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REAL FUNCTIONS

On Borel Classes of Sets of Fréchet Subdifferentiability

by

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Summary. We study possible Borel classes of sets of Fréchet subdifferentiability of continuous functions on reflexive spaces.

1. Introduction and main result. Our terminology follows [2, 6]. We recall the most important definitions and notation in Sections 1 and 2. In this paper, all normed linear spaces are supposed to be real.

Let X be a normed linear space and f be a real function on X. For $x \in X$, we define the *Fréchet subdifferential* of f at x by

$$\partial f(x) = \left\{ u \in X^* : \liminf_{y \to x} \frac{f(y) - f(x) - u(y-x)}{\|y - x\|} \geq 0 \right\}.$$

Any element of $\partial f(x)$ is called a *Fréchet subgradient* of f at x. We say that x is a *point of Fréchet subdifferentiability* of f if $\partial f(x) \neq \emptyset$. The set of all points of Fréchet subdifferentiability of f is denoted by $S(f)$.

First, we recall a known result in this area.

THEOREM 1.1 (Holický, Laczkovich). *Let f be a lower semicontinuous function on a normed linear space X with reflexive completion. Then $S(f)$ is a Σ_4^0 set.*

The proof of this theorem can be found in [3]. Note that the set of Fréchet subdifferentiability of a continuous function on a normed linear space X may not be Borel if the completion of X is not reflexive (see [3, Theorem 1.3]).

The main result of the paper follows. It says that the result of Holický and Laczkovich is "best possible".

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THEOREM 1.2. *Let X be a normed linear space with* $\dim X \geq 3$. *Then there is a continuous real function f on X such that $S(f)$ is Σ_4^0-complete.*

This theorem will be proved in Section 3.

REMARK 1.3. 1) If f is a continuous function on $\mathbb{R}$, then $S(f)$ is a Π_3^0 set by the classical result that the Dini derivatives of f are of Baire class 2 (see, e.g., [1]). On the other hand, by a result of Zahorski (see, e.g., [5]), there is a Lipschitz function f on $\mathbb{R}$ such that the set $D(f)$ of all points of differentiability of f is Π_3^0-complete. Since $D(f) = S(f) \cap S(-f)$, at least one of the sets $S(f), S(-f)$ is Π_3^0-complete.

If $f : \mathbb{R}^2 \to \mathbb{R}$ is lower semicontinuous, then $S(f)$ is Σ_4^0 (by Theorem 1.1) and can be Π_3^0-complete (by the result of Zahorski). We do not know anything more about the situation in $\mathbb{R}^2$.

2) The set of Fréchet subdifferentiability of a Lipschitz function f on a space with reflexive completion is a Π_3^0 set. This follows from the proof of Theorem 1.2 in [3] and from the observation that the norms of Fréchet subgradients of f are uniformly bounded by the Lipschitz constant (and thus $S(f) = \bigcap_{k=1}^{\infty} \bigcup_{(n_1,\ldots,n_k)\in\mathbb{N}^k} A^K_{n_1,\ldots,n_k}$ for some $K \in \mathbb{N}$, where $A^K_{n_1,\ldots,n_k} = \bigcup_{\|u\|\leq K} \bigcap_{i=1}^{k} \{x \in X : \|y-x\| < 1/n_i \Rightarrow f(y)-f(x) \geq u(y-x)-i^{-1}\|y-x\|\}$). Together with the above-mentioned result of Zahorski, this says that Π_3^0 is the smallest Borel class which contains the set of Fréchet subdifferentiability of each Lipschitz function on a reflexive space.

3) Let $g : X \to [-\infty, \infty)$ be a lower semicontinuous function, where X is a space with reflexive completion. Then the set $G = \{x \in X : g(x) > -\infty\}$ is open, and $S(g) \subset G$ can be defined in the same way as $S(f)$ for finite f. By the method of Holický and Laczkovich, $S(g) \in \Sigma_4^0$.

4) Let X be a space with reflexive completion and $f : X \to \mathbb{R}$ be Σ_α^0-measurable (i.e., $f^{-1}(U) \in \Sigma_\alpha^0$ whenever $U \subset \mathbb{R}$ is open). One may ask whether $S(f)$ is Borel, or even of which Borel class it is. By an observation of Šmídek (see [4]), $S(f) = S(g) \cap \{x \in X : f(x) = g(x)\}$, where g is the greatest lower semicontinuous minorant of f. So $S(f)$ is the intersection of a Σ_4^0 set and a Π_α^0 set.

