BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 55, No. 3, 2007

GENERAL TOPOLOGY

Uniformly Movable Categories and Uniform Movability of Topological Spaces

by

P. S. GEVORGYAN and I. POP

Presented by Czesław BESSAGA

Summary. A categorical generalization of the notion of movability from inverse systems and shape theory was given by the first author who defined the notion of movable category and used it to interpret the movability of topological spaces. In this paper the authors define the notion of uniformly movable category and prove that a topological space is uniformly movable in the sense of shape theory if and only if its comma category in the homotopy category **HTop** over the subcategory **HPol** of polyhedra is uniformly movable. This is a weakened version of the categorical notion of uniform movability introduced by the second author.

1. Introduction. The notion of movability for metric compacta was introduced by K. Borsuk [1] as an important shape invariant. The movable spaces are a generalization of spaces having the shape of ANR's. The movability assumption allows a series of important results in algebraic topology (like the Whitehead and Hurewicz theorems) to remain valid with the homotopy pro-groups replaced by the corresponding shape groups. The term "movability" comes from the geometric interpretation of the definition in the compact case: if X is a compactum lying in a space $M \in$ AR, one says that X is movable if for every neighborhood U of X in M there exists a neighborhood $V \subset U$ of X such that for every neighborhood $W \subset U$ of X there is a homotopy $H : V \times [0,1] \to U$ such that $H(x,0) = x$ and $H(x,1) \in W$ for every $x \in V$. One shows that the choice of $M \in$ AR is irrelevant [1]. After the notion of movability had been expressed in terms of ANR-systems, for arbitrary topological spaces, [4], it became clear that one could define it in

2000 *Mathematics Subject Classification*: 54C56, 55P55, 18A25.
Key words and phrases: shape theory, uniformly movable inverse system (space), uniformly movable category.

arbitrary pro-categories. The definitions of a movable object in an arbitrary pro-category and of uniform movability were given by M. Moszyńska [6]. Uniform movability is important in the study of mono- and epi-morphisms in pro-categories and in the study of the shape of pointed spaces. In the book of Mardešić & Segal [5] all these approaches and applications of various types of movability are discussed.

A categorical generalization of the notion of movability from inverse systems and shape theory was given by the first author of the present paper who defined the notion of movable category and interpreted the movability of topological spaces in terms of this property [2].

A concept of uniform movability for a category was introduced by the second author in [7]. In that paper a category $\mathcal{K}$ is called uniformly movable with respect to a subcategory $\mathcal{K}'$ if there exists a pair (F,φ) with $F:\mathcal{K}\to\mathcal{K}$ a covariant functor and $\varphi: F\to 1_{\mathcal{K}}$ a natural transformation such that every morphism $f\in\mathcal{K}(Y,X)$ with $Y\in\mathcal{K}'$ admits a morphism $G(f)\in\mathcal{K}(F(X),Y)$ satisfying $f\circ G(f)=\varphi(X)$ and such that the correspondence $f\mapsto G(f)$ is natural in the sense that a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "u"] & X \arrow[d, "v"] \\ Y' \arrow[r, "f'"] & X' \end{tikzcd}$$

in the category $\mathcal{K}$, with $u:Y\to Y'$ a morphism in $\mathcal{K}'$, induces the equality $G(f')\circ F(v)=u\circ G(f)$. In the case $\mathcal{K}'=\mathcal{K}$ the category $\mathcal{K}$ was simply named uniformly movable. The pair (F,φ) was called a uniform movability pair of $\mathcal{K}$ and the morphism $G(f)$ a uniform movability factor of f.

This definition is good and suggestive from the functorial point of view but it is too strong if one has in mind movability of topological spaces. The uniform movability of the comma category of a space X in **HTop** over **HPol** implies the uniform movability of X ([7, Cor. 1]). However, the converse was proved only under two supplementary conditions on the space X ([7, Cor. 2]). In the present paper we give a weakened version of this definition that permits a characterization of the uniform movability of an arbitrary space X via the uniform movability of the comma category of X.

2. Uniformly movable categories. Let $\mathcal{K}$ be an arbitrary category.

Definition 1 ([2], [3]). We say that an object X of $\mathcal{K}$ is *movable* if there are an object $M(X)\in\mathcal{K}$ and a morphism $m_X: M(X)\to X$ in $\mathcal{K}$ such that for any object $Y\in\mathcal{K}$ and any morphism $p:Y\to X$ in $\mathcal{K}$ there exists a morphism $u(p): M(X)\to Y$ which makes the diagram

$$\begin{tikzcd} & M(X) \arrow[dl, "m_X"'] \arrow[dd, "u(p)"] \\ X & \\ & Y \arrow[ul, "p"] \end{tikzcd}$$

Diagram 1

commutative, i.e., $p \circ u(p) = m_X$.

A category $\mathcal{K}$ is called *movable* if any object of $\mathcal{K}$ is movable.

DEFINITION 2. We say that an object X of the category $\mathcal{K}$ is *uniformly movable* if there are an object $M(X) \in \mathcal{K}$ and a morphism $m_X : M(X) \to X$ in $\mathcal{K}$ that satisfy the following conditions:

1. for any object $Y \in \mathcal{K}$ and any morphism $p : Y \to X$ in $\mathcal{K}$ there exists a morphism $u(p) : M(X) \to Y$ in $\mathcal{K}$ which makes Diagram 1 commutative,
2. for all morphisms $p : Y \to X$, $q : Z \to X$ and $r : Z \to Y$ in $\mathcal{K}$ such that $p \circ r = q$, Diagram 2 below is commutative:

$$\begin{tikzcd} Y \arrow[dr, "p"] & & \\ & X & M(X) \arrow[l, "m_X"'] \arrow[ull, "u(p)"'] \arrow[dll, "u(q)"] \\ Z \arrow[uu, "r"] \arrow[ur, "q"'] & & \end{tikzcd}$$

Diagram 2

i.e., $u(p) = r \circ u(q)$.

