First of all recall that for any irreducible algebraic set Z, $k[Z]$ and $k(Z)$ mean, respectively, the ring of regular functions on Z and the field of rational functions on Z. Recall also that a mapping $f : V \to W$ is called *finite* if $k[V]$ is integral over $f^*(k[W])$, where $f^* : k[W] \ni h \mapsto h \circ f \in k[V]$, and that a polynomial mapping $f : V \to \mathbb{C}^n$ is called an *embedding* if $f(V) = \overline{f(V)}$ and f is an isomorphism onto its image. We will need the following well known

Lemma 2.1 (e.g. [2]). *If* $X \subset \mathbb{C}^n$ *is a closed algebraic smooth set*, $\dim X = k$ *and* $n > 2k+1$, *then we can change the coordinates in such a way that the projection*

$$\phi : X \ni (x, y) \mapsto (0, y) \in 0 \times \mathbb{C}^{2k+1}$$

is an embedding.

By Lemma 2.1 we can assume that the projections

$$\varphi_1 : V \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, 0) \in \mathbf{k}^{n-1} \times 0 \subset \mathbf{k}^n$$

and

$$\varphi_2 : W \ni (y_1, \ldots, y_{n+1}) \mapsto (y_1, \ldots, y_{n-1}, 0, 0) \in \mathbf{k}^{n-1} \times 0 \subset \mathbf{k}^{n+1}$$

are embeddings. In this situation it is very easy and elementary to write down extensions $\Phi_1 \in \operatorname{Aut}(\mathbf{k}^n)$, $\Phi_2 \in \operatorname{Aut}(\mathbf{k}^{n+1})$ of φ_1 and φ_2, respectively. Indeed, for $(x_1, \ldots, x_{n-1}) \in \varphi_1(V)$, we have $\varphi_1^{-1}(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, P(x_1, \ldots, x_{n-1}))$ for some polynomial P. Thus

$$\Phi_1 : \mathbf{k}^n \ni (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, x_n - P(x_1, \ldots, x_{n-1})) \in \mathbf{k}^n$$

is the desired extension of φ_1. Similarly we construct Φ_2.

Consider the mapping $\widetilde{f} = \varphi_2 \circ f \circ \varphi_1^{-1} : \widetilde{V} \to \widetilde{W}$, where $\widetilde{V} = \varphi_1(V)$ and $\widetilde{W} = \varphi_2(W)$. Since $\widetilde{V} \subset \mathbf{k}^{n-1} \times 0 \subset \mathbf{k}^n$ and $\widetilde{W} \subset \mathbf{k}^{n-1} \times 0 \subset \mathbf{k}^{n+1}$, there exists a finite mapping $\widetilde{F} = (\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, 0, 0) : \mathbf{k}^{n-1} \times 0 \to \mathbf{k}^{n-1} \times 0$ such that $\widetilde{F}|_{\widetilde{V}} = \widetilde{f}$ [10].

Let $h = a_1 x_1 + \cdots + a_{n-1} x_{n-1}$, where $a_1, \ldots, a_{n-1} \in \mathbf{k}$, be a primitive element of $\mathbf{k}(\mathbf{k}^{n-1}) = \mathbf{k}(x_1, \ldots, x_{n-1})$ over $(\widetilde{F})^*(\mathbf{k}(\mathbf{k}^{n-1})) = \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1})$ (see e.g. [11, Theorem A.7.1]).

Put $\widehat{F} = (\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h) : \mathbf{k}^n \to \mathbf{k}^{n+1}$. Then $\widehat{F}$ is a finite mapping. Indeed, since $\mathbf{k}[x_1, \ldots, x_{n-1}]$ is integral over $(\widetilde{F})^*(\mathbf{k}[\mathbf{k}^{n-1}]) = \mathbf{k}[\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}]$ (because $\widetilde{F} : \mathbf{k}^{n-1} \times 0 \to \mathbf{k}^{n-1} \times 0$ is finite), $x_1, \ldots, x_{n-1}$ are integral over $\mathbf{k}[\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}]$, and so over $\mathbf{k}[\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h]$. Obviously x_n is integral over $\mathbf{k}[\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h]$ too. This means that $\mathbf{k}[x_1, \ldots, x_n]$ is integral over $\mathbf{k}[\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h]$, i.e. $\widehat{F}$ is finite.

Also, $\widehat{F} : \mathbf{k}^n \to \widehat{F}(\mathbf{k}^n)$ is a birational mapping, because

$$\mathbf{k}(x_1, \ldots, x_n) = \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h) = (\widehat{F})^*(\mathbf{k}(\mathbf{k}^{n+1})).$$

Indeed, we have

$$h = \frac{x_n h}{x_n} \in \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h).$$

Thus $\mathbf{k}(x_1, \ldots, x_{n-1}) = \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, h) \subset \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h)$. Since also $x_n \in \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h)$, it follows that $\mathbf{k}(x_1, \ldots, x_n) \subset \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h)$, and so

$$\mathbf{k}(x_1, \ldots, x_n) = \mathbf{k}(\widetilde{F}_1, \ldots, \widetilde{F}_{n-1}, x_n, x_n h).$$

Finally, because $x_n \in I(\widetilde{V})$, where $I(\widetilde{V})$ is the ideal of polynomials vanishing on $\widetilde{V}$, we have $\widehat{F}(x) = (\widetilde{F}_1(x), \ldots, \widetilde{F}_{n-1}(x), 0, 0) = \widetilde{f}(x)$ for $x \in \widetilde{V}$. Thus $\widehat{F}$ is a birational finite extension of $\widetilde{f}$. Now it is easy to see that $\Phi_2^{-1} \circ \widehat{F} \circ \Phi_1$ is a birational finite extension of f.

References

[1] M. Artin, *Algebraization of formal moduli: II. Existence of modifications*, Ann. of Math. (2) 91 (1970), 88–135.

[2] Z. Jelonek, *The extension of regular and rational embeddings*, Math. Ann. 277 (1987), 113–120.

[3] M. Karaś, *An estimate of the geometric degree of an extensions of some polynomial proper mappings*, Univ. Iagel. Acta Math. 35 (1997), 131–135.

[4] —, *Geometric degree of finite extension of projections*, ibid. 37 (1999), 109–119.

[5] —, *Birational finite extensions*, J. Pure Appl. Algebra 148 (2000), 251–253.

[6] —, *Finite extensions of mappings from a smooth variety*, Ann. Polon. Math. 75 (2000), 79–86.

[7] —, *Finite extensions of mappings of finite sets*, Bull. Polish Acad. Sci. Math. 50 (2002), 237–239.

[8] —, *Geometric degree of finite extensions of mappings from a smooth variety*, J. Pure Appl. Algebra 212 (2008), 1145–1148.

[9] —, *A note on geometric degree of finite extensions of mappings from a smooth variety*, Bull. Polish Acad. Sci. Math. 56 (2008), 105–108.

[10] M. Kwieciński, *Extending finite mappings to affine space*, J. Pure Appl. Algebra 76 (1991), 151–153.

[11] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, PWN, Warszawa, 1988.

[12] V. Srinivas, *On the embedding dimension of the affine variety*, Math. Ann. 289 (1991), 125–132.

Marek Karaś
Instytut Matematyki
Uniwersytet Jagielloński
Reymonta 4
30-059 Kraków, Poland
E-mail: Marek.Karas@im.uj.edu.pl

Received January 5, 2009;
received in final form June 4, 2009 (7897)