2. Some elements of descriptive set theory. Let us recall some definitions and notation. A topological space is called *Polish* if it is separable and completely metrizable.

Given a topological space M, we use $\Sigma_\alpha^0(M)$ and $\Pi_\alpha^0(M)$, where $\alpha < \omega_1$, for the Borel classes (see [2]). What is most important for us is that Σ_4^0 is $F_{\sigma\delta\sigma}$ and Π_3^0 is $F_{\sigma\delta}$ in the classical notation. We say that $A \subset M$ is Σ_α^0-*hard* (resp. Π_α^0-*hard*) if, for every zero-dimensional Polish space P and $B \in \Sigma_\alpha^0(P)$ (resp. $B \in \Pi_\alpha^0(P)$), there exists a continuous mapping $f : P \to M$ such that

$f^{-1}(A) = B$. We say that A is Σ^0_α-*complete* (resp. Π^0_α-*complete*) if A is Σ^0_α-hard and $A \in \Sigma^0_\alpha(M)$ (resp. Π^0_α-hard and $A \in \Pi^0_\alpha(M)$).

Let P be a Polish space. It is known that being Σ^0_α-complete in P amounts to being an element of $\Sigma^0_\alpha(P) \setminus \Pi^0_\alpha(P)$ (and similarly for Σ^0_α and Π^0_α interchanged). For example, a subset of $\mathbb{R}^3$ is Σ^0_4-complete if and only if it is $F_{\sigma\delta\sigma}$, but not $G_{\delta\sigma\delta}$.

By $\forall^\infty$ we mean "for all but finitely many".

Lemma 2.1 (cf. [2, Exercise 23.3]). *The set*

$$D = \big\{\nu \in \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} : \forall^\infty k \in \mathbb{N}\ \forall m \in \mathbb{N}\ \forall^\infty l \in \mathbb{N}\ \nu(k,l,m) = 0\big\}$$

is Σ^0_4*-hard in* $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$.

Proof. By [2, 23.A], the set

$$E = \big\{\sigma \in \{0,1\}^{\mathbb{N}\times\mathbb{N}} : \forall m \in \mathbb{N}\ \forall^\infty l \in \mathbb{N}\ \sigma(l,m) = 0\big\}$$

is Π^0_3-complete in $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$. Let P be a zero-dimensional Polish space and $B \in \Sigma^0_4(P)$. Then $B = \bigcup_{k=1}^\infty B_k$ for some $B_1, B_2, \ldots \in \Pi^0_3(P)$. Since the class Π^0_3 is closed under finite unions, we may suppose that $B_1 \subset B_2 \subset \cdots$. For every $k \in \mathbb{N}$, there exists a continuous mapping $f_k : P \to \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ such that $f_k^{-1}(E) = B_k$. We define

$$f(p)(k,l,m) = f_k(p)(l,m), \qquad p \in P,\ k,l,m \in \mathbb{N}.$$

It is easy to check that $f : P \to \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ is continuous and that $f^{-1}(D) = B$, which proves the lemma. ∎

3. Proof of Theorem 1.2. In this section, by *c-Lipschitz* we mean Lipschitz with constant c.

Lemma 3.1. *There are continuous functions* $\chi_{k,l} : \mathbb{R} \to [0,1]$, $k,l \in \mathbb{N}$, *such that*

(a) $\chi_{k,l}(x) \ge \chi_{k+1,l}(x)$ *for every* $k,l \in \mathbb{N}$, $x \in \mathbb{R}$,
(b) $\chi_{k,l}$ *is* l*-Lipschitz for every* $k,l \in \mathbb{N}$,
(c) *the set*

$$\bigcup_{k=1}^\infty \big\{x \in \mathbb{R} : \lim_{l\to\infty} \chi_{k,l}(x) = 0\big\}$$

is Σ^0_4*-hard in* $\mathbb{R}$.