A category $\mathcal{K}$ is called *uniformly movable* if any object of $\mathcal{K}$ is uniformly movable.

We call m_X a (*uniform*) *movability morphism* of X and $u(p)$ a (*uniform*) *movability factor* of p.

It is evident that every uniformly movable object is movable. The following example shows that the converse is not true.

EXAMPLE 1. Let $\mathbf{Set}^\circ$ be the category of nonempty sets. Then each singleton is a movable object which is not uniformly movable.

Indeed, let $\{*\}$ be a singleton. Then a movability morphism for $\{*\}$ can be any constant map $M(*) \to \{*\}$. For an arbitrary map $q : Z \to \{*\}$ we can take for $u(q) : M(*) \to Z$ any map. Thus $\{*\}$ is movable. But if $p : Y \to \{*\}$

is another map we can write $q = p \circ r$ and $q = p \circ r'$ for any maps $r, r' : Z \to Y$. Let $x_* \in M(*)$ and suppose that $r(u(q)(x_*)) \neq r'(u(q)(x_*))$ (suppose that Y has cardinality at least two). Then the relations $u(p) = r \circ u(q)$ and $u(p) = r' \circ u(q)$ are incompatible. Therefore $\{*\}$ is not uniformly movable.

Remark 1. If $\mathcal{K}$ is a uniformly movable category in the sense of [7] (see also the introduction) then $\mathcal{K}$ is also uniformly movable in the sense of Definition 2.

Indeed, let (F, φ) be a uniform movability pair of $\mathcal{K}$, and $G(f)$ a uniform movability factor of a morphism $f \in \mathcal{K}(X, Y)$ (see introduction or Definition 1 in [7]). With the notations of Definition 2, if X is an object in $\mathcal{K}$, and $p : Y \to X$ is a morphism in $\mathcal{K}$, we take $M(X) = F(X)$, $m_X = \varphi(X)$ and $u(p) = G(p)$. Now the relation $p \circ G(p) = \varphi(X)$ translates as $p \circ u(p) = m_X$, which is condition 1 from Definition 2. Then for some morphisms $p : Y \to X$, $q : Z \to X$ and $r : Z \to Y$ such that $p \circ r = q$, we can consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "p"] & X \\ Z \arrow[u, "r"] \arrow[r, "q"] & X \arrow[u, "1_X"'] \end{tikzcd}$$

with $r \circ G(p) = G(q)$, which translates as $r \circ u(p) = u(q)$, which is condition 2 from Definition 2.

This remark permits us to take over a series of examples from [7].

Proposition 1. *Every category $\mathcal{K}$ with null morphisms is uniformly movable.*

Proof. Fix an object $X_0 \in \mathcal{K}$. For each $X \in \mathcal{K}$ put $M(X) = X_0$ and let $m_X = 0_{X_0 X} : X_0 \to X$ be the null morphism from X_0 to X; for an arbitrary morphism $p : Y \to X$, set $u(p) = 0_{X_0 Y}$. Now it is not difficult to verify conditions 1 and 2 of Definition 2. ∎

In particular, the category $\mathbf{Set}_*$ of pointed sets is uniformly movable.

Proposition 2. *Every category $\mathcal{K}$ with an initial object O is uniformly movable.*

Proof. For each object $X \in \mathcal{K}$, put $M(X) = O$ and let $m_X : O \to X$ be the only element in $\mathcal{K}(O, X)$; for any morphism $p : Y \to X$, let $u(p) : O \to Y$ be the only element in $\mathcal{K}(O, Y)$. Now it is not difficult to verify conditions 1 and 2 of Definition 2. ∎

In particular, the categories **Set** of all sets and maps and **Gr** of all groups and homomorphisms are uniformly movable.

Proposition 3. *An object dominated by a uniformly movable object is uniformly movable.*

Proof. Let X be a uniformly movable object in a category $\mathcal{K}$ and Y an object dominated by X. Let $f : X \to Y$ and $g : Y \to X$ be morphisms with $f \circ g = 1_Y$. Set $M(Y) = M(X)$ and $m_Y = f \circ m_X$. For any morphism $p : Z \to Y$ define $u(p) = u(g \circ p)$. Then the relation $(g \circ p) \circ u(g \circ p) = m_X$ implies $(f \circ g) \circ p \circ u(g \circ p) = f \circ m_X$ and thus $p \circ u(p) = m_Y$, which is condition 1 of Definition 2. Now let morphisms $p : Z \to Y$, $q : U \to Y$, $r : U \to Z$ satisfy $q = p \circ r$. Then $g \circ q = (g \circ p) \circ r$, which implies $u(g \circ p) = r \circ u(g \circ q)$, i.e., condition 2 of Definition 2 holds. ∎

Definition 3 ([3]). We say that a category $\mathcal{L}$ is *weakly functorially dominated* by a category $\mathcal{K}$ if there are functors $J : \mathcal{L} \to \mathcal{K}$ and $D : \mathcal{K} \to \mathcal{L}$ and a natural transformation $\psi : D \circ J \to 1_{\mathcal{L}}$.

The following proposition is similar to Theorem 2 from [3].

