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MEASURE AND INTEGRATION

Upper Estimate of Concentration and Thin Dimensions of Measures

by

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The idea of this article originated during the last visit of Professor Andrzej Lasota in L'Aquila, December 2005. Unfortunately, his untimely death made a common conclusion of our nice discussions impossible. We have completed this paper as a token of respect for our Master and Friend.

Summary. We show upper estimates of the concentration and thin dimensions of measures invariant with respect to families of transformations. These estimates are proved under the assumption that the transformations have a squeezing property which is more general than the Lipschitz condition. These results are in the spirit of a paper by A. Lasota and J. Traple [Chaos Solitons Fractals 28 (2006)] and generalize the classical Moran formula.

1. Introduction. The concept of dimension of sets and measures is a basic tool in diverse branches of mathematics. For example, it is an important characteristic of attractors generated by iterated function systems. Various notions of dimension have been proposed: Hausdorff dimension, fractal dimension, correlation dimension, information dimension, capacity, entropy. These concepts have been widely investigated and used, but unfortunately, all of them are rather hard to calculate.

Recently two other concepts of dimension have been proposed by A. Lasota: the concentration dimension and the thin dimension. These dimensions are often easier to calculate and provide a natural and intrinsic estimation of the Hausdorff dimension and the fractal dimension of sets (see [13], [11]). The concentration dimension is defined by using the Lévy concentration function

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and it is strongly related to the Hausdorff dimension. The thin dimension is based on the notion of the thin function, which is a kind of anti-concentration function. This dimension is related to the fractal dimension.

Since the calculation of dimensions is rather difficult, it is important to find their estimates. Undoubtedly the most elegant and popular such result is the so called Moran formula (see [16]). Suppose that we have an IFS $\{w_i : i = 1, \dots, N\}$, where all functions w_i are strictly contractive with Lipschitz constants L_i. It is well known that such a system admits an attractor K. Then the Hausdorff dimension of K is less than or equal to d, where d is the unique solution of the *Moran equation*

$$\sum_{i=1}^{n} L_i^d = 1. \tag{1}$$

If all w_i are similarities with scaling factor L_i, then the Hausdorff dimension of K is equal to d. The above result has been generalized in various directions.

We will find an upper estimate of the concentration dimension of a measure invariant with respect to an iterated function system with a squeezing property, and an upper estimate of the thin dimension of a measure invariant with respect to an iterated function system with condensation.

2. Notation and auxiliary results. Let (X, ρ) be a separable metric space. By $B(x, r)$ (resp. $B^\circ(x, r)$) we denote the closed (resp. open) ball with center at $x \in X$ and radius $r > 0$. For $A \subset X$ and $x \in X$, $\overline{A}$ stands for the closure of A, $\operatorname{diam} A$ for the diameter of A, ∂A for the boundary of A and $\rho(x, A)$ for the distance from x to A. Occasionally we write $|A|$ in place of $\operatorname{diam} A$. As usual, we denote by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{N}$ the sets of reals, of all nonnegative reals and of all positive integers respectively.

By $\mathcal{B}(X)$ we denote the σ-algebra of Borel subsets of X and by $\mathcal{M}(X)$ the family of all finite Borel measures on X. Finally, $\mathcal{M}_1(X)$ denotes the space of all measures $\mu \in \mathcal{M}(X)$ such that $\mu(X) = 1$.

For $\mu \in \mathcal{M}(X)$ the *support* of μ is defined by

$$\operatorname{supp} \mu = \{x \in X : \mu(B(x, \varepsilon)) > 0 \text{ for every } \varepsilon > 0\}.$$

It is easy to verify that for every fixed $\varepsilon > 0$ the function $X \ni x \mapsto \mu(B^\circ(x, \varepsilon))$ is lower semicontinuous. Moreover, if (X, ρ) is separable, then $\sup\{\mu(B^\circ(x, \varepsilon)) : x \in X\} > 0$.

Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of subsets of X. The *lower limit* $\operatorname{Li} A_n$ and the *upper limit* $\operatorname{Ls} A_n$ are defined by the following conditions: A point x belongs to $\operatorname{Li} A_n$ if there is a sequence$\{x_n\}$ converging to x such that $x_n \in A_n$, while x belongs to $\operatorname{Ls} A_n$ if there is a sequence $\{x_{n_k}\}$ converging to x such that $x_{n_k} \in A_{n_k}$ for $k \in \mathbb{N}$. If $\operatorname{Li} A_n = \operatorname{Ls} A_n$, we denote this limit set by $\operatorname{Lt} A_n$ and call it the *topological* or *Kuratowski limit* of $\{A_n\}$.