Proof. We define functions $n_{k,l} : \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \to \mathbb{N}$ and $\varphi_{k,l}, \phi_{k,l} : \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \to [0,1]$ for $k,l \in \mathbb{N}$ by

$$\begin{aligned} n_{k,l}(\nu) &= \min(\{m \in \mathbb{N} : \nu(k,l,m) = 1\} \cup \{l\}), \\ \varphi_{k,l}(\nu) &= \frac{1}{n_{k,l}(\nu)}, \\ \phi_{k,l} &= \begin{cases} 0, & k > l, \\ \displaystyle\sum_{j=k}^{l} 2^{-j} \varphi_{j,l}, & k \le l. \end{cases} \end{aligned}$$

For $k \in \mathbb{N}$ and $\nu \in \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$, we verify the following two equivalences:

$$\lim_{l\to\infty} \varphi_{k,l}(\nu) = 0 \;\Leftrightarrow\; \lim_{l\to\infty} n_{k,l}(\nu) = \infty \;\Leftrightarrow\; \forall m \in \mathbb{N}\ \forall^\infty l \in \mathbb{N} : \nu(k,l,m) = 0.$$

The first equivalence is obvious; let us prove the other one. Assume that $\lim_{l\to\infty} n_{k,l}(\nu) = \infty$. For given $m \in \mathbb{N}$, we have to find $p \in \mathbb{N}$ such that $\nu(k,l,m) = 0$ for every $l \ge p$. We choose $p \in \mathbb{N}$ such that $n_{k,l}(\nu) > m$ for every $l \ge p$. By the definition of $n_{k,l}$, $\nu(k,l,j) = 0$ for $1 \le j \le m$ and $l \ge p$, which gives the implication "$\Rightarrow$". Now, suppose

$$\forall m \in \mathbb{N}\ \forall^\infty l \in \mathbb{N} : \nu(k,l,m) = 0.$$

For given m, we have to find $p \in \mathbb{N}$ such that $n_{k,l}(\nu) > m$ for every $l \ge p$. If $l > m$, then by the definition of $n_{k,l}$ we have $n_{k,l}(\nu) > m$ whenever $\nu(k,l,i) = 0$ for $1 \le i \le m$. So it is enough to choose $p \in \mathbb{N}$ such that $p > m$ and $\nu(k,l,i) = 0$ for $l \ge p$ and for $1 \le i \le m$. This proves the implication "$\Leftarrow$", and the second equivalence is also proved.

Now, we are going to prove that

$$\bigcup_{k=1}^{\infty} \Big\{\nu \in \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} : \lim_{l\to\infty} \phi_{k,l}(\nu) = 0\Big\} = D$$

for the set D from Lemma 2.1. Indeed, for $\nu \in \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$, we have

$$\begin{aligned} \nu \in D &\Leftrightarrow \exists k_0 \in \mathbb{N}\ \forall k \ge k_0\ \forall m \in \mathbb{N}\ \forall^\infty l \in \mathbb{N} : \nu(k,l,m) = 0 \\ &\Leftrightarrow \exists k_0 \in \mathbb{N}\ \forall k \ge k_0 : \lim_{l\to\infty} \varphi_{k,l}(\nu) = 0 \\ &\Leftrightarrow \exists k_0 \in \mathbb{N} : \lim_{l\to\infty} \sum_{j=k_0}^{\infty} 2^{-j} \varphi_{j,l}(\nu) = 0 \\ &\Leftrightarrow \exists k_0 \in \mathbb{N} : \lim_{l\to\infty} \phi_{k_0,l}(\nu) = 0. \end{aligned}$$

Let $\pi : \mathbb{N}\times\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ be a bijection. We define a homeomorphism h between $\{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by

$$h : (\alpha_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}} \mapsto (\alpha_{\pi(k,l,m)})_{(k,l,m)\in\mathbb{N}\times\mathbb{N}\times\mathbb{N}}.$$

Consider the following metric on $\{0,1\}^{\mathbb{N}}$:

$$\varrho(\nu,\nu') = \max(\{3^{-n} : \nu(n) \ne \nu'(n)\} \cup \{0\}), \quad \nu,\nu' \in \{0,1\}^{\mathbb{N}}.$$

Let us check that

(1) $\forall l \in \mathbb{N}\ \exists L_l > 0\ \forall k \in \mathbb{N} : \phi_{k,l} \circ h$ is L_l-Lipschitz on $(\{0,1\}^{\mathbb{N}}, \varrho)$.

