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FUNCTIONAL ANALYSIS

Quotients of Continuous Convex Functions on Nonreflexive Banach Spaces

by

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Summary. On each nonreflexive Banach space X there exists a positive continuous convex function f such that $1/f$ is not a d.c. function (i.e., a difference of two continuous convex functions). This result together with known ones implies that X is reflexive if and only if each everywhere defined quotient of two continuous convex functions is a d.c. function. Our construction also gives a stronger version of Klee's result concerning renormings of nonreflexive spaces and non-norm-attaining functionals.

A function on a Banach space X is called a *d.c. function* if it can be represented as a difference of two continuous convex functions (all functions considered in this note are real-valued). Thus the system of all d.c. functions on X is the smallest vector space containing all continuous convex functions. Moreover, it is well known (see, e.g., [3, III.2]), and not difficult to show, that it is also closed with respect to taking products and pointwise maxima; hence it is even an algebra and a lattice. While an everywhere defined quotient g/f of two d.c. functions on a finite-dimensional Banach space is always d.c. (cf. [2, Corollary]), the situation is completely different for infinite-dimensional spaces: by [7, Corollary 5.6], on each infinite-dimensional Banach space there exists a positive d.c. function such that $1/f$ is not d.c.

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The following natural question arises:

Is the quotient g/f of two continuous convex functions on X d.c. if $f \neq 0$?

Quite surprisingly, the answer is affirmative for all reflexive spaces X; indeed, it is proved in [7, Remark 3.5(i)] that $1/f$ ($f \neq 0$ continuous and convex) is d.c. on X whenever X is reflexive. The main aim of this note is to show that the above question has a negative answer for each nonreflexive Banach space X.

The following criterion for non-d.c. functions (cf. [7, Lemma 5.1]) suggests how to construct a counterexample.

LEMMA 1. *Let X be a Banach space and $h: X \to \mathbb{R}$ be a function. If there exist sets $M \subset X$ of arbitrarily small diameter such that h is unbounded on M, then h is not a d.c. function.*

If there exists a continuous convex function f on X such that

(1) $f > 0$, and there exist sets M of arbitrarily small diameters with $\inf f(M) = 0$,

then $1/f$ is not a d.c. function by Lemma 1. (Of course, such an f cannot exist if X is reflexive since, in this case, f attains its minimum on any closed ball.)

To construct f, it might seem natural to proceed by finding an $x^* \in X^*$ such that

(2) x^* does not attain its norm, and there exist sets $M \subset B_X$ of arbitrarily small diameter such that $\sup x^*(M) = \|x^*\|_*$.

Indeed, if we had such an x^*, it would be sufficient to put $f(x) := \|x\| - \|x^*\|_*$ if $x^*(x) = \|x^*\|_*$, and to extend f to the whole X so that f is constant on each line parallel to a fixed vector $v \in X$ such that $x^*(v) \neq 0$. While it is not difficult to check that no such x^* exists in the classical nonreflexive spaces c_0 and ℓ_1 (with their canonical norms), it is possible to prove (see below) that such an x^* always exists after a suitable equivalent renorming of (a nonreflexive) X.

However, we proceed in a different order. First, using James' sequential characterization of nonreflexivity, we construct a continuous convex function f on X, satisfying (1), as a distance function from a certain bounded convex set in $X \oplus \mathbb{R}$. Using this f, we easily prove our main Theorem 4, which also gives a modification of the well known characterization of nonreflexive spaces by monotone sequences of closed convex sets. Then, using the existence of such f on each hyperplane of X, we show that, if X is nonreflexive, each nonzero functional $x^* \in X^*$ satisfies (2) with respect to a suitable equivalent norm on X. This last assertion is the content of Proposition 5 which we believe to be of independent interest since it improves the following result of Klee [5]: each nonzero bounded linear functional on a nonreflexive Banach space X is non-norm-attaining for some equivalent norm on X.

Let us start by fixing some notations. We consider only Banach spaces over the reals. We denote by B_X or $B_{(X,\|\cdot\|)}$ the closed unit ball in a Banach space X endowed with a norm $\|\cdot\|$. By $\|\cdot\|_*$ we denote the corresponding dual norm on X^* (the topological dual of X).

