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REAL FUNCTIONS

Explicit Construction of Piecewise Affine Mappings with Constraints

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Summary. We construct explicitly piecewise affine mappings $u : \mathbb{R}^n \to \mathbb{R}^n$ with affine boundary data satisfying the constraint $\operatorname{div} u = 0$. As an application of the construction we give short and direct proofs of the main approximation lemmas with constraints in convex integration theory. Our approach provides direct proofs avoiding approximation by smooth mappings and works in all dimensions $n \geq 2$. After a slight modification of our construction, the constraint $\operatorname{div} u = 0$ can be turned into $\det Du = 1$, giving new examples of piecewise affine mappings u with $\det Du = 1$.

1. Introduction. The convex integration method is a tool to construct Lipschitz mappings $u : \Omega \to \mathbb{R}^m$ satisfying the differential inclusion $Du \in K$, where K is a given set of matrices and Ω is an open set in $\mathbb{R}^n$. The method relies on approximating the set K using a sequence of piecewise affine mappings u_n (i.e. mappings that are continuous on Ω and affine on some sets $\Omega_1, \Omega_2, \ldots$ that are open, disjoint and their union has measure equal to the measure of Ω) in the following way.

Let $F \in \mathbb{R}^{m\times n}$. Set $u_1(x) = Fx$ and proceed inductively: Assume that u_n is a piecewise affine mapping defined on Ω such that $u_n(x) = Fx$ for $x \in \partial\Omega$. Denote by Ω_{ni} $(i = 1, 2, \ldots)$ the subsets of Ω on which u_n is affine. To obtain u_{n+1}, replace u_n on each Ω_{ni} by a piecewise affine mapping φ_i such that φ_i and u_n agree on $\partial\Omega_{ni}$. We thus obtain a new piecewise affine mapping u_{n+1} defined on Ω.

The advantage of this approximation is that the sequence Du_n is a martingale, hence it converges strongly and almost everywhere, provided it is

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bounded. Therefore if the sequence $Du_n(x)$ approaches a point of K, then automatically the pointwise limit of $Du_n(x)$ coincides almost everywhere with the gradient Du of a Lipschitz mapping u satisfying $Du \in K$.

For applications of this procedure as well as a parallel approach based on Baire's Theorem see [Na54], [Ku55], [Gr73], [Gr86], [CDK07], [AFS04], [KMS03], [Mu99], [MS01], [MRS05], [Po10], [Sy06], [Zh06], [LS09], [DM97], [DM99], [Ki01], [Ki02].

However, this type of approximation does not provide too much freedom. Since the minors are Null-Lagrangians, the sequence $\mathbf{M}(Du_n)$ is also a martingale ($\mathbf{M}(X)$ is the vector of all minors of the matrix X). So if for instance $\det X = 1$ (with $m = n$) for each $X \in K$, then our approximating sequence u_n of piecewise affine mappings must also satisfy $\det Du_n(x) = 1$.

In paper [MS99] S. Müller and V. Šverák have proved that there are sufficiently many piecewise affine mappings u_n with $\det Du_n(x) = 1$ to ensure approximation along rank-one lines. This result is known as the main approximation lemma in convex integration theory (actually S. Müller and V. Šverák considered a more general case, where the determinant is replaced by a fixed subdeterminant of size ≥ 2). Since in the case without constraints one usually moves along rank-one lines, their result shows that adding the extra constraint $\det X = 1$ does not limit the freedom in the approximation process.

The proof in [MS99] consists of two steps. First, one proves the existence of an appropriate sequence u_n of *smooth* mappings with $\det Du_n(x) = 1$. Then one modifies carefully the sequence u_n, preserving the constraint $\det X = 1$, to obtain piecewise affine mappings. This procedure leads to very complicated mappings that are hard to control or to deduce further useful properties. In the same paper, S. Müller and V. Šverák remark [MS99, Remark after Theorem 6.1] that in dimension 2 a direct construction of piecewise affine mappings u_n with 20 gradients is possible. Influenced by the unpublished notes of S. Müller and V. Šverák, S. Conti and F. Theil [CT05] presented a direct construction in dimension 2 using only 5 gradients and 12 regions. Recently, S. Conti [Co08] has extended the construction to higher dimensions, based on the results in dimension 2.