It is enough to prove that there exists $L_{k,l} > 0$ such that $\varphi_{k,l} \circ h$ is $L_{k,l}$-Lipschitz for every $k, l \in \mathbb{N}$ because then we can take $L_l = \sum_{j=1}^{l} 2^{-j} L_{j,l}$. We put

$$L_{k,l} = \max\left\{3^{\pi(k,l,m)} : 1 \le m \le l\right\}, \quad k, l \in \mathbb{N}.$$

Let $\nu, \nu' \in \{0,1\}^{\mathbb{N}}$. If $\varrho(\nu,\nu') \ge 1/L_{k,l}$, then $|(\varphi_{k,l} \circ h)(\nu) - (\varphi_{k,l} \circ h)(\nu')| \le 1 \le L_{k,l}\varrho(\nu,\nu')$. If $\varrho(\nu,\nu') < 1/L_{k,l}$ (i.e., $\varrho(\nu,\nu') < 3^{-\pi(k,l,m)}$ for $1 \le m \le l$), then, by the definition of ϱ, $\nu(\pi(k,l,m)) = \nu'(\pi(k,l,m))$ (i.e., $h(\nu)(k,l,m) = h(\nu')(k,l,m)$) for $1 \le m \le l$, and, by the definitions of $n_{k,l}$ and $\varphi_{k,l}$, $n_{k,l}(h(\nu)) = n_{k,l}(h(\nu'))$ and $\varphi_{k,l}(h(\nu)) = \varphi_{k,l}(h(\nu'))$. So the choice of $L_{k,l}$ works, and (1) is proved.

Now, define $g : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ by

$$g(\nu) = 2\sum_{k=1}^{\infty} \left(\frac{1}{3}\right)^{k} \nu(k), \quad \nu \in \{0,1\}^{\mathbb{N}}.$$

One can easily check that

$$\frac{1}{3}\,|g(\nu) - g(\nu')| \le \varrho(\nu,\nu') \le |g(\nu) - g(\nu')|, \quad \nu, \nu' \in \{0,1\}^{\mathbb{N}}.$$

Set $C = g(\{0,1\}^{\mathbb{N}})$. We see that g is a homeomorphism of $\{0,1\}^{\mathbb{N}}$ onto C. Consequently, the set

$$\bigcup_{k=1}^{\infty} \left\{x \in C : \lim_{l\to\infty} (\phi_{k,l} \circ h \circ g^{-1})(x) = 0\right\} = g(h^{-1}(D))$$

is Σ^0_4-hard in C by Lemma 2.1.

Since g^{-1} is 1-Lipschitz, the function $\phi_{k,l} \circ h \circ g^{-1}$ is L_l-Lipschitz for $k, l \in \mathbb{N}$. We can extend these functions from C to $\mathbb{R}$ by

$$\chi'_{k,l} = \sup\left\{u : \mathbb{R} \to [0,1] :\ u \text{ is } L_l\text{-Lipschitz},\ u \le \phi_{k,l} \circ h \circ g^{-1} \text{ on } C\right\}.$$

We now prove that the following conditions hold:

(a′) $\chi'_{k,l}(x) \ge \chi'_{k+1,l}(x)$ for every $k, l \in \mathbb{N},\ x \in \mathbb{R}$,

(b′) $\chi'_{k,l}$ is L_l-Lipschitz for every $k, l \in \mathbb{N}$,

(c′) the set

$$\bigcup_{k=1}^{\infty} \left\{x \in \mathbb{R} : \lim_{l\to\infty} \chi'_{k,l}(x) = 0\right\}$$

is Σ^0_4-hard in $\mathbb{R}$.

Let $k, l \in \mathbb{N}$. Obviously, $\phi_{k,l} \ge \phi_{k+1,l}$ on $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$. Thus, $\phi_{k,l} \circ h \circ g^{-1} \ge \phi_{k+1,l} \circ h \circ g^{-1}$ on C. Hence $\chi'_{k,l} \ge \chi'_{k+1,l}$ by the definitions of $\chi'_{k,l}$ and

$\chi'_{k+1,l}$. So (a′) holds. Since the supremum of any non-empty system of c-Lipschitz functions with a uniform upper bound at one point is c-Lipschitz, (b′) holds. Finally, since $g(h^{-1}(D)) = C \cap \bigcup_{k=1}^{\infty}\{x \in \mathbb{R} : \lim_{l\to\infty} \chi'_{k,l}(x) = 0\}$ is Σ^0_4-hard in C, (c′) holds.