In what follows, we consider $X \oplus \mathbb{R}$ equipped with the maximum norm, and we identify $x \in X$ with $(x, 0) \in X \oplus \mathbb{R}$ (and so X with $X \times \{0\}$).

Lemma 2. *Let X be a nonreflexive Banach space. Then there exists a nonempty bounded convex set $C \subset X \oplus \mathbb{R}$ such that*

(a) $\varphi(x) := \operatorname{dist}(x, C) > 0$ *for every* $x \in X$,
(b) *for each* $\varepsilon > 0$ *there is a set* $M_\varepsilon \subset X$ *with* $\operatorname{diam} M_\varepsilon < \varepsilon$ *and* $\inf \varphi(M_\varepsilon) = 0$.

Proof. Since X is nonreflexive, by [4, Theorem 1] (see, e.g., [1, Theorem 10.3] or [6, Theorem 1.13.4] for simpler proofs) there exist unit vectors $\{e_i\}_{i=1}^\infty$ in X and unit functionals $\{e_i^*\}_{i=1}^\infty$ in X^* such that

$$e_i^*(e_j) = 0 \quad \text{if } i > j, \qquad e_i^*(e_j) > 1/2 \quad \text{if } i \le j. \tag{3}$$

Set $e_\infty := (0, 1) \in X \oplus \mathbb{R}$, and let $f_i \in (X \oplus \mathbb{R})^*$ be the extension of e_i^* for which $f_i(e_\infty) = 1$. Clearly $\|f_i\|_* = 2$. For $0 < k < n$ in $\mathbb{N}$, we define

$$x_{k,n} := 2e_k + \frac{2}{k}\, e_n + \frac{1}{n}\, e_\infty.$$

Clearly

$$f_i(x_{k,n}) \ge 1 \qquad \text{for } 1 \le i \le k, \tag{4}$$

$$f_i(x_{k,n}) \ge \frac{1}{k} \qquad \text{for } k < i \le n, \tag{5}$$

$$f_i(x_{k,n}) = \frac{1}{n} \qquad \text{for } i > n. \tag{6}$$

We define

$$C := \operatorname{conv}\{x_{k,n} : 0 < k < n,\ k, n \in \mathbb{N}\}, \qquad X_0 := \overline{\operatorname{span}}\{e_j : j \in \mathbb{N}\}.$$

To prove (a), we need to show $\overline{C} \cap X = \emptyset$. Since clearly $\overline{C} \cap X \subset X_0$, it is sufficient to show that $\overline{C} \cap X_0 = \emptyset$. So, suppose to the contrary that an $x_0 \in \overline{C} \cap X_0$ is given. As $\|f_i\|_* = 2$ and $\lim_{i\to\infty} f_i(e_j) = 0$ for each $j \in \mathbb{N}$, it is easy to check that $\lim_{i\to\infty} f_i(x) = 0$ for every $x \in X_0$. So, we may find natural numbers $i_1 < i_2 < i_3$ such that

$$f_{i_1}(x_0) < \frac{1}{3}, \qquad i_1 f_{i_2}(x_0) < \frac{1}{3}, \qquad i_2 f_{i_3}(x_0) < \frac{1}{3}. \tag{7}$$

Since $x_0 \in \overline{C}$ and f_{i_1}, f_{i_2}, f_{i_3} are continuous, we can find $c \in C$ so close to

x_0 that

$$(8)\qquad f_{i_1}(c) < \frac{1}{3}, \quad i_1 f_{i_2}(c) < \frac{1}{3}, \quad i_2 f_{i_3}(c) < \frac{1}{3}.$$

Since $c \in C$, we can assign to each (k,n) with $1 \le k < n$ a number $\alpha_{k,n} \ge 0$ so that $\sum \alpha_{k,n} = 1$, the set $\{(k,n) : \alpha_{k,n} \ne 0\}$ is finite, and $c = \sum \alpha_{k,n} x_{k,n}$.