Moreover, the authors remarked in [MS99] that some other constraints like $\operatorname{div} u = 0$ or $Du = (Du)^T$ can be treated with an analogous method. In [Ki02] B. Kirchheim confirms this by providing a detailed proof for the case $Du = (Du)^T$. The proof goes along the same lines as in [MS99] and therefore the piecewise affine mappings obtained are complicated and hard to control.

In [CT05] S. Conti and F. Theil remarked that the method of [MS99] is too complicated to deduce further properties of the mappings (like additional constraints). Therefore, it seems to be important to devise direct constructions of constrained piecewise affine mappings with simple geometry, allowing further modifications for various applications.

The goal of the present note is to give an explicit construction of piecewise affine mappings having affine boundary data and preserving the constraint $\operatorname{div} u = a$. Then we use this result to give a simple proof of the corresponding approximation lemma, without the passage through smooth mappings.

In dimension 2 the approximation lemma for the constraint $\operatorname{div} u = 0$, which after a suitable change of variables is equivalent to $Du = (Du)^T$, is used in the solution to the 5-gradient problem (see [Ki02], [Po10]). Since some results concerning the 5-gradient problem are still unknown (see [Po10]), it seems to be important to learn more about the structure of the piecewise affine mappings u respecting the condition $\operatorname{div} u = 0$, even though it is linear.

The linearity of the constraint $\operatorname{div} u = 0$ does not help in the direct construction. In fact, we present a slight modification of our construction, which turns the condition $\operatorname{div} u = 0$ into $\det Du = 1$. In this way we obtain new examples of piecewise affine mappings preserving the constraint $\det Du = 1$, other than those constructed in [Co08], [CT05].

2. The construction of piecewise affine maps with constraints in dimension 2. Our construction of piecewise affine mappings u with the constraint $\det Du = 1$ or $\operatorname{div} u = 0$ in arbitrary dimension is based on a construction in dimension 2.

In this section we assume that Ω is an open domain in $\mathbb{R}^2$.

LEMMA 2.1. *For each $\varepsilon > 0$ there exists a piecewise affine mapping $u : \Omega \to \mathbb{R}^2$ such that*

(1) $u(x) = x$ *for* $x \in \partial\Omega$,
(2) $\det Du(x) = 1$ *for* $x \in \Omega$,
(3) $0 < \|Du - \mathrm{Id}\|_\infty < \varepsilon$.

Proof. By the Vitali covering theorem we may assume that Ω is an equilateral triangle $A_0A_1A_2$, whose center is O (identified with the center of the coordinate system).

We divide the triangle $A_0A_1A_2$ into seven regions as follows. Let M be the midpoint of the side A_1A_2 (Fig. 1). Let $X_0X_1X_2$ be an equilateral triangle with center O lying inside the triangle $A_0A_1A_2$ and such that X_0 is close to the line segment OM, but does not lie on it.

For each nonempty subset I of $\{0, 1, 2\}$ let $\mathcal{T}_I$ be the triangle with vertices X_i and A_j, where $i \in I$ and $j \in \{0, 1, 2\} \setminus I$. In this way the triangle $A_0A_1A_2$ is cut into seven triangles $\mathcal{T}_I$.

Now we divide the triangle $A_0A_1A_2$ into diferent seven regions in a similar way: Let Y_0 be the point symmetric to X_0 with respect to the line segment OM and let $Y_0Y_1Y_2$ be the equilateral triangle with center 0 (Fig. 2). Then the triangle $Y_0Y_1Y_2$ is congruent to $X_0X_1X_2$ ($Y_0Y_1Y_2$ can also be obtained from $X_0X_1X_2$ by rotation about O through the angle $2\angle X_0OM$).

For each nonempty subset I of $\{0,1,2\}$ let $\mathcal{S}_I$ be the triangle with vertices Y_i and A_j, where $i \in I$ and $j \in \{0,1,2\} \setminus I$. In this way the triangle $A_0A_1A_2$ is cut into seven triangles $\mathcal{S}_I$.