We choose an increasing sequence of natural numbers $1 \le s_1 < s_2 < \cdots$ such that $s_i \ge L_i$ for $i \in \mathbb{N}$. For $k, l \in \mathbb{N}$, we define

$$\chi_{k,l} = \begin{cases} 1, & 1 \le l < s_1, \\ \chi'_{k,i}, & s_i \le l < s_{i+1},\ i \in \mathbb{N}, \end{cases}$$

which completes the proof. ∎

Proof of Theorem 1.2. It is enough to construct a function f with the required properties on $\mathbb{R}^3$ (in the general case, X can be expressed as the topological sum $\mathbb{R}^3 \oplus Y$ for some closed subspace Y of X, and if we define $F(x+y) = f(x)$ for $x \in \mathbb{R}^3$ and $y \in Y$, then $S(F) = S(f) + Y$ would also be Σ^0_4-complete). In the proof, we use $|\cdot|$ for the Euclidean norm on $\mathbb{R}^n, n = 2, 3$.

Let $\chi_{k,l} : \mathbb{R} \to [0,1]$, $k, l \in \mathbb{N}$, be as in Lemma 3.1. We define functions $f, f_l : \mathbb{R}^3 \to \mathbb{R}$, $l \in \mathbb{N}$, by

$$f_l(x,y,z) = \max\{(k-1)y - \chi_{k,l}(x)|(y,z)| : 1 \le k \le l\}, \quad (x,y,z) \in \mathbb{R}^3,$$

$$f(x,y,z) = \begin{cases} 0, & y = z = 0, \\ \dfrac{(l+1)^{-2} - |(y,z)|}{(l+1)^{-2} - (l+2)^{-2}}\, f_{l+1}(x,y,z) & \\ \qquad + \dfrac{|(y,z)| - (l+2)^{-2}}{(l+1)^{-2} - (l+2)^{-2}}\, f_l(x,y,z), & \\ & (l+2)^{-2} \le |(y,z)| < (l+1)^{-2}, \\ f_1(x,y,z), & 1/4 \le |(y,z)|. \end{cases}$$

Obviously, the functions $f_l,\ l \in \mathbb{N}$, are continuous and the function f is continuous on $\{(x,y,z) \in \mathbb{R}^3 : (l+2)^{-2} \le |(y,z)| < (l+1)^{-2}\},\ l \in \mathbb{N}$, and on $\{(x,y,z) \in \mathbb{R}^3 : 1/4 \le |(y,z)|\}$. To prove that f is continuous on the union of these sets (i.e., on $\{(x,y,z) \in \mathbb{R}^3 : |(y,z)| > 0\}$), we have to check that for $l \in \mathbb{N}$ and $(x_0,y_0,z_0) \in \mathbb{R}^3$ with $|(y_0,z_0)| = (l+1)^{-2}$,

$$\lim_{\substack{(x,y,z)\to(x_0,y_0,z_0) \\ (l+2)^{-2} \le |(y,z)| < (l+1)^{-2}}} f(x,y,z) = f(x_0,y_0,z_0).$$

This holds because both sides of the equality are equal to $f_l(x_0,y_0,z_0)$. The proof of the continuity of f will be completed if we verify that

$$|f(x,y,z)| \le \sqrt{|(y,z)|} \quad \text{for } (x,y,z) \in \mathbb{R}^3 \text{ with } |(y,z)| < 1/4$$

(and thus that f is continuous at each $(x,0,0)$ for $x \in \mathbb{R}$). Let $(x,y,z) \in \mathbb{R}^3$ and $|(y,z)| < 1/4$. We may suppose that $|(y,z)| > 0$. Let $l \in \mathbb{N}$ be such that