Using (4), (5), and (6) in turn, we obtain

$$(9)\qquad f_{i_1}(c) = \sum \alpha_{k,n} f_{i_1}(x_{k,n}) \ge \sum_{\substack{k \ge i_1 \\ n > k}} \alpha_{k,n}\,,$$

$$(10)\qquad f_{i_2}(c) = \sum \alpha_{k,n} f_{i_2}(x_{k,n}) \ge \sum_{\substack{k < i_1 \\ n \ge i_2}} \frac{1}{k}\,\alpha_{k,n} \ge \frac{1}{i_1} \sum_{\substack{k < i_1 \\ n \ge i_2}} \alpha_{k,n}\,,$$

$$(11)\qquad f_{i_3}(c) = \sum \alpha_{k,n} f_{i_3}(x_{k,n}) \ge \sum_{\substack{k < i_1 \\ n < i_2}} \frac{1}{n}\,\alpha_{k,n} \ge \frac{1}{i_2} \sum_{\substack{k < i_1 \\ n < i_2}} \alpha_{k,n}\,.$$

Using (9), (10), (11) and (8), we easily obtain $\sum \alpha_{k,n} < 1$, which is a contradiction.

To prove (b), consider an arbitrary $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ with $4/k_0 < \varepsilon$ and set $M_\varepsilon := \{2e_{k_0} + (2/k_0)e_n : n > k_0\}$. Then clearly $\operatorname{diam} M_\varepsilon \le 4/k_0 < \varepsilon$. The other desired property of M_ε also holds, since, for each $n > k_0$,

$$\begin{aligned}\inf \varphi(M_\varepsilon) &= \operatorname{dist}(M_\varepsilon, C) \\ &\le \|(2e_{k_0} + (2/k_0)e_n) - (2e_{k_0} + (2/k_0)e_n + (1/n)e_\infty)\| = 1/n. \ \blacksquare\end{aligned}$$

Remark 3.

(i) To obtain C with the weaker property $\inf_{x \in X} \varphi(x) = 0$ instead of (b) in Lemma 2, it is sufficient to put $C := \operatorname{conv}\{2e_k + (1/k)e_\infty : k \in \mathbb{N}\}$, and the proof becomes simpler.

(ii) Set $C := \operatorname{conv}\{2e_k + (2/k)e_n + (2/n)e_m + (1/m)e_\infty : 0 < k < n < m,\ k,n,m \in \mathbb{N}\}$. An easy modification of the proof of Lemma 2 gives the following property which is slightly stronger than (b):

(b^2) *there exist sets $M \subset X$ of arbitrarily small diameter such that M contains sets A of arbitrarily small diameter with* $\inf \varphi(A) = 0$.

(Analogously, using indices $0 < k_1 < \cdots < k_{p+1}$ in the definition of C, it is possible to obtain the corresponding iterated property (b^p).)

Now, we are ready to state the following main result of the present paper.

Theorem 4. *The following properties of a Banach space X are equivalent.*

(a) *X is nonreflexive.*
(b) *There is a continuous convex function $f: X \to (0,\infty)$ such that $1/f$ is not representable as a difference of two continuous convex functions.*
(c) *There is a decreasing sequence $\{C_n\}_{n=1}^{\infty}$ of bounded closed convex subsets of X such that*

$$\bigcap_{n=1}^{\infty} C_n = \emptyset, \qquad \bigcap_{n=1}^{\infty} (C_n + \varepsilon B_X) \neq \emptyset \quad \textit{for every } \varepsilon > 0.$$

Proof. If X is nonreflexive, take $f := \varphi$ where φ is as in Lemma 2. By Lemma 1, $1/f$ is not d.c. on X. On the other hand, if X is reflexive and f is a positive continuous convex function, then $1/f$ is d.c. on X by [7, Remark 3.5(i)]. Thus (a) and (b) are equivalent.

Let us show that (a) and (c) are equivalent. If X is nonreflexive, let φ be again the function from Lemma 2. The sets $C_n := \{x \in X : \varphi(x) \le 1/n\}$, $n \in \mathbb{N}$, are nonempty, closed, convex, bounded (since the set C in Lemma 2 is bounded) and their intersection is empty. Let $\varepsilon > 0$. By the properties of φ, there exists $x \in X$ such that, for each n, there is $y \in B(x,\varepsilon)$ with $\varphi(y) \le 1/n$, i.e. $y \in C_n$. In other words, $x \in \bigcap_{n=1}^{\infty}(C_n + \varepsilon B_X)$. Hence (a) implies (c). On the other hand, if X is reflexive, then each decreasing sequence $\{C_n\}$ of nonempty closed bounded convex subsets of X has a nonempty intersection, since each C_n is weakly compact. ∎

Let us conclude our paper with the promised strengthening of a result from [5].