Proposition 4. *If a category $\mathcal{L}$ is weakly functorially dominated by a uniformly movable category $\mathcal{K}$ then $\mathcal{L}$ is also uniformly movable.*

Proof. For an object $X \in \mathcal{L}$, set $M(X) = D(M(J(X)))$ and $m_X = \psi(X) \circ D(m_{J(X)}) : M(X) \to X$. If $p : Y \to X$ is a morphism in $\mathcal{L}$, put $u(p) = \psi(Y) \circ D(u(J(p))) : M(X) \to Y$. Now we verify the conditions from Definition 2. For condition 1 we have

$$\begin{aligned} p \circ u(p) &= [p \circ \psi(Y)] \circ D(u(J(p))) = [\psi(X) \circ D(J(p))] \circ D(u(J(p))) \\ &= \psi(X) \circ [J(p) \circ D(u(J(p)))] = \psi(X) \circ D(m_{J(X)}) = m_X. \end{aligned}$$

For condition 2, if $p : Y \to X$, $q : Z \to X$ and $r : Z \to Y$ are morphisms in $\mathcal{L}$ satisfying $p \circ r = q$, then $J(p) \circ J(r) = J(q)$. This implies $J(r) \circ u(J(q)) = u(J(p))$ and by applying the functor D we deduce $D(J(r)) \circ D(u(J(q))) = D(u(J(p)))$, so $\psi(Y) \circ D(J(r)) \circ D(u(J(q))) = \psi(Y) \circ D(u(J(p))$ and hence $r \circ \psi(Z) \circ D(u(J(q))) = u(p)$, that is, $r \circ u(q) = u(p)$. ∎

In particular, Proposition 4 applies if $\mathcal{L}$ is functorial dominated by $\mathcal{K}$, i.e., $D \circ J = 1_{\mathcal{L}}$:

Corollary 1. *If a category $\mathcal{L}$ is functorial dominated by a uniformly movable category $\mathcal{K}$ then $\mathcal{L}$ is also uniformly movable.*

Proposition 5. *A product $\mathcal{K} = \prod_{i \in I} \mathcal{K}_i$ of categories is uniformly movable if and only if every category $\mathcal{K}_i$, $i \in I$, is uniformly movable.*

Proof. Suppose $\mathcal{K} = \prod_{i \in I} \mathcal{K}_i$ is uniformly movable. For fixed $i_0 \in I$ and any $i \in I$, $i \neq i_0$, select an object $X_i^0 \in \mathcal{K}_i$. Then consider the following functors: the projection $P_{i_0} : \mathcal{K} \to \mathcal{K}_{i_0}$ and $J_{i_0} : \mathcal{K}_{i_0} \to \mathcal{K}$ defined by $J_{i_0}(X_{i_0}) = (X_i')_{i \in I}$, where $X_{i_0}' = X_{i_0}$ and $X_i' = X_i^0$, $i \neq i_0$, and for a morphism $f : X_{i_0} \to Y_{i_0}$ in $\mathcal{K}_{i_0}$, $J_{i_0}(f) = (f_i')_{i \in I} : J_{i_0}(X_{i_0}) \to J_{i_0}(Y_{i_0})$ is given by $f_i' = 1_{X_i^0}$, if $i \neq i_0$ and $f_{i_0}' = f$. Then $P_{i_0} \circ J_{i_0} = 1_{\mathcal{K}_{i_0}}$ and we can apply Corollary 1.

Conversely, suppose that all categories $\mathcal{K}_i, i \in I$, are uniformly movable. To prove that $\mathcal{K} = \prod_{i\in I} \mathcal{K}_i$ is also uniformly movable, for $X = (X_i)_{i\in I}$, define $M(X) = (M(X_i))_{i\in I}$ and $m_X = (m_{X_i})_{i\in I}$, and if $p = (p_i)_{i\in I} : (X_i)_{i\in I} \to (Y_i)_{i\in I}$, then set $u(p) = (u(p_i))_{i\in I}$. Then the conditions of Definition 2 are immediately verified. ∎

Now let us consider a category $\mathcal{K}$ with pull-back diagrams. This means that for any pair of morphisms $f : X \to Z$ and $g : Y \to Z$ there exists a commutative diagram

$$\begin{tikzcd} X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "f"'] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

called a pull-back diagram, such that for any morphisms $u_X : U \to X$ and $u_Y : U \to Y$ satisfying $f \circ u_X = g \circ u_Y$ there is a unique morphism $u_X \times_Z u_Y : U \to X \times_Z Y$ such that $p_X \circ (u_X \times_Z u_Y) = u_X$ and $p_Y \circ (u_X \times_Z u_Y) = u_Y$.

Proposition 6. *Let $\mathcal{K}$ be a category with pull-back diagrams. If an object $Z \in \mathcal{K}$ is uniformly movable then for any morphisms $f : X \to Z$ and $g : Y \to Z$,*

$$(*) \qquad u(f) \times_Z u(g) = u(f \circ p_X) = u(g \circ p_Y).$$

Proof. Consider the morphism $u(f) : M(Z) \to X$ with $f \circ u(f) = m_Z$ and $u(g) : M(Z) \to Y$ with $g \circ u(g) = m_Z$ (Diagram 3). since $f \circ u(f) = g \circ u(g)$ we can consider the morphism $u(f) \times_Z u(g) : M(Z) \to X \times_Z Y$. Setting $t = f \circ p_X = g \circ p_Y$, we obtain another morphism $u(t) : M(Z) \to X \times_Z Y$. But by condition 2 of Definition 2, the relations $t = f \circ p_X = g \circ p_Y$ imply $u(f) = p_X \circ u(t)$ and $u(g) = p_Y \circ u$. These relations and the uniqueness of $u(f) \times_Z u(g)$ prove that $u(t) = u(f) \times_Z u(g)$, i.e., $(*)$. ∎

$$\begin{tikzcd} M(Z) \arrow[dr] \arrow[drr, "u(f)"] \arrow[ddr, "u(g)"'] & & \\ & X \times_Z Y \arrow[r, "p_X"] \arrow[d, "p_Y"] & X \arrow[d, "f"'] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Diagram 3

Remark 2. The dual notions to movability and uniform movability can be defined. An object X of a category $\mathcal{K}$ is (*uniformly*) *co-movable* if there are an object $M(X) \in \mathcal{K}$ and a morphism $m_X^0 : M(X) \leftarrow X$ in $\mathcal{K}$ that satisfy the following condition(s): for any object $Y \in \mathcal{K}$ and any morphism

$p : X \to Y$ in $\mathcal{K}$ there exists a morphism $u^0(p) : M(X) \leftarrow Y$ in $\mathcal{K}$ satisfying $m^0_X = u^0(p) \circ p$ (and if $p : X \to Y$, $q : X \to Z$ and $r : Y \to Z$ are morphisms in $\mathcal{K}$ such that $q = r \circ p$, then $u^0(p) = u^0(q) \circ r$). A category $\mathcal{K}$ is called (*uniformly*) *co-movable* if all its objects are (uniformly) co-movable. This is equivalent to the fact that the dual category of $\mathcal{K}$ is (uniformly) movable.