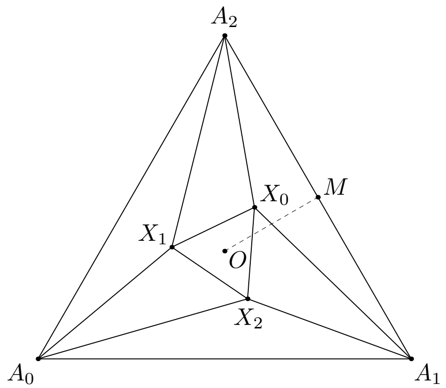

Fig. 1

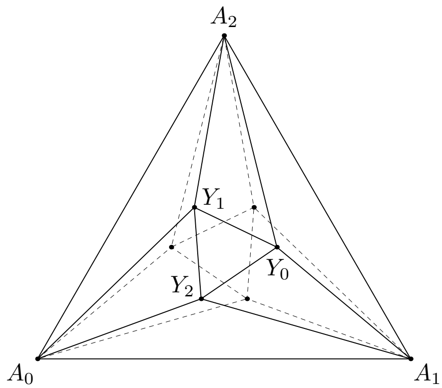

Fig. 2

Let now u be the piecewise affine mapping which for each nonempty subset I of $\{0,1,2\}$ takes the triangle $\mathcal{T}_I$ to $\mathcal{S}_I$ in such a way that the vertex X_i goes to Y_i and the vertex A_j stays fixed.

Then the mapping u satisfies our requirements. Indeed, (1) is obviously satisfied and (2) follows from the observation that the triangles $\mathcal{T}_I$ and $\mathcal{S}_I$ are congruent (and hence have equal areas), and the mapping u does not change the orientation of $\mathcal{T}_I$. Finally (3) holds if we choose X_0 close enough to the line segment OM. ■

Remark 2.2. A similar construction can be done for a square $A_0A_1A_2A_3$ instead of the equilateral triangle $A_0A_1A_2$ (see Figs. 3 and 4). Then the corresponding piecewise affine mapping u has only five different gradients.

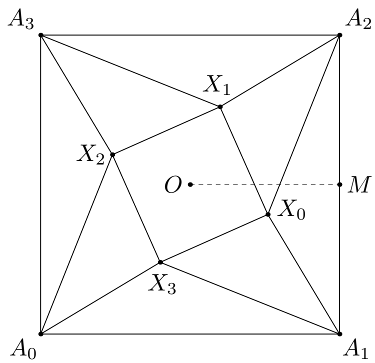

Fig. 3

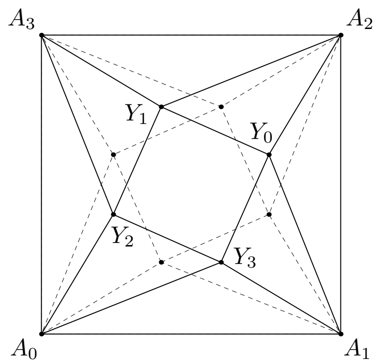

Fig. 4

A similar construction can also be done to obtain an analogous result for the affine constraint $\operatorname{div} u = 2$.

LEMMA 2.3. *For each $\varepsilon > 0$, there exists a piecewise affine mapping $u : \Omega \to \mathbb{R}^2$ such that*

(1) $u(x) = x$ *for* $x \in \partial\Omega$,
(2) $\operatorname{div} u = 2$ *for* $x \in \Omega$,
(3) $0 < \|Du - \mathrm{Id}\|_\infty < \varepsilon$.

Proof. Without loss of generality we may assume that Ω is an equilateral triangle $A_0A_1A_2$ with center 0 and such that A_0A_1 is parallel to the x-axis (Fig. 5).

Denote by H_α the linear mapping on $\mathbb{R}^2$ which is the rotation through the angle α composed with the homothety with the scale $1/\cos\alpha$, i.e.

$$H_\alpha = \frac{1}{\cos\alpha}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & -\tan\alpha \\ \tan\alpha & 1 \end{pmatrix}.$$

Fix $0 < \mu < 1/2$ and define $X_i = -\mu A_i$ and $Y_i = H_\alpha X_i$ for $i = 0, 1, 2$. For each nonempty subset I of $\{0, 1, 2\}$ let $\mathcal{T}_I$ be the triangle with vertices X_i and A_j, where $i \in I$ and $j \in \{0, 1, 2\} \setminus I$. Similarly, denote by $\mathcal{S}_I$ the triangle with vertices Y_i and A_j, where $i \in I$ and $j \in \{0, 1, 2\} \setminus I$. In this way the triangle $A_0A_1A_2$ is cut into seven triangles $\mathcal{T}_I$ (Fig. 5) and also into seven triangles $\mathcal{S}_I$ (Fig. 6).