PROPOSITION 5. *Let Y be a nonreflexive Banach space and $0 \neq y^* \in Y^*$. Then there exists an equivalent norm $|\cdot|$ on Y such that*

(a) *y^* does not attain its norm on $B_{(Y,|\cdot|)}$,*
(b) *for each $\varepsilon > 0$, there is $M_\varepsilon \subset B_{(Y,|\cdot|)}$ such that* $\operatorname{diam} M_\varepsilon < \varepsilon$ *and* $\sup y^*(M_\varepsilon) = |y^*|_*$.

Proof. Set $X := \{y \in Y : \ y^*(y) = 0\}$ and choose $e \in Y$ with $y^*(e) = 1$. Up to renorming, we may suppose that the norm on Y satisfies

$$\|y\| = \max\{\|y - y^*(y)e\|, |y^*(y)|\} \quad \text{ for all } y \in Y.$$

In this way we may identify Y with $X \oplus_\infty \mathbb{R}$ so that $y^*((x,t)) = t$ for $(x,t) \in X \times \mathbb{R}$.

As Y is not reflexive, neither is X. Let φ be the function on X given by Lemma 2. Choose $\alpha > \varphi(0)$ and set

$$A = \{x \in X : \varphi(x) < \alpha\}.$$

By the properties of φ the set A is bounded. Therefore we can choose $r > 0$ such that $A \subset B(0,r)$. Choose $\beta > \sup \varphi(B(0,r))$; this is possible as φ is

1-Lipschitz. Further, define

$$D = \{(x,t) \in X \times \mathbb{R} : x \in B(0,r),\, t = \varphi(x) - \beta\},$$
$$C = \overline{\mathrm{conv}}(D \cup (-D)).$$

Then C is clearly a bounded closed convex symmetric set. Further, $0 \in \operatorname{int} C$, as $0 \in A$ and $A \times (\alpha-\beta, \beta-\alpha) \subset C$. It follows that there exists an equivalent norm $|\cdot|$ on $X \times \mathbb{R}$ such that C is the closed unit ball in this norm. We will show that this norm has the required properties.

We have

$$-|y^*|_* = \inf y^*(C) = \inf y^*(D \cup (-D)) = \inf y^*(D)$$
$$= \inf\{\varphi(x) - \beta : x \in B(0,r)\} = -\beta,$$

as clearly $\inf \varphi(B(0,r)) = \inf \varphi(X) = 0$. Thus $|y^*|_* = \beta$.

Next we show that y^* does not attain its norm on C. Suppose it does. Then there is a point $z = (x_0, -\beta) \in C$ (recall that $y^*((x,t)) = t$). Note that

$$C \subset \{(x,t) \in X \times \mathbb{R} : x \in B(0,r) \;\&\; t \geq \varphi(x) - \beta\}.$$

The reason is that the set on the right hand side is closed and convex and it contains both D and $-D$. It follows that z belongs to the set on the right hand side, i.e. $-\beta \geq \varphi(x_0) - \beta$. So $\varphi(x_0) \leq 0$, a contradiction.

It remains to show (b). Let $\varepsilon > 0$ be given. By the properties of φ we can choose a set $P_\varepsilon \subset A$ such that $\operatorname{diam} P_\varepsilon < \varepsilon$ and $\inf \varphi(P_\varepsilon) = 0$. (Note that $\varphi \geq \alpha$ outside A.) Now set

$$P^*_\varepsilon := \{(x,t) \in X \times \mathbb{R} : x \in P_\varepsilon,\, t = \varphi(x) - \beta\}.$$

Then clearly $P^*_\varepsilon \subset C$ and

$$\inf_{z \in P^*_\varepsilon} y^*(z) = -\beta = -|y^*|_*.$$

As φ is 1-Lipschitz with respect to $\|\cdot\|$, we see that $\|\cdot\|$-$\operatorname{diam} P^*_\varepsilon < \varepsilon$. Set $M_\varepsilon := -P^*_{\varepsilon/K}$, where $K > 0$ is such that $|\cdot| \leq K\|\cdot\|$ on $X \times \mathbb{R}$. Then M_ε has all required properties and the proof is complete. ∎

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