$(l+2)^{-2} \le |(y,z)| < (l+1)^{-2}$. Since $f(x,y,z)$ is a convex combination of $f_l(x,y,z)$ and $f_{l+1}(x,y,z)$, it is enough to check that

$$|(y,z)| \le j^{-2} \;\Rightarrow\; |f_j(x,y,z)| \le \sqrt{|(y,z)|}$$

for $j \in \mathbb{N}$ (and thus $|f_l(x,y,z)| \le \sqrt{|(y,z)|}$ and $|f_{l+1}(x,y,z)| \le \sqrt{|(y,z)|}$). Let $j \in \mathbb{N}$ be such that $|(y,z)| \le j^{-2}$. Using the definition of f_j (and the fact that the ranges of $\chi_{k,j}$ are subsets of $[0,1]$), we get $|f_j(x,y,z)| \le j|(y,z)|$. We have

$$|f_j(x,y,z)| \le j|(y,z)| \le |(y,z)|^{-1/2}|(y,z)| = \sqrt{|(y,z)|},$$

and the continuity of f is proved.

Let us proceed to the investigation of $S(f)$. By Theorem 1.1, $S(f)$ is Σ^0_4. By the property (c) of the system $\{\chi_{k,l}\}_{k,l\in\mathbb{N}}$, to prove that $S(f)$ is Σ^0_4-complete, it is sufficient to prove that, for $a \in \mathbb{R}$,

$$(a,0,0) \in S(f) \;\Leftrightarrow\; \exists k \in \mathbb{N} : \lim_{l\to\infty} \chi_{k,l}(a) = 0.$$

Let us prove the implication "$\Rightarrow$". Suppose $\limsup_{l\to\infty} \chi_{k,l}(a) > 0$ for every $k \in \mathbb{N}$ and let $u \in (\mathbb{R}^3)^*$. We have to check that u is not a Fréchet subgradient of f at $(a,0,0)$. Suppose the opposite, i.e., $u \in \partial f(a,0,0)$. Let $\lambda \in \mathbb{R}$. By the definition of f_l, $l \in \mathbb{N}$, we have $f_l(a,0,\lambda) \le 0$. Consequently, $f(a,0,\lambda) \le 0$. We have

$$0 \le \liminf_{\lambda\to 0} \frac{f(a,0,\lambda) - u(0,0,\lambda)}{|(0,0,\lambda)|} \le \liminf_{\lambda\to 0} \frac{-u(0,0,\lambda)}{|\lambda|} = -|u(0,0,1)|.$$

So $u(0,0,1) = 0$ and

$$u(0,y,z) = cy, \quad y,z \in \mathbb{R},$$

where $c = u(0,1,0)$. We choose $n \in \mathbb{N}$ such that $n \ge c+1$. There exists $\varepsilon > 0$ such that $(c+1)\varepsilon < \limsup_{l\to\infty} \chi_{n,l}(a)$. If we define

$$p_l = (l+1)^{-2}(0, -\varepsilon, \sqrt{1-\varepsilon^2}), \quad l \in \mathbb{N},$$

and use the property (a), we have

$$\begin{aligned}
\frac{f((a,0,0)+p_l) - u(p_l)}{|p_l|} &= \frac{f_l((a,0,0)+p_l) - u(p_l)}{|p_l|} \\
&= \frac{1}{(l+1)^{-2}} \left(\max\{(1-k)(l+1)^{-2}\varepsilon - \chi_{k,l}(a)(l+1)^{-2} : k \le l\}\right) + c\varepsilon \\
&\le \sup\{(1-k)\varepsilon - \chi_{k,l}(a) : k \in \mathbb{N}\} + c\varepsilon \\
&\le \max\{\max\{(1-k)\varepsilon - \chi_{k,l}(a) : 1 \le k \le n\} + c\varepsilon, -n\varepsilon + c\varepsilon\} \\
&\le \max\{c\varepsilon - \chi_{n,l}(a), -\varepsilon\}.
\end{aligned}$$

By the choice of ε, for every $l_0 \in \mathbb{N}$, there exists $l \geq l_0$ such that $\chi_{n,l}(a) \geq (c+1)\varepsilon$, i.e., $c\varepsilon - \chi_{n,l}(a) \leq -\varepsilon$. Consequently,

$$\frac{1}{|p_l|}\big(f((a,0,0)+p_l) - u(p_l)\big) \leq -\varepsilon$$

for such l. Since $p_l \to (0,0,0)$,

$$\liminf_{(x,y,z)\to(a,0,0)} \frac{1}{|(x-a,y,z)|}\big(f(x,y,z) - u(x-a,y,z)\big) \leq -\varepsilon,$$

which contradicts the fact that u is a Fréchet subgradient of f at $(a,0,0)$. So the implication "$\Rightarrow$" is proved.