3. Main result. Recall that if $\mathcal{T}$ is a category with a subcategory $\mathcal{P}$, and $X \in \mathcal{T}$, then the *comma category* of X over $\mathcal{P}$, denoted by $X_{\mathcal{P}}$, has as objects all morphisms $p : X \to P$ in $\mathcal{T}$ with $P \in \mathcal{P}$, and as morphisms $(X \xrightarrow{p} P) \to (X \xrightarrow{p'} P')$ all morphisms $u : P \to P'$ in $\mathcal{P}$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dd, "u"] \\ X \arrow[ur, "p"] \arrow[dr, "p'"'] & \\ & P' \end{tikzcd}$$

Now recall from [5, Ch. II, §6, 7] the notions of movability and uniform movability in terms of inverse systems.

Let $\mathcal{T}$ be a category. Then an object $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{T}$ is said to be *movable* provided every $\lambda \in \Lambda$ admits an $m(\lambda) \geq \lambda$ (called a *movability index* of λ) such that for any $\lambda'' \geq \lambda$ there is a morphism $r^\lambda : X_{m(\lambda)} \to X_{\lambda''}$ of $\mathcal{T}$ which satisfies

$$p_{\lambda\lambda''} \circ r^\lambda = p_{\lambda,m(\lambda)},$$

i.e., makes the following diagram commutative:

$$\begin{tikzcd} & X_{m(\lambda)} \arrow[dl, "p_{\lambda,m(\lambda)}"'] \arrow[dd, "r^\lambda"] \\ X_\lambda & \\ & X_{\lambda''} \arrow[ul, "p_{\lambda,\lambda''}"] \end{tikzcd}$$

The essential feature of this condition is that $p_{\lambda,m(\lambda)}$ factors through $X_{\lambda''}$ for λ'' arbitrarily large (note that r^λ is not a bonding morphism).

Then an object $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of pro-$\mathcal{T}$ is *uniformly movable* if every $\lambda \in \Lambda$ admits an $m(\lambda) \geq \lambda$ (called a *uniform movability index* of λ) such that there is a morphism $\mathbf{r}(\lambda) : \mathbf{X}_{m(\lambda)} \to \mathbf{X}$ in pro-$\mathcal{T}$ satisfying

$$\mathbf{p}_\lambda \circ \mathbf{r}(\lambda) = p_{\lambda,m(\lambda)},$$

where $\mathbf{p}_\lambda : \mathbf{X} \to X_\lambda$ is the morphism of pro-$\mathcal{T}$ given by 1_{X_λ}, i.e., $\mathbf{p}_\lambda$ is the restriction of $\mathbf{X}$ to X_λ. Consequently, $p_{\lambda,m(\lambda)}$ factors through $\mathbf{X}$. Note that $\mathbf{r}(\lambda)$ determines for every $\nu \in \Lambda$ a morphism

$$\mathbf{r}(\lambda)^\nu : X_{m(\lambda)} \to X_\nu$$

in $\mathcal{T}$ such that

$$p_{\nu\nu'} \circ \mathbf{r}(\lambda)^{\nu'} = \mathbf{r}(\lambda)^\nu \quad \text{if } \nu \le \nu', \quad \text{and} \quad \mathbf{r}(\lambda)^\lambda = p_{\lambda,m(\lambda)}.$$

In particular, for any $\nu \ge \lambda$ one obtains $p_{\lambda,m(\lambda)} = \mathbf{r}(\lambda)^\lambda = p_{\lambda\nu} \circ \mathbf{r}(\lambda)^\nu$, so that uniform movability implies movability.

We also mention that movability and uniform movability for inverse systems are preserved by isomorphisms of such systems [5, pp. 159–161].

Another notion which we need in this section is that of expansion system of an object.

If $\mathcal{T}$ is a category and $\mathcal{P}$ a subcategory of $\mathcal{T}$, then for an object X of $\mathcal{T}$, a $\mathcal{P}$-*expansion* of X is a morphism in pro-$\mathcal{T}$ of X (as rudimentary system) to an inverse system $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ in $\mathcal{P}$, $\mathbf{p} : X \to \mathbf{X}$, with the following universal property:

For any inverse system $\mathbf{Y} = (Y_\mu, q_{\mu\mu'}, M)$ in $\mathcal{P}$ (called a $\mathcal{P}$-*system*) and any morphism $\mathbf{h} : X \to \mathbf{Y}$ in pro-$\mathcal{T}$, there exists a unique morphism $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{T}$ such that $\mathbf{h} = \mathbf{f} \circ \mathbf{p}$, i.e., the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\mathbf{p}"] \arrow[dr, "\mathbf{h}"'] & \mathbf{X} \arrow[d, "\mathbf{f}"] \\ & \mathbf{Y} \end{tikzcd}$$

If $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{p}' : X \to \mathbf{X}'$ are two $\mathcal{P}$-expansions of the same object X, then there is a unique isomorphism $\mathbf{i} : \mathbf{X} \to \mathbf{X}'$ such that $\mathbf{i} \circ \mathbf{p} = \mathbf{p}'$. This isomorphism is called the *natural isomorphism.*

The subcategory $\mathcal{P}$ is called a *dense* subcategory of $\mathcal{T}$ provided every object $X \in \mathcal{T}$ admits a $\mathcal{P}$-expansion $\mathbf{p} : X \to \mathbf{X}$.