By an *iterated function system* (briefly IFS) we mean a family of continuous functions

$$w_i : X \to X, \quad i \in I,$$

where $I = \{1, \ldots, N\}$.

Given an IFS $\{w_i : i \in I\}$ we define

$$F(A) = \overline{\bigcup_{i \in I} w_i(A)} \quad \text{for } A \subset X. \tag{2}$$

A closed set A_0 such that $F(A_0) = A_0$ is called *invariant* with respect to the IFS $\{w_i : i \in I\}$.

If there exists a closed set A_0 such that $\operatorname{Lt} F^n(A) = A_0$ for every nonempty bounded subset A of X, then the IFS $\{w_i : i \in I\}$ is called *asymptotically stable.* The set A_0 is uniquely defined and it is called the *attractor* or *fractal* (in the sense of Barnsley). Observe that in the case when X is a compact space, the topological limit coincides with the Hausdorff limit. Note also that the function F, in general, is not continuous with respect to the topological limit.

It is well known that if all w_i are strictly contractive, then there exists a unique compact set K such that

$$F(K) = \bigcup_{i=1}^{N} w_i(K).$$

Moreover, for every compact set $A \subset X$, $F^n(A) \to K$ in the Hausdorff distance. In the classical theory of iterated function systems the set K is called the *attractor* or *fractal* corresponding to the IFS $\{w_i : i \in I\}$ (see [1]). If all w_i are lipschitzian function with Lipschitz constant L_i, then $\dim_H K \leq d$, where d is the unique solution of the Moran equation (1). If all w_i are similarities with scaling factor L_i, then $\dim_H K = d$.

We say that an IFS $\{w_i : i \in I\}$ is *regular* if there exists a nonempty subset $I_0 \subset I$ such that the IFS $\{w_i : i \in I_0\}$ is asymptotically stable. The attractor of the subsystem $\{w_i : i \in I_0\}$ is called a *nucleus* of the system $\{w_i : i \in I\}$.

Let $\{w_i : i \in I\}$ be a regular IFS and let A_0 be a nucleus of this system. Define

$$A_* = \overline{\bigcup_{i \in I} F^n(A_0)}, \tag{3}$$

where F is given by (2).

Proposition 1 ([12]). *Let $\{w_i : i \in I\}$ be a regular IFS and let A_* be the corresponding semifractal defined by* (3). *Then*:

(i) A_* *does not depend on the choice of nucleus*;
(ii) A_* *is the smallest set invariant with respect to IFS*;
(iii) $\operatorname{Lt} F^n(A) = A_*$ *for every nonempty subset* A *of* A_*.

An operator $P : \mathcal{M}_1 \to \mathcal{M}_1$ is called *Markov* if:

- $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for $\lambda_1, \lambda_2 \in \mathbb{R}_+$, $\mu_1, \mu_2 \in \mathcal{M}_1$,
- $P\mu(X) = \mu(X)$.

A measure $\mu^* \in \mathcal{M}_1$ is called *invariant* with respect to P if $P\mu_* = \mu_*$. The operator P is called *asymtotically stable* if it admits an invariant measure μ_* and $\{P^n\mu\}$ converges weakly to μ_* (i.e $\int_X f\,dP^n\mu \to \int_X f\,d\mu$ for every continuous function $f : X \to \mathbb{R}$).

The family $\{(w_i, p_i) : i \in I\}$, where $w_i : X \to X$, $p_i : X \to (0,1)$, $i \in I$, are continuous functions and $\sum_{i\in I} p_i(x) = 1$ for all $x \in X$, is called an *IFS with probabilities.*

Given an IFS $\{(w_i, p_i) : i \in I\}$ we can define a Markov operator $P : \mathcal{M}_1 \to \mathcal{M}_1$ by

$$P\mu(A) = \sum_{i\in I} \int_{w_i^{-1}(A)} p_i(x)\,\mu(dx), \qquad A \in \mathcal{B}(X).$$

We say that a measure μ is *invariant with respect to the IFS* $\{(w_i, p_i) : i \in I\}$ if it is invariant with respect to the corresponding Markov operator. Similarly, IFS is called *asymptotically stable* if P is asymptotically stable. In particular, if all functions p_i are constant the invariant measure satisfies the condition

$$\mu(A) = \sum_{i\in I} p_i \mu(w_i^{-1}(A)), \qquad A \in \mathcal{B}(X). \tag{4}$$

It is well known that if all functions w_i are lipschitzian with Lipschitz constants L_i the probabilities p_i are constant, and if

$$\sum_{i\in I} p_i L_i < 1, \tag{5}$$

then the IFS $\{(w_i, p_i) : i \in I\}$ is asymptotically stable.