Let now u be the piecewise affine mapping, which takes $\mathcal{T}_I$ to $\mathcal{S}_I$ in such a way that the vertex X_i goes to Y_i and the vertex A_j stays fixed.

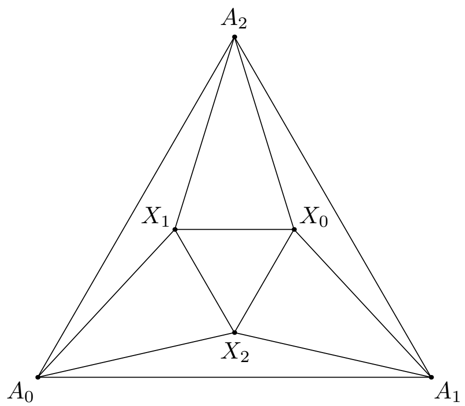

Fig. 5

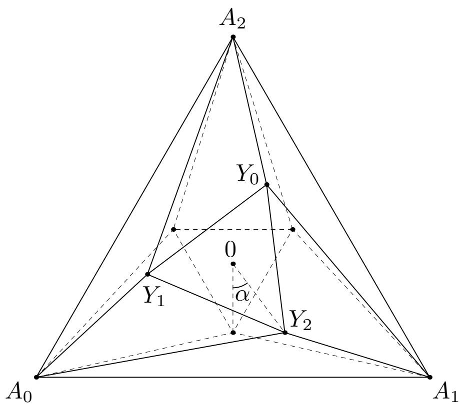

Fig. 6

Then the mapping u satisfies our requirements. Indeed, (1) follows directly from the definition, and (4) is satisfied for sufficiently small values of α. To see (2) denote by u_I the mapping u restricted to the triangle $\mathcal{T}_I$.

Then

$$Du_{\{0,1,2\}} = H_\alpha \quad \text{and} \quad Du_{\{2\}} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix},$$

so $\operatorname{tr}(Du_{\{0,1,2\}}) = \operatorname{tr}(Du_{\{2\}}) = 2$. Moreover, if R_α denotes the rotation through α, then

$$Du_{\{1\}} = R^{-1}_{2\pi/3} Du_{\{2\}} R_{2\pi/3}, \quad Du_{\{0\}} = R^{-1}_{-2\pi/3} Du_{\{2\}} R_{-2\pi/3},$$

which gives $\operatorname{tr}(Du_{\{0\}}) = \operatorname{tr}(Du_{\{1\}}) = \operatorname{tr}(Du_{\{2\}}) = 2$. Moreover,

$$Du_{\{2,0\}} = R^{-1}_{-2\pi/3} Du_{\{1,2\}} R_{-2\pi/3}, \quad Du_{\{0,1\}} = R^{-1}_{2\pi/3} Du_{\{1,2\}} R_{2\pi/3},$$

which gives $\operatorname{tr}(Du_{\{2,0\}}) = \operatorname{tr}(Du_{\{0,1\}}) = \operatorname{tr}(Du_{\{1,2\}}) = a$. Since trace is a Null-Lagrangian and since $\operatorname{tr}(\mathrm{Id}) = 2$, we immediately obtain $a = 2$. ∎

REMARK 2.4. In the proofs of both lemmas the gradient Du consists of seven matrices $C_1, \ldots, C_7$. For each C_i, set $c_i = |\{x \in \Omega : Du(x) = C_i\}|/|\Omega|$. It is visible from the above construction that the numbers c_i may depend on ε. However, taking X_0 sufficiently close to the midpoint of OM we have $c_i > 1/17$. This observation will be used in the proof of the next two propositions.

Denote by δ_{ij} the matrix with the (i, j) entry equal to 1 and the other entries zero.

Using the above lemmas and a scaling argument, one can obtain special cases of the main approximation lemmas.