If $\mathbf{p} : X \to \mathbf{X}$, $\mathbf{p}' : X \to \mathbf{X}'$ and $\mathbf{q} : Y \to \mathbf{Y}$, $\mathbf{q}' : Y \to \mathbf{Y}'$ are $\mathcal{P}$-expansions, then two morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$, $\mathbf{f}'$: $\mathbf{X}' \to \mathbf{Y}'$ in pro-$\mathcal{T}$ are equivalent, $\mathbf{f} \sim \mathbf{f}'$, provided the following diagram in pro-$\mathcal{T}$ commutes:

$$\begin{tikzcd} \mathbf{X} \arrow[r, "\mathbf{i}"] \arrow[d, "\mathbf{f}"] & \mathbf{X}' \arrow[d, "\mathbf{f}'"] \\ \mathbf{Y} \arrow[r, "\mathbf{j}"] & \mathbf{Y}' \end{tikzcd}$$

Now if $\mathcal{P}$ is a dense subcategory of $\mathcal{T}$, then the *shape category* for $(\mathcal{T}, \mathcal{P})$, denoted by $\mathrm{Sh}_{(\mathcal{T},\mathcal{P})}$, has as objects all the objects of $\mathcal{T}$, and morphisms

$F : X \to Y$ are $\sim$ equivalence classes of morphisms $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ in pro-$\mathcal{T}$, for some $\mathcal{P}$-expansions $\mathbf{p} : X \to \mathbf{X}$ and $\mathbf{q} : Y \to \mathbf{Y}$.

Now, an object $X \in \mathcal{T}$ is called *movable* (*uniformly movable*) *in* $\mathrm{Sh}_{(\mathcal{T},\mathcal{P})}$ or simply *movable* (*uniformly movable*) if it has a movable (uniformly movable) $\mathcal{P}$-expansion. This definition is correct since the properties of movability and uniform movability for inverse systems are invariant with respect to isomorphisms in pro-$\mathcal{P}$.

If **HTop** is the homotopy category of topological spaces, then the homotopy category **HPol** of polyhedra is a dense subcategory of **HTop**, and a topological space X is called (*uniformly*) *movable* if X is **HPol**-(uniformly) movable.

Now we can establish the main theorem.

THEOREM 1. *Let $\mathcal{T}$ be a category, $\mathcal{P}$ a subcategory of $\mathcal{T}$, and let $X \in \mathcal{T}$ be any object and $\mathbf{p} = (p_\lambda) : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ a $\mathcal{P}$-expansion of X. Then $\mathbf{X}$ is a uniformly movable inverse system if and only if the comma category $X_{\mathcal{P}}$ of X in $\mathcal{T}$ over $\mathcal{P}$ is uniformly movable.*

Proof. Suppose that $X_{\mathcal{P}}$ is a uniformly movable category. If $\lambda \in \Lambda$, consider $p_\lambda : X \to X_\lambda$ as an object of $X_{\mathcal{P}}$. There are an object $M(p_\lambda) = f' : X \to Q'$ in $X_{\mathcal{P}}$ and a morphism $m_{p_\lambda} = \eta : Q' \to X_\lambda$ in $X_{\mathcal{P}}$ satisfying the conditions of Definition 2 (see Diagram 4 below).

By property (AE1) of a $\mathcal{P}$-expansion [5, Ch. I, §2.1, Th. 1], there is a $\widetilde{\lambda} \in \Lambda$, $\widetilde{\lambda} \geq \lambda$, and an $\widetilde{f}' : X_{\widetilde{\lambda}} \to Q'$ such that

(1) $$f' = \widetilde{f}' \circ p_{\widetilde{\lambda}}.$$

Then

(2) $$\eta \circ \widetilde{f}' \circ p_{\widetilde{\lambda}} = \eta \circ f' = p_\lambda = p_{\lambda\widetilde{\lambda}} \circ p_{\widetilde{\lambda}}.$$

$$\begin{tikzcd}
& & Q' \arrow[dl, "\eta"'] \arrow[ddd, "u(p_{\lambda\lambda''})"] \\
X_\lambda \arrow[urr, "\tilde{f}'"] \arrow[d, "p_{\lambda\tilde{\lambda}}"'] & X_{\lambda'} \arrow[l, "p_{\lambda\tilde{\lambda}}"] & \\
X_\lambda & X \arrow[l, "p_\lambda"'] \arrow[u, "p_{\lambda'}"'] \arrow[uur, "f'"'] \arrow[dr, "p_{\lambda''}"] & \\
& & X_{\lambda''} \arrow[uull, "p_{\lambda\lambda'}"]
\end{tikzcd}$$

Diagram 4

From (2) and property (AE2) [5, Ch. I, §2.1, Th. 1], we deduce the existence of $\lambda' \in \Lambda$, $\lambda' \geq \widetilde{\lambda}$, for which

$$p_{\lambda\widetilde{\lambda}} \circ p_{\widetilde{\lambda}\lambda'} = \eta \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'}. \tag{3}$$