Proposition 2 ([12]). *Assume that the IFS* $\{(w_i, p_i) : i \in I\}$ *is asymptotically stable and the IFS* $\{w_i : i \in I\}$ *is regular. Moreover, assume that* $p_i(x) > 0$ *for* $x \in X$ *and* $i \in I$. *Then*

$$A_* = \operatorname{supp} \mu_*,$$

where A_* *is a semifractal of the IFS* $\{w_i : i \in I\}$ *and* μ_* *is an invariant measure with respect to the IFS* $\{(w_i, p_i) : i \in I\}$.

In the theory of iterated function systems normally the functions under consideration are supposed to be contractions or more generally lipschitzian.

Here we assume that they have the so called *squeezing property*. This property has been frequently used in the theory of differential equations (see [2]–[10], [15]–[17]) and more recently in the theory of iterated function systems (see [7, 8, 9, 14, 19, 20]).

Squeezing Property. Let A be a nonempty bounded subset of X. We say that a family of functions $\{w_i : i \in I\}$, $w_i : A \to A$, has the *squeezing property* if there exist nonexpansive mappings $P_i : A \to \mathbb{R}^{k_i}$ and constants $L_i \in (0,1)$ and $c_i \geq 0$ for $i \in I$ such that

$$\rho(w_i(x), w_i(y)) \leq \max\{L_i \rho(x,y), c_i \|P_i(x) - P_i(y)\|_i\} \quad \text{for } x, y \in X,$$

where $\|\cdot\|_i$ denotes the norm in $\mathbb{R}^{k_i}$.

The following covering property of Euclidean spaces is essential for further results:

Covering Property. Let $L \in (0,1)$, $c > 0$ and $(\mathbb{R}^k, \|\cdot\|)$ be given. Then there exists an integer $m \geq 1$ such that for every set $B \subset \mathbb{R}^k$ with $\operatorname{diam} B \leq c$ there exist Borel sets $\Delta_1, \ldots, \Delta_m$ such that

$$B \subset \bigcup_{j=1}^{m} \Delta_j \quad \text{and} \quad |\Delta_j| \leq L \quad \text{for } j = 1, \ldots, m.$$

Furthermore, in the case when $\operatorname{diam} B \leq cr$ we can find sets Δ_j, $j = 1, \ldots, m$, such that

$$B \subset \bigcup_{j=1}^{m} \Delta_j \quad \text{and} \quad |\Delta_j| \leq Lr \quad \text{for } j = 1, \ldots, m.$$

Lemma 1. *Let X be a metric space. Suppose that a family $\{w_i : i \in I\}$ has the squeezing property. Let $r > 0$. For every $i \in \{1, \ldots, N\}$ there exists m_i (chosen according to the Covering Property) such that for every $B \subset X$ with $|B| \leq r$ there are sets $D_1^i, \ldots, D_{m_i}^i$ such that*

$$w_i(B) \subset \bigcup_{j=1}^{m_i} D_j^i \quad \textit{and} \quad |D_j^i| < L_i r \quad \textit{for } j = 1, \ldots, m_i. \tag{6}$$

Proof. The proof is implicitly contained in the first step of the proof of Theorem 2.1 in [14]. For the convenience of the reader we give the main idea, thus making our exposition self-contained. For $i \in \{1, \ldots, N\}$ let m_i be the integer chosen according to the Covering Property and related to the constants c_i, L_i and the space $\mathbb{R}^{k_i}$. Let B be a Borel set such that $|B| \leq r$. Recall that $P_i(B) \subset \mathbb{R}^{k_i}$. Since $|c_i P_i(B)| \leq c_i r$, by the Covering Property there exist sets $\{\Delta_1, \ldots, \Delta_{m_i}\}$ such that $c_i P_i(B) \subset \bigcup_{j=1}^{m_i} \Delta_j$ and $|\Delta_j| \leq r L_i$ for $j = 1, \ldots, m_i$. Define

$$D_j^i = w_i^{-1}((c_i P_i)^{-1}(\Delta_j) \cap B), \qquad j = 1, \ldots, m_i.$$

Simple calculation shows that such sets satisfy condition (6).