PROPOSITION 2.5. *Assume* $0 < \lambda < 1$. *Then for each* $\varepsilon > 0$ *there exists a piecewise affine mapping* $u : \Omega \to \mathbb{R}^2$ *such that*

(1) $u(x) = x$ *on* $\partial\Omega$,
(2) $\det Du(x) = 1$ *a.e. on* Ω,
(3) $\operatorname{dist}(Du(x), [A, B]) < \varepsilon$ *a.e. on* Ω, *where* $A = \mathrm{Id} - (1 - \lambda)\delta_{12}$ *and* $B = \mathrm{Id} + \lambda\delta_{12}$,
(4) *the measure of the set* $Z = \{x \in \Omega : \operatorname{dist}(Du(x), \{A, B\}) > \varepsilon\}$ *is less than or equal to* $\frac{16}{17}|\Omega|$.

The same proof applies if the nonlinear constraint $\det X = 1$ is replaced by the affine one, $\operatorname{tr}(X) = 2$.

PROPOSITION 2.6. *Assume* $0 < \lambda < 1$. *Then for each* $\varepsilon > 0$ *there exists a piecewise affine mapping* $u : \Omega \to \mathbb{R}^2$ *such that* $\operatorname{div} u(x) = 2$ *a.e. on* Ω *and conditions* (1), (3), (4) *of Proposition* 2.5 *hold.*

Proof of Propositions 2.5 and 2.6. By the Vitali covering theorem it is enough to prove the results for Ω being a fixed triangle (possibly depending on ε).

Fix $\varepsilon > 0$ and without loss of generality assume that $\varepsilon^2 < \min(\lambda, 1-\lambda)$. According to Lemma 2.1 (or Lemma 2.3, respectively), there exists a piecewise affine mapping v defined on an equilateral triangle Ω in $\mathbb{R}^2$ such that conditions (1), (2) of Lemma 2.1 (or Lemma 2.3, respectively) are satisfied and

$$|Dv(x) - \mathrm{Id}| < \varepsilon^2 \quad (x \in \Omega), \tag{2.1}$$

where we assume that $|\cdot|$ is the l^∞-norm in $\mathbb{R}^{n\times n}$. Define

$$a_+ = \max\left\{ \frac{1}{\lambda} \cdot \frac{\partial v_1}{\partial x_2}(x) : \frac{\partial v_1}{\partial x_2}(x) \geq 0,\, x \in \Omega \right\},$$
$$a_- = \max\left\{ \frac{1}{1-\lambda} \cdot \left|\frac{\partial v_1}{\partial x_2}(x)\right| : \frac{\partial v_1}{\partial x_2}(x) \leq 0,\, x \in \Omega \right\}.$$

Finally, define $a = \max(a_+, a_-)$. Then using (2.1) and the inequality $\varepsilon^2 < \min(\lambda, 1-\lambda)$ we infer that $0 < a < 1$.

Assume now that $a = a_+$; the case $a = a_-$ can be treated analogously. Then there exists a triangle $\mathcal{T}_I$ such that

$$a = \frac{1}{\lambda} \cdot \frac{\partial v_1}{\partial x_2}(x) \quad \text{for } x \in \mathcal{T}_I.$$

Let $S = \mathrm{diag}(a^{1/2}, a^{-1/2})$. We prove that the piecewise affine mapping

$$u(y) = (S^{-1} \circ v \circ S)(y) \tag{2.2}$$

defined on the triangle $\Omega_1 = S^{-1}\Omega$ satisfies our requirements.

Indeed (1) and (2) follow directly from the definition of u. So we concentrate on (3) and (4). From (2.2) we have, for $x = Sy$,

$$\frac{\partial u_1}{\partial x_1}(y) = \frac{\partial v_1}{\partial x_1}(x), \qquad \frac{\partial u_2}{\partial x_2}(y) = \frac{\partial v_2}{\partial x_2}(x),$$
$$\frac{\partial u_1}{\partial x_2}(y) = \frac{1}{a} \cdot \frac{\partial v_1}{\partial x_2}(x), \qquad \frac{\partial u_2}{\partial x_1}(y) = a \cdot \frac{\partial v_2}{\partial x_1}(x).$$

Therefore using 2.1 we obtain

$$\left|\frac{\partial u_i}{\partial x_i}(y) - 1\right| = \left|\frac{\partial v_i}{\partial v_i}(x) - 1\right| < \varepsilon^2 < \varepsilon \quad (i = 1, 2). \tag{2.3}$$