Now we show that λ' is a uniform movability index of λ, i.e., we have to define a morphism $\mathbf{r} = (r^{\lambda''}) : X_{\lambda'} \to \mathbf{X}$ in pro-$\mathcal{T}$, with $r^{\lambda''} : X_{\lambda'} \to X_{\lambda''}$, $\lambda'' \in \Lambda$, satisfying the condition

$$\mathbf{p}_\lambda \circ \mathbf{r} = p_{\lambda\lambda'}. \tag{4}$$

Let $\lambda'' \in \Lambda$ be arbitrary, with $\lambda'' \geq \lambda$. For the object $p_{\lambda''} : X \to X_{\lambda''}$ and the morphism $p_{\lambda\lambda''} : X_{\lambda''} \to X_\lambda$ of the comma category $X_{\mathcal{P}}$, there exists a morphism $u(p_{\lambda\lambda''}) : Q' \to X_{\lambda''}$ which satisfies the equality

$$\eta = p_{\lambda\lambda''} \circ u(p_{\lambda\lambda''}) \tag{5}$$

(see Definition 2). Observe that if $\lambda'' = \lambda$ we get

$$u(p_{\lambda\lambda}) = \eta. \tag{6}$$

Now define

$$r^{\lambda''} = u(p_{\lambda\lambda''}) \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} : X_{\lambda'} \to X_{\lambda''}. \tag{7}$$

By (7), (6) and (3), we have

$$r^\lambda = u(p_{\lambda\lambda}) \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} = \eta \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} = p_{\lambda\widetilde{\lambda}} \circ p_{\widetilde{\lambda}\lambda'} = p_{\lambda\lambda'}. \tag{8}$$

By applying (3), (5) and (7), we get

$$p_{\lambda\lambda'} = p_{\lambda\widetilde{\lambda}} \circ p_{\widetilde{\lambda}\lambda'} = \eta \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} = p_{\lambda\lambda''} \circ u(p_{\lambda\lambda''}) \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} = p_{\lambda\lambda''} \circ r^{\lambda''}.$$

Thus, for any $\lambda'' \in \Lambda$, $\lambda'' \geq \lambda$,

$$p_{\lambda\lambda'} = p_{\lambda\lambda''} \circ r^{\lambda''}. \tag{9}$$

Now, for $\lambda'' \in \Lambda$ with $\lambda'' < \lambda < \lambda'$ define $r^{\lambda''} = p_{\lambda''\lambda'}$.

In order to show that $\mathbf{r} = (r^{\lambda''}) : X_{\lambda'} \to \mathbf{X}$ is a morphism in pro-$\mathcal{T}$ which satisfies (4), we have to prove that for any $\lambda'' < \lambda'''$,

$$p_{\lambda''\lambda'''} \circ r^{\lambda'''} = r^{\lambda''}. \tag{10}$$

Let $\lambda \leq \lambda'' < \lambda'''$. In view of (7), we have

$$r^{\lambda'''} = u(p_{\lambda\lambda'''}) \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} : X_{\lambda'} \to X_{\lambda'''}. \tag{11}$$

Since $p_{\lambda\lambda'''} = p_{\lambda\lambda''} \circ p_{\lambda''\lambda'''}$, by condition 2 from Definition 2, we have

$$u(p_{\lambda\lambda''}) = p_{\lambda''\lambda'''} \circ u(p_{\lambda\lambda'''}). \tag{12}$$

By applying (11), (12), (7), we get

$$p_{\lambda''\lambda'''} \circ r^{\lambda'''} = p_{\lambda''\lambda'''} \circ u(p_{\lambda\lambda'''}) \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} = u(p_{\lambda\lambda''}) \circ \widetilde{f}' \circ p_{\widetilde{\lambda}\lambda'} = r^{\lambda''}.$$

So, $\mathbf{r} = (r^{\lambda''}) : X_{\lambda'} \to \mathbf{X}$ is a morphism in pro-$\mathcal{T}$ which satisfies (4), and thus X is a uniformly movable in the sense of shape theory.

To prove the converse, suppose $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ is a uniformly movable inverse system. Consider an object $f : X \to Q$ of the comma category $X_{\mathcal{P}}$ (see Diagram 5). By condition (AE1) there exist $\lambda \in \Lambda$ and $f_\lambda : X_\lambda \to Q$ in $\mathcal{P}$ such that

$$f = f_\lambda \circ p_\lambda. \tag{13}$$

Consider a corresponding uniform movability index $\lambda' \in \Lambda$, $\lambda' \geq \lambda$. From (13) we get

$$f = f_\lambda \circ p_{\lambda\lambda'} \circ p_{\lambda'}. \tag{14}$$

$$\begin{tikzcd}
 & X_\lambda \arrow[dl, "f_\lambda"'] & & X_{\lambda'} \arrow[ll, "p_{\lambda\lambda'}"'] \arrow[dr, "r^{\lambda'''}"] & \\
Q & & X \arrow[ll, "f"'] \arrow[ul, "p_\lambda"'] \arrow[ur, "p_{\lambda'}"] \arrow[rr, "p_{\lambda'''}"] \arrow[dl, "f''"'] \arrow[dr, "p_{\lambda''}"] & & X_{\lambda'''} \arrow[dl, "p_{\lambda''\lambda'''}"] \\
 & Q'' \arrow[ul, "\eta'"] & & X_{\lambda''} \arrow[ll, "f_{\lambda''}"] &
\end{tikzcd}$$

Diagram 5

Now let us prove that the object $M(f) := p_{\lambda'} : X \to X_{\lambda'}$ and the morphism

$$m_f := f_\lambda \circ p_{\lambda\lambda'} : X_{\lambda'} \to Q \tag{15}$$

are as required in the definition of the uniform movability for $X_{\mathcal{P}}$. Indeed, let $f'' : X \to Q''$ be an arbitrary object and $\eta' : Q'' \to Q$ an arbitrary morphism in $X_{\mathcal{P}}$, i.e.,

$$f = \eta' \circ f''. \tag{16}$$

There exist $\lambda'' \in \Lambda$, $\lambda'' \geq \lambda$, and $f_{\lambda''} : X_{\lambda''} \to Q''$ such that

$$f'' = f_{\lambda''} \circ p_{\lambda''}. \tag{17}$$

It is clear that

$$f_\lambda \circ p_{\lambda\lambda''} \circ p_{\lambda''} = \eta' \circ f_{\lambda''} \circ p_{\lambda''}.$$