Lemma 2. *Let $\alpha_i, \beta_i, L_i \in (0,1)$ for $i \in J$ be given, where J is an arbitrary set of indices. Let μ be a probability measure and let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a bounded increasing function. Suppose that for some $a > 0$ the function Φ satisfies the following condition*:

$$\Phi(r) \ge \sup_{i\in J} \alpha_i \Phi(r/L_i) \quad \text{for } r \in (0,a).$$

Then there exists $c > 0$ such that

$$\Phi(r) \ge cr^s \quad \text{for } r \in (0,a),$$

where

$$s = \min_{i\in J} \frac{\log \alpha_i}{\log L_i}.$$

For the proof of this lemma we refer to [13].

Lemma 3. *Let $L_i, p_i \in (0,1)$ for $i \in 1,\ldots,N$ and $\alpha > 0$ be given. Let functions $\varphi, \psi : (0,\infty) \to [0,\infty)$ be such that*

$$\psi(r) \ge \min\{\alpha, p_1\psi(r/L_1), \ldots, p_N\psi(r/L_N)\} \quad \text{for } 0 < r \le r_0, \tag{7}$$

$$\varphi(r) \le \min\{\alpha, p_1\varphi(r/L_1), \ldots, p_N\varphi(r/L_N)\} \quad \text{for } 0 < r \le r_0. \tag{8}$$

Suppose that

$$\varphi(r) \le \psi(r) \tag{9}$$

for $r \in [lr_0, r_0]$, where $l = \min\{L_1,\ldots,L_N\}$. Then inequality (9) *holds for every $r \in (0, r_0]$.*

Proof. Let $L = \max\{L_1,\ldots,L_N\}$. Using an induction argument one can show that for every $n \in \mathbb{N}$, inequality (9) holds for $r \in [L^{n+1}lr_0, r_0]$. Since $L^{n+1} \to 0$ as $n \to \infty$, the statement follows.

3. Upper bound for upper concentration dimension. Given a Borel measure $\mu \in \mathcal{M}_1$ the *upper concentration dimension* of μ is given by the formula

$$\overline{\dim}_L \mu = \limsup_{r\to 0} \frac{\log Q_\mu(r)}{\log r},$$

where

$$Q_\mu(r) = \sup\{\mu(A) : A \in \mathcal{B}(X),\ |A| \le r\} \quad \text{for } r > 0$$

is the well known Lévy concentration function. The values of Q_μ are always positive (see [11, Remark 2.1]).

Theorem 1. *Assume that a system $\{(w_i, p_i) : i \in I\}$ has the Squeezing Property. Let μ be an invariant measure with respect to this system. Further, for each couple of numbers L_i, c_i, $i \in I$, let the integer m_i be chosen according* *to the Covering Property. Then*

$$\overline{\dim}_L \mu \le \min_{i\in I} \frac{\log(p_i/m_i)}{\log L_i}.$$

Proof. Denote by $\tilde{\mu}$ the outer measure generated by μ, i.e.

$$\tilde{\mu}(E) = \inf\{\mu(A) : A \in \mathcal{B}(X),\, E \subset A\} \quad \text{for } E \subset X.$$

Using the relation (4) it is easy to verify that

$$\tilde{\mu}(E) \ge \sum_{i\in I} p_i \tilde{\mu}(w_i^{-1}(E)), \qquad E \subset X.$$