Moreover, if $\frac{\partial u_1}{\partial x_2}(y) > 0$, then we obtain

$$\frac{\partial u_1}{\partial x_2}(y) = \frac{1}{a} \cdot \frac{\partial v_1}{\partial x_2}(x) \leq \lambda, \tag{2.4}$$

and the equality in (2.4) holds for $x \in \mathcal{T}_I$. On the other hand, if $\frac{\partial u_1}{\partial x_2}(y) \leq 0$, then we obtain

$$\left|\frac{\partial u_1}{\partial x_2}(y)\right| = \frac{1}{a} \cdot \left|\frac{\partial v_1}{\partial x_2}(x)\right| \leq 1 - \lambda. \tag{2.5}$$

Finally, we have

$$\left|\frac{\partial u_2}{\partial x_1}(y)\right| = a \cdot \left|\frac{\partial v_2}{\partial x_1}(x)\right| < \varepsilon. \tag{2.6}$$

Inequalities (2.3)–(2.6) directly give condition (3). Moreover, the equality in (2.4) for $x \in \mathcal{T}_I$ implies property (4) if X_0 is chosen to be sufficiently close to the midpoint of OM (see Remark 2.4). ∎

3. Application: The approximation lemmas with constraints. In this section we use Propositions 2.5 and 2.6 to present direct proofs of the main approximation lemmas in convex integration theory. One of them preserves the constraint $\det Du = a \neq 0$, while the other deals with $\operatorname{div} u = a$. Both cases are treated in an arbitrary dimension.

THEOREM 3.1 (S. Müller, V. Šverák [MS99]). *Let $A, B \in \mathbb{R}^{n\times n}$ be such that* $\operatorname{rank}(B - A) = 1$ *and* $\det A = \det B = a \neq 0$. *Let moreover $F = \lambda A + (1-\lambda)B$, where $\lambda \in (0,1)$. Then for each $\varepsilon > 0$ there exists a piecewise affine mapping u defined on Ω and having the following properties:*

(1) *$u(x) = Fx$ on $\partial\Omega$,*
(2) *$\det Du(x) = a$ a.e. on Ω,*
(3) *$\operatorname{dist}(Du(x), [A, B]) < \varepsilon$ a.e. on Ω,*
(4) *the measure of the set $Z = \{x \in \Omega : \operatorname{dist}(Du(x), \{A, B\}) > \varepsilon\}$ is less than or equal to $c|\Omega|$, where $0 < c < 1$ is a constant depending only on the dimension n.*

REMARKS. The original result of S. Müller and V. Šverák is more general. It deals with a fixed minor (subdeterminant) of order ≥ 2 instead of the determinant. Also, condition (4) is a bit different: it says that the measure of the set Z is less than ε. However, the above weaker condition (4) is easier to obtain and it is still sufficient for an application in convex integration theory (see [Po10, Appendix]).

THEOREM 3.2 (S. Müller, V. Šverák [MS99]). *Let $A, B \in \mathbb{R}^{n\times n}$ be such that* $\operatorname{rank}(B-A) = 1$ *and* $\operatorname{tr} A = \operatorname{tr} B = a$. *Let moreover $F = \lambda A + (1-\lambda)B$, where $\lambda \in (0,1)$. Then for each $\varepsilon > 0$ there exists a piecewise affine mapping u defined on Ω and having the following properties:*

(1) *$u(x) = Fx$ on $\partial\Omega$,*
(2) *$\operatorname{div} u(x) = a$ a.e. on Ω,*
(3) *$\operatorname{dist}(Du(x), [A, B]) < \varepsilon$ a.e. on Ω,*
(4) *the measure of the set $Z = \{x \in \Omega : \operatorname{dist}(Du(x), \{A, B\}) > \varepsilon\}$ is less than or equal to $c|\Omega|$, where $0 < c < 1$ is a constant depending only on the dimension n.*

Proof of Theorems 3.1 and 3.2. By the Vitali covering theorem, it is enough to prove the statement for a fixed polyhedron in $\mathbb{R}^n$.

STEP 1. Assume that $F = \mathrm{Id}$ and $B - A = \delta_{12}$. If $n = 2$, then the conclusion follows directly from Proposition 2.5. More precisely, there exists a triangle $\Omega = A_0A_1A_2$ divided into the triangles $\mathcal{T}_1, \dots, \mathcal{T}_7$ and also into the triangles $\mathcal{S}_1, \dots, \mathcal{S}_7$ and a piecewise affine mapping v that is affine on each $\mathcal{T}_i$, takes $\mathcal{T}_i$ onto $\mathcal{S}_i$ and satisfies (1)–(4).