Therefore, according to condition (AE2), we can find $\lambda''' \in \Lambda$, $\lambda''' \geq \lambda''$, such that

$$f_\lambda \circ p_{\lambda\lambda''} \circ p_{\lambda''\lambda'''} = \eta' \circ f_{\lambda''} \circ p_{\lambda''\lambda'''}. \tag{18}$$

By the uniform movability of $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, there exists a morphism $\mathbf{r} = (r^{\lambda'''}) : X_{\lambda'} \to \mathbf{X}$ in pro-$\mathcal{T}$ such that $\mathbf{p}_\lambda \circ \mathbf{r} = p_{\lambda\lambda'}$, i.e., the mapping $r^{\lambda'''} : X_{\lambda'} \to X_{\lambda'''}$ satisfies

$$p_{\lambda\lambda'} = p_{\lambda\lambda'''} \circ r^{\lambda'''}. \tag{19}$$

Define

$$u(\eta') = f_{\lambda''} \circ p_{\lambda''\lambda'''} \circ r^{\lambda'''} : X_{\lambda'} \to Q. \tag{20}$$

By (18)–(20), and (15) we get

$$\eta' \circ u(\eta') = \eta' \circ f_{\lambda''} \circ p_{\lambda''\lambda'''} \circ r^{\lambda'''} = f_\lambda \circ p_{\lambda\lambda''} \circ p_{\lambda''\lambda'''} \circ r^{\lambda'''} = f_\lambda \circ p_{\lambda\lambda'} = m_f. \tag{21}$$

Thus, the first condition of uniform movability of $X_{\mathcal{P}}$ is proved.

To prove the second condition of uniform movability, let $\widetilde{f}'' : X \to \widetilde{Q}''$ be an arbitrary object and $\widetilde{\eta}' : \widetilde{Q}'' \to Q$, $\varphi : \widetilde{Q}'' \to Q''$ be some morphisms of $X_{\mathcal{P}}$ such that

$$\widetilde{\eta}' = \eta' \circ \varphi. \tag{22}$$

Since $\varphi : \widetilde{Q}'' \to Q''$ is a morphism in $X_{\mathcal{P}}$,

$$f'' = \varphi \circ \widetilde{f}''. \tag{23}$$

In analogy with the construction of $u(\eta')$ (see (20)) $u(\widetilde{\eta}')$ can be written as

$$u(\widetilde{\eta}') = f_{\widetilde{\lambda}''} \circ p_{\widetilde{\lambda}''\widetilde{\lambda}'''} \circ r^{\widetilde{\lambda}'''}. \tag{24}$$

Moreover, (see (17)),

$$\widetilde{f}'' = f_{\widetilde{\lambda}''} \circ p_{\widetilde{\lambda}''}. \tag{25}$$

It remains to show that

$$u(\eta') = \varphi \circ u(\widetilde{\eta}'). \tag{26}$$

Let $\lambda_0 \in \Lambda$ be such that $\lambda_0 \geq \lambda'''$ and $\lambda_0 \geq \widetilde{\lambda}'''$ (we know that $(\Lambda, \leq)$ is a directed set). Taking into account (23) and (25) it is not difficult to see that

$$f_{\lambda''} \circ p_{\lambda''\lambda'''} \circ p_{\lambda'''\lambda_0} \circ p_{\lambda_0} = \varphi \circ f_{\widetilde{\lambda}''} \circ p_{\widetilde{\lambda}''\widetilde{\lambda}'''} \circ p_{\widetilde{\lambda}'''\lambda_0} \circ p_{\lambda_0}.$$

Hence, by the property (AE2) of the $\mathcal{P}$-expansion $\mathbf{p} = (p_\lambda) : X \to \mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, we can find $\lambda_1 \in \Lambda$, $\lambda_1 \geq \lambda_0$, such that

$$f_{\lambda''} \circ p_{\lambda''\lambda'''} \circ p_{\lambda'''\lambda_0} \circ p_{\lambda_0\lambda_1} = \varphi \circ f_{\widetilde{\lambda}''} \circ p_{\widetilde{\lambda}''\widetilde{\lambda}'''} \circ p_{\widetilde{\lambda}'''\lambda_0} \circ p_{\lambda_0\lambda_1}. \tag{27}$$

Since $\mathbf{r} = (r^{\lambda'''}) : X_{\lambda'} \to \mathbf{X}$ is a morphism of inverse systems, and $\lambda_1 \geq \lambda'''$, $\lambda_1 \geq \widetilde{\lambda}'''$, we have

$$r^{\lambda'''} = p_{\lambda'''\lambda_0} \circ p_{\lambda_0\lambda_1} \circ r^{\lambda_1}, \tag{28}$$

$$r^{\widetilde{\lambda}'''} = p_{\widetilde{\lambda}'''\lambda_0} \circ p_{\lambda_0\lambda_1} \circ r^{\lambda_1}. \tag{29}$$

By (20), (24) and (27)–(29), we get (26):

$$\begin{aligned}\varphi \circ u(\widetilde{\eta}') &= \varphi \circ f_{\widetilde{\lambda}''} \circ p_{\widetilde{\lambda}''\widetilde{\lambda}'''} \circ r^{\widetilde{\lambda}'''} = \varphi \circ f_{\widetilde{\lambda}''} \circ p_{\widetilde{\lambda}''\widetilde{\lambda}'''} \circ p_{\widetilde{\lambda}'''\lambda_0} \circ p_{\lambda_0\lambda_1} \circ r^{\lambda_1} \\ &= f_{\lambda''} \circ p_{\lambda''\lambda'''} \circ p_{\lambda'''\lambda_0} \circ p_{\lambda_0\lambda_1} \circ r^{\lambda_1} = f_{\lambda''} \circ p_{\lambda''\lambda'''} \circ r^{\lambda'''} = u(\eta'). \quad \blacksquare\end{aligned}$$

Now by theorems on dense subcategories (see [5, Th. 2, Ch. I, §4.1; Ths. 6 and 7, Ch. I, §4.3]), we get the following corollaries.