Fix $r_0 > 0$ and let $r \in (0, r_0]$. Let $B \in \mathcal{B}(X)$ be an arbitrary set such that $|B| < r$. From the above inequality and the inclusion $w_i^{-1}(w_i(B)) \supseteq B$ it follows that

$$\tilde{\mu}(w_i(B)) \ge p_i \tilde{\mu}(w_i^{-1}(w_i(B))) \ge p_i \mu(B).$$

Let $D_1^i, \ldots, D_{m_i}^i$ be given by Lemma 1. Obviously

$$\sum_{j=1}^{m_i} \mu(D_j^i) \ge \tilde{\mu}(w_i(B)) \ge p_i \mu(B).$$

Recall that $|D_j^i| < L_i r$ for $j = 1, \ldots, m_i$. From the definition of the Lévy concentration function Q_μ it follows that

$$\sum_{j=1}^{m_i} Q_\mu(L_i r) \ge p_i \mu(B).$$

Since $Q_\mu(L_i r)$ does not depend on the index j we have

$$m_i Q_\mu(L_i r) \ge p_i \mu(B).$$

The last inequality is true for every $B \subset X$ with $|B| < r$. Consequently,

$$m_i Q_\mu(L_i r) \ge p_i Q_\mu(r).$$

Evidently this condition is equivalent to

$$Q_\mu(r) \ge \frac{p_i}{m_i}\, Q_\mu(r/L_i).$$

Since $i \in I$ is arbitrary, it follows that

$$Q_\mu(r) \ge \sup_{i\in I} \frac{p_i}{m_i}\, Q_\mu(r/L_i), \qquad r \in (0, r_0].$$

According to Lemma 2 there exists $c > 0$ such that

$$Q_\mu(r) \ge c r^s, \qquad r \in (0, r_0],$$

where

$$s = \min_{i\in I} \frac{\log(p_i/m_i)}{\log L_i}.$$

Consequently,

$$\overline{\dim}_L \mu \le \min_{i\in I} \frac{\log(p_i/m_i)}{\log L_i}.$$

Theorem 1 immediately yields

COROLLARY 1. *Let d be the unique positive solution of the equation*

$$\sum_{i\in I} m_i L_i^d = 1.$$

Put $p_i = m_i L_i^d$ for $i \in I$ and consider the IFS $\{(w_i, p_i) : i \in I\}$. Suppose that a measure μ is invariant with respect to this system. Then

$$\overline{\dim}_L \mu \le d.$$

In Theorem 1 we assume that all L_i are strictly positive and this condition is essentially used in the proof. Now we will show that the statement of Theorem 1 remains true if some of the constants L_i are zero.

THEOREM 2. *Let an IFS $\{(w_i, p_i) : i \in \{1, \ldots, N + M\}\}$ be given. Assume that there exist nonexpansive maps $P_i : X \to \mathbb{R}^{k_i}$ for $i = 1, \ldots, N+M$, constants $L_i \in (0, 1)$ for $i = 1, \ldots, N$ and constants $c_i \ge 0$ for $i = 1, \ldots, N + M$ such that*

$$\rho(w_i(x), w_i(y)) \le \max\{L_i \rho(x, y), c_i \|P_i(x) - P_i(y)\|_i\}, \qquad x, y \in X,$$

for $i = 1, \ldots, N$ and

$$\rho(w_i(x), w_i(y)) \le c_i \|P_i(x) - P_i(y)\|_i, \qquad x, y \in X,$$

for $i = N + 1, \ldots, N + M$. Suppose that μ is an invariant measure with respect to the given system. Then

$$\overline{\dim}_L \mu \le \min\left\{\frac{\log(p_1/m_1)}{\log L_1}, \ldots, \frac{\log(p_N/m_N)}{\log L_N}, k_{N+1}, \ldots, k_{N+M}\right\}.$$

Proof. We may assume that all coefficients c_i for $i > N$ are positive integers. Fix an integer $n \ge 2$ and define $L_i = 1/n$ for $i = N+1, \ldots, N+M$. For given c_i and $L_i = 1/n$, $i > N$, the integer number m_i, required by the Covering Property, is equal to $q_i(c_i n)^{k_i}$, where q_i is a constant depending on the norm in $\mathbb{R}^{k_i}$. For example, if $\mathbb{R}^{k_i}$ is endowed with the supremum norm, then $q_i = 1$. Observe that the completed IFS $\{(w_i, p_i) : i \in \{1, \ldots, N + M\}\}$ has the Squeezing Property. Applying Theorem 1 we obtain

$$\overline{\dim}_L \mu \le \min\left\{\frac{\log(p_1/m_1)}{\log L_i}, \ldots, \frac{\log(p_{N+M}/m_{N+M})}{\log L_{N+M}}\right\}. \tag{10}$$

For $i > N$, we have

$$\frac{\log(p_i/m_i)}{\log L_i} = \frac{\log(p_i q_i) - k_i \log n - k_i \log c_i}{-\log n}.$$

Letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} \frac{\log(p_i/m_i)}{\log L_i} = k_i \quad \text{for } i = N+1, \ldots, N+M.$$

From the last observation and inequality (10) the statement of Theorem 2 follows immediately.