Let now $n = 3$. Place the triangle $A_0A_1A_2$, together with its partitions $\{\mathcal{T}_1, \dots, \mathcal{T}_7\}$ and $\{\mathcal{S}_1, \dots, \mathcal{S}_7\}$ on the plane $x_3 = 0$ in such a way that the point $(0,0,0)$ lies inside the triangle $A_0A_1A_2$. Define $A_3 = (0,0,1)$ and $A_4 = (0,0,-1)$. Then the tetrahedrons $\mathcal{T}_i^+ = \mathrm{conv}(\mathcal{T}_i, A_3)$ and $\mathcal{T}_i^- = \mathrm{conv}(\mathcal{T}_i, A_4)$ with $i = 1, \dots, 7$ determine a triangulation of $\Omega = \mathrm{conv}(A_0, A_1, \dots, A_4)$ into 14 parts. Similarly, the tetrahedrons $\mathcal{S}_i^+ = \mathrm{conv}(\mathcal{S}_i, A_3)$ and $\mathcal{S}_i^- = \mathrm{conv}(\mathcal{S}_i, A_4)$ with $i = 1, \dots, 7$ determine another partition of Ω into 14 parts.

Let now u be the piecewise affine mapping, which is affine on each $\mathcal{T}_i^+$, $\mathcal{T}_i^-$ and which takes $\mathcal{T}_i^+$, $\mathcal{T}_i^-$ onto $\mathcal{S}_i^+$, $\mathcal{S}_i^-$, respectively, in such a way that $u(x) = v(x)$ for $x \in \mathcal{T}_i$ and $u(A_3) = A_3$, $u(A_4) = A_4$.

Then the mapping u satisfies our requirements for $n = 3$. Indeed, since $u(A_j) = A_j$, (1) is satisfied. To see (2), observe that the gradient of u at any point takes each vector (x, y, z) to (u, v, z) (in other words, the gradient of u at each point does not change the last coordinate of any vector). This yields

$$\frac{\partial u_3}{\partial x_1}(x) = \frac{\partial u_3}{\partial x_2}(x) = 0 \quad \text{and} \quad \frac{\partial u_3}{\partial x_3}(x) = 1 \qquad (x \in \Omega),$$

from which (2) follows.

Moreover, choosing the triangle $A_0A_1A_2$ small enough we obtain

$$\left|\frac{\partial u_1}{\partial x_3}(x)\right| < \varepsilon \quad \text{and} \quad \left|\frac{\partial u_2}{\partial x_3}(x)\right| < \varepsilon \qquad (x \in \Omega).$$

This together with the conclusion for $n = 2$ gives (3) and (4).

We use the same procedure to pass from an arbitrary dimension n to the dimension $n+1$. As a result we obtain a convex polyhedron $\Omega = \mathrm{conv}(A_0, A_1, \dots, A_{2n-2})$ in $\mathbb{R}^n$ divided into $7 \cdot 2^{n-2}$ regions and a piecewise affine mapping $u : \Omega \to \mathbb{R}^n$ satisfying assumptions (1)–(4).

STEP 2. Assume that $F = \mathrm{Id}$ and A, B are arbitrary. By the Jordan decomposition theorem we can find an invertible matrix T such that $B - A = T^{-1}\alpha\delta_{11}T$ ($\alpha \in \mathbb{R}$, $\alpha \neq 0$) or $B - A = T^{-1}\delta_{12}T$. Then for each real number t we have $\mathrm{Id} + t(B - A) = T^{-1}(\mathrm{Id} + t\alpha\delta_{11})T$ or $\mathrm{Id} + t(B - A) = T^{-1}(\mathrm{Id} + t\delta_{12})T$, respectively. Since for each $t \in \mathbb{R}$ we have $\det(\mathrm{Id} + t(B - A)) = 1$ or $\mathrm{tr}(\mathrm{Id} + t(B - A)) = n$, the former case is impossible.

For a fixed $\varepsilon > 0$ we find a mapping $v : \Omega \to \