COROLLARY 2. *Let $\mathcal{T}$ be a category and $\mathcal{P}$ a dense subcategory. An object $X \in \mathcal{T}$ is uniformly movable in the sense of shape theory if and only if the comma category $X_{\mathcal{P}}$ of X in $\mathcal{T}$ over $\mathcal{P}$ is uniformly movable.*

COROLLARY 3. *If $\mathcal{P}$ is a dense subcategory of a category $\mathcal{T}$ then any object $P \in \mathcal{P}$ is uniformly movable.*

Proof. The comma category $P_{\mathcal{P}}$ has as initial object the identity morphism $1_P : P \to P$. By Proposition 2, this implies that $P_{\mathcal{P}}$ is a uniformly movable category and we can apply Corollary 2. ∎

COROLLARY 4. *A topological space X is uniformly movable if and only if its comma category $X_{\mathbf{HPol}}$ in the category **HTop** over the subcategory **HPol** is uniformly movable.*

In particular, polyhedra and ANR's are uniformly movable spaces.

COROLLARY 5. *A pair (X, X_0) of topological spaces is uniformly movable if and only if its comma category $(X, X_0)_{\mathbf{HPol}^2}$ in the homotopy category of pairs $\mathbf{HTop}^2$ over the homotopy subcategory of polyhedral pairs $\mathbf{HPol}^2$ is uniformly movable.*

In particular, a pointed space $(X, *)$ is uniformly movable if and only if its comma category $(X, *)_{\mathbf{HPol}_*}$ in the pointed homotopy category $\mathbf{HTop}_*$ over the pointed homotopy subcategory of polyhedra $\mathbf{HPol}_*$ is uniformly movable.

All pointed polyhedra and pointed ANR's are uniformly movable.

REMARK 3. In [2] a theorem similar to Theorem 1 was stated for movable spaces. Precisely, it was proved that a topological space X is movable if and only if its comma category $X_{\mathbf{HPol}}$ in **HTop** over **HPol** is movable. Now we can use the fact that there are movable objects which are not uniformly movable (see [5, p. 255]) to conclude that there are movable categories which are not uniformly movable.

By our Theorem 1 and Theorem 4 from [4, p. 173] we have the following particular case.

COROLLARY 6. *Let $\mathcal{T}$ be a category, $\mathcal{P}$ a subcategory, and $X \in \mathcal{T}$. Suppose that X has as a $\mathcal{P}$-expansion an inverse sequence $\mathbf{p} : X \to \mathbf{X} = (X_n, p_{n,n+1})$. Then the comma category $X_{\mathcal{P}}$ is uniformly movable if and only if it is movable.*

REMARK 4. As specified in the introduction, if we take into consideration the more restrictive definition for the uniform movability of a category given in [7], in order to prove the uniform movability of the comma category $X_{\mathcal{P}}$ of an object X, two supplementary conditions were added to the uniform movability of a $\mathcal{P}$-expansion $\mathbf{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of X, namely:

(G1) If $m(\lambda)$ is the uniform movability index of λ, then $p_{\lambda,m(\lambda)} : X_{m(\lambda)} \to X_\lambda$ is a $\mathcal{P}$-monomorphism, that is, if $p_{\lambda,m(\lambda)} \circ u = p_{\lambda,m(\lambda)} \circ v$, where $u, v : P \to X_{m(\lambda)}$ are two morphisms in the subcategory $\mathcal{P}$, then $u = v$.

(G2) If $\lambda, \lambda' \in \Lambda$, then there exists a $\lambda^* \in \Lambda$ with $\lambda^* \geq m(\lambda), m(\lambda')$ such that the following diagram commutes:

$$\begin{tikzcd}
& X_{m(\lambda)} \arrow[dr, "\mathbf{r}_\lambda"] & \\
X_{\lambda^*} \arrow[ur, "p_{m(\lambda),\lambda^*}"] \arrow[dr, "p_{m(\lambda'),\lambda^*}"'] & & \mathbf{X} \\
& X_{m(\lambda')} \arrow[ur, "\mathbf{r}_{\lambda'}"'] &
\end{tikzcd}$$

A space admitting such a $\mathcal{P}$-expansion was called *$\mathcal{P}$-global uniformly movable*. An example is the Warsaw circle [7, Ex. 9].

References

[1] K. Borsuk, *On movable compacta*, Fund. Math. 66 (1969/70), 137–146.

[2] P. S. Gevorgyan, *Movable categories*, Glasnik Mat. Ser. III 38 (58) (2003), 177–183.

[3] —, *Movable categories and movability of topological spaces*, to appear.

[4] S. Mardešić and J. Segal, *Movable compacta and ANR-systems*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 18 (1970), 649–654.

[5] —, —, *Shape Theory. The Inverse System Approach*, North-Holland, 1982.

[6] M. Moszyńska, *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), 125–144.

[7] I. Pop, *A categorical notion of movability*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Ser. Noua Mat. 49 (2003), 327–341.

P. S. Gevorgyan
Department of Higher Mathematics
Moscow Power Engineering Institute (Technical University)
Krasnokazarmennaya, 14
111250 Moscow, Russia
E-mail: gevorgyan@mech.math.msu.su
pgev@yandex.ru

I. Pop
Faculty of Mathematics
"Al. I. Cuza" University
700505 Iaşi, Romania
E-mail: ioanpop@uaic.ro

Received February 17, 2007;
received in final form April 26, 2007 (7584)