4. Upper bound for thin dimension. Given $\mu \in \mathcal{M}_1(X)$ we define the *lower* and *upper thin dimensions* by the formulas

$$\underline{\dim}_T\, \mu = \liminf_{r\to 0} \frac{\log T_\mu(r)}{\log r}, \qquad \overline{\dim}_T\, \mu = \limsup_{r\to 0} \frac{\log T_\mu(r)}{\log r},$$

where

$$T_\mu(r) = \inf\{\mu(B(x,r)) : x \in \operatorname{supp}\mu\} \quad \text{for } r > 0.$$

The function $T_\mu : (0,\infty) \to [0,1]$ will be called the *thin function* corresponding to the measure μ. Note that the values $T_\mu(r)$ are positive if $\operatorname{supp}\mu$ is a compact set. In general $T_\mu(r)$ is only nonnegative and we adopt the convention that $\log 0 = -\infty$.

We are going to find upper bounds on the thin dimension of the invariant measure for so called condensation systems (see [1]). Let now (X,ϱ) denote a complete, separable metric space.

Let $w_i : X \to X$, $i = 1,\ldots,N$, be a sequence of functions satisfying the Lipschitz conditions

$$\varrho(w_i(x), w_i(y)) \le L_i \varrho(x,y) \quad \text{for } x, y \in X,\ i = 1,\ldots,N. \tag{11}$$

Let C be an arbitrary finite subset of X and let $w_0 : X \to C$ be a given function. Further, let $(p_0, p_1, \ldots, p_N)$ be a probability vector, i.e.

$$\sum_{i=0}^{N} p_i = 1, \qquad p_k > 0 \quad \text{for } k = 0,\ldots,N. \tag{12}$$

THEOREM 3. *Suppose that the IFS* $\{(w_i,p_i) : i \in \{0,\ldots,N\}\}$ *satisfies conditions* (11) *and* (12). *Assume that* $w_0 : X \to C$, *where* C *is a finite subset of* X. *Let* μ *be an invariant measure of the system* $\{(w_i,p_i) : i \in \{0,\ldots,N\}\}$, *i.e.*

$$\mu(A) = \sum_{i=0}^{N} p_i \mu(w_i^{-1}(A)) \quad \text{for } A \in \mathcal{B}(X). \tag{13}$$

Further, assume that $\operatorname{supp}\mu$ *is compact. Then*

$$\overline{\dim}_T\, \mu \le \max_{1\le i\le N} \frac{\log p_i}{\log L_i}.$$

Proof. Standard calculation shows that for every measurable function $w : X \to X$ we have

$$\operatorname{supp}(\mu \circ w^{-1}) \subset \overline{w(\operatorname{supp}\mu)}. \tag{14}$$

From (13) it follows that

$$\operatorname{supp}\mu = \bigcup_{i=0}^{N} \operatorname{supp}(\mu \circ w_i^{-1}),$$

and by (14),

$$\operatorname{supp}\mu \subset \operatorname{supp}(\mu \circ w_0^{-1}) \cup \overline{w_1(\operatorname{supp}\mu)} \cup \cdots \cup \overline{w_N(\operatorname{supp}\mu)}.$$

Since $K = \operatorname{supp}\mu$ is compact, the last inclusion can be rewritten in the form

$$K \subset \operatorname{supp}(\mu \circ w_0^{-1}) \cup w_1(K) \cup \cdots \cup w_N(K). \tag{15}$$

Fix $r_0 > 0$ and let $r \in (0, r_0)$. Since the function $X \ni x \mapsto \mu(B^\circ(x, r))$ is lower semicontinuous, by the Weierstrass theorem there exists $\tilde{x} \in K$ such that

$$\mu(B^\circ(\tilde{x}, r)) = \inf\{\mu(B^\circ(x, r)) : x \in K\}.$$

Consequently,

$$T_\mu(r) = \mu(B^\circ(\tilde{x}, r)).$$

By (15) the point $\tilde{x}$ belongs to $\bigcup_{i=1}^{N} w_i(K)$ or to $\operatorname{supp}(\mu \circ w_0^{-1})$.