Now, let us prove "$\Leftarrow$". We have to find a Fréchet subgradient of f at $(a,0,0)$ assuming that there exists $k \in \mathbb{N}$ such that $\lim_{l\to\infty} \chi_{k,l}(a) = 0$. Let us fix such a k. We claim that

$$u(x,y,z) = (k-1)y, \quad (x,y,z) \in \mathbb{R}^3,$$

is the required Fréchet subgradient. Let $\varepsilon > 0$ be given. We can choose $l_0 \in \mathbb{N}$ such that $\chi_{k,l}(a) \leq \varepsilon/2$ for every $l \geq l_0$. We choose $\delta > 0$ such that

$$\delta < 1/4, \quad \delta^{1/2} \leq \varepsilon, \quad \delta^{1/6} \leq \varepsilon/2, \quad \delta < (l_0+1)^{-2}, \quad \delta < (k+1)^{-2}.$$

Let $(x,y,z) \in \mathbb{R}^3$ and $0 < |(x-a,y,z)| \leq \delta$. We now check that

$$\frac{f(x,y,z) - u(x-a,y,z)}{|(x-a,y,z)|} \geq -\varepsilon.$$

Clearly, this holds if $(y,z) = 0$. So we may suppose that $|(y,z)| > 0$. For $l \geq k$, by the definition of f_l,

$$f_l(x,y,z) - (k-1)y \geq -\chi_{k,l}(x)|(y,z)|.$$

Since $0 < |(y,z)| \leq \delta < 1/4$, we have $(l+2)^{-2} \leq |(y,z)| < (l+1)^{-2}$ for some $l \in \mathbb{N}$. Since $(l+2)^{-2} \leq |(y,z)| \leq \delta < (k+1)^{-2}$, it follows that $l \geq k$. Since $f(x,y,z)$ is a convex combination of $f_l(x,y,z)$ and $f_{l+1}(x,y,z)$, it follows that

$$(2) \qquad f(x,y,z) - u(x-a,y,z) \geq -\max\{\chi_{k,l}(x)|(y,z)|, \chi_{k,l+1}(x)|(y,z)|\}.$$

If $|(y,z)| \leq |x-a|^{3/2}$, using (2), we have

$$\frac{f(x,y,z) - u(x-a,y,z)}{|(x-a,y,z)|} \geq -\frac{|(y,z)|}{|(x-a,y,z)|} \geq -|x-a|^{1/2} \geq -\delta^{1/2} \geq -\varepsilon.$$

In the other case (i.e., if $|(y,z)| > |x-a|^{3/2}$), by (b) and by the fact that $l \geq l_0$ ($(l+2)^{-2} \leq |(y,z)| \leq \delta < (l_0+1)^{-2}$), using (2) again, we have

$$\begin{aligned}\frac{f(x,y,z) - u(x-a,y,z)}{|(x-a,y,z)|} &\geq -\frac{\max\{\chi_{k,l}(x)|(y,z)|, \chi_{k,l+1}(x)|(y,z)|\}}{|(x-a,y,z)|}\\ &\geq -\max\{\chi_{k,l}(x), \chi_{k,l+1}(x)\}\end{aligned}$$

$$\geq -\max\{\chi_{k,l}(a), \chi_{k,l+1}(a)\} - (l+1)|x-a|$$
$$> -\varepsilon/2 - |(y,z)|^{-1/2}|(y,z)|^{2/3}$$
$$\geq -\varepsilon/2 - \delta^{1/6} \geq -\varepsilon.$$

So, for given $\varepsilon > 0$, we have found $\delta > 0$ such that

$$0 < |(x-a,y,z)| \leq \delta \;\Rightarrow\; \frac{f(x,y,z) - u(x-a,y,z)}{|(x-a,y,z)|} \geq -\varepsilon.$$

This means that u is a Fréchet subgradient of f at $(a,0,0)$, and the implication "$\Leftarrow$" is proved. ∎

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