Suppose first that $\tilde{x} \in \bigcup_{i=1}^{N} w_i(K)$. This means that $\tilde{x} = w_i(y)$ for some $y \in K$. On the other hand, observe that

$$w_i^{-1}(B^\circ(\tilde{x}, r)) \supset B^\circ(y, r/L_i). \tag{16}$$

Indeed, for $z \in B^\circ(y, r/L_i)$, we have

$$\rho(\tilde{x}, w_i(z)) = \rho(w_i(y), w_i(z)) < L_i \rho(y, z) < r,$$

which means that $w_i(z) \in B^\circ(\tilde{x}, r)$. By (13) and (16) we have

$$\mu(B^\circ(\tilde{x}, r)) \geq p_i \mu(w_i^{-1}(B^\circ(\tilde{x}, r))) \geq p_i \mu(B^\circ(y, r/L_i)) \geq p_i T_\mu(r/L_i).$$

Consequently,

$$T_\mu(r) \geq p_i T_\mu(r/L_i) \quad \text{for } r \in (0, r_0]. \tag{17}$$

Suppose now that $\tilde{x} \in \operatorname{supp}(\mu \circ w_0^{-1})$. Then (14) yields $\operatorname{supp}(\mu \circ w_0^{-1}) \subset \overline{w_0(K)}$. But $w_0(K)$ is a finite set. Consequently, so is $\operatorname{supp}(\mu \circ w_0^{-1})$, say $\{c_1, \ldots, c_m\}$. Obviously $(\mu \circ w_0^{-1})(\{c_j\}) > 0$ for $j = 1, \ldots, m$. Let

$$\alpha_\mu = \min_{1 \leq j \leq m} (\mu \circ w_0^{-1})(\{c_j\}).$$

Since $\tilde{x} \in \{c_1, \ldots, c_m\}$ we have

$$(\mu \circ w_0^{-1})(B^\circ(\tilde{x}, r)) \geq \alpha_\mu \quad \text{for } r \in (0, r_0],$$

and by (13),

$$\mu(B^\circ(\tilde{x},r)) \geq p_0(\mu\circ w_0^{-1})(B^\circ(\tilde{x},r)) \geq p_0\alpha_\mu \quad \text{for } r\in(0,r_0].$$

Finally, from the last inequality and conditions (17) it follows that

$$T_\mu(r) \geq \min\{p_0\alpha_\mu, p_1T_\mu(r/L_1),\ldots,p_NT_\mu(r/L_N)\}, \quad r\in(0,r_0]. \tag{18}$$

The last inequality says that the function $\psi(r) = T_\mu(r)$ satisfies condition (7) with $p_0\alpha_\mu$ in place of α.

Now consider the function

$$\varphi(r) = cr^s, \quad \text{where} \quad 0 < c \leq p_0\alpha_\mu \quad \text{and} \quad s \geq \max_{1\leq i\leq N} \frac{\log p_i}{\log L_i}.$$

It is easy to see that φ satisfies condition (8).

Observe that the smallest value of ψ in the interval $[lr_0, r_0]$ is $T_\mu(lr_0)$. On the other hand, the largest value of φ in $[lr_0, r_0]$ is $\varphi(r_0) = cr_0^s$. Thus, for a sufficiently small constant c, inequality (9) holds in $[lr_0, r_0]$, and by Lemma 3 this inequality holds for every $r\in(0,r_0]$.

Obviously

$$\overline{\dim}_T\,\mu = \limsup_{r\to 0}\frac{\log T_\mu(r)}{\log r} \leq \limsup_{r\to 0}\frac{\log\varphi(r)}{\log r} = s.$$

Hence the statement of Theorem 3 follows immediately.

The example below shows that the statement of Theorem 3 fails to hold without the assumption that C is a finite subset of X.

Example 1. Consider the set

$$X = [0,13/8]\cup\{a_n : n = 0,1,\ldots\}\cup\{9/4\},$$

where

$$a_0 = 2 \quad \text{and} \quad a_n = 2 + \frac{3}{2\pi^2}\sum_{k=1}^{n}\frac{1}{k^2} \quad \text{for } n\geq 1.$$

On the space X consider the functions

$$w_1(x) = \frac{1}{2}\,x, \quad w_2(x) = \frac{1}{2}\,x + \frac{1}{2} \quad \text{for } x\in X,$$

and

$$w_0(x) = \begin{cases} 2 & \text{for } x\in[0,13/8],\\ a_n & \text{for } x = a_{n-1},\ n = 1,2,\ldots,\\ 9/4 & \text{for } x = 9/4.\end{cases}$$

Since $\sum_{k=1}^\infty(1/k^2) = \pi^2/6$, the sequence $\{a_n\}$ converges to $9/4$. Obviously the functions w_1 and w_2 are similarities with scaling factor $L_1 = L_2 = 1/2$ and w_0 is a Lipschitzian function with Lipschitz constant $L_0 = 1$.

Observe that the IFS $\{(w_i, p_i) : i = 0, 1, 2\}$, where $p_i = 1/3$, $i = 0, 1, 2$, satisfies condition (5) and so it is asymptotically stable. Let μ_* be the corresponding invariant measure. Obviously, IFS $\{w_i : i = 0, 1, 2\}$ is regular. Let K be the corresponding semifractal. According to Proposition 2 we have $K = \operatorname{supp} \mu_*$. Clearly,

$$K = w_0(K) \cup w_1(K) \cup w_2(K).$$

Set $D = \overline{\{a_n : n = 0, 1, \ldots\}}$. From (3) it follows easily that $D \subset K$. In addition we have

$$w_0(K) = D, \qquad w_1(K) \subset [0, 9/8], \qquad w_2(K) \subset [1/2, 13/8].$$

Thus $w_0^{-1}(w_1(K)) = \emptyset$ and $w_0^{-1}(w_2(K)) = \emptyset$. Consequently,

$$(\mu_* \circ w_0^{-1})(w_1(K)) = 0 \quad \text{and} \quad (\mu_* \circ w_0^{-1})(w_2(K)) = 0. \tag{19}$$

In particular, since $\operatorname{supp} \mu_* = K$ and $w_0^{-1}(w_0(K)) \supset K$ we have

$$(\mu_* \circ w_0^{-1})(w_0(K)) = 1.$$

Using (4), (19) and the last equality we obtain

$$\begin{aligned}\mu_*(w_0(K)) &= \frac{1}{3}\,\mu_*(w_0^{-1}(w_0(K))) + \frac{1}{3}\,\mu_*(w_1^{-1}(w_0(K))) + \frac{1}{3}\,\mu_*(w_2^{-1}(w_0(K))) \\ &= \frac{1}{3}.\end{aligned}$$

Observe also that for $n \ge 2$,

$$w_0^n(K) = \overline{\{a_{n-1}, a_n, \ldots\}}$$

and

$$w_0^{-1}(w_0^n(K)) = \overline{\{a_{n-2}, a_{n-1}, \ldots\}} = w_0^{n-1}(K).$$

By an induction argument it is easy to verify that

$$\begin{aligned}\mu_*(w_0^n(K)) &= \frac{1}{3}\,\mu_*(w_0^{-1}(w_0^n(K))) + \frac{1}{3}\,\mu_*(w_1^{-1}(w_0^n(K))) + \frac{1}{3}\,\mu_*(w_2^{-1}(w_0^n(K))) \\ &= \frac{1}{3}\,\mu_*(w_0^{-1}(w_0^n(K))) = \frac{1}{3^n}.\end{aligned}$$

Thus

$$\begin{aligned}\mu_*(\{a_n\}) &= \mu_*(w_0^{n+1}(K) \setminus w_0^{n+2}(K)) \\ &= \mu_*(w_0^{n+1}(K)) - \mu_*(w_0^{n+2}(K)) \\ &= 1/3^{n+1} - 1/3^{n+2} = 2/3^{n+2}.\end{aligned}$$

Finally, observe that for $r_n = 3/(4\pi^2 n^2)$ we have $B^\circ(a_n, r_n) = \{a_n\}$. Consequently,

$$T_{\mu_*}(r_n) \le \mu_*(\{a_n\}) \le 2/3^{n+2}.$$

It follows that

$$\frac{\log T_{\mu_*}(r_n)}{\log r_n} \geq \frac{\log(2/3^{n+2})}{\log r_n} = \frac{\log 2 - (n+2)\log 3}{\log(3/4\pi^2) - 2\log n}.$$

Letting $n \to \infty$ we obtain

$$\underline{\dim}_T \mu_* = \infty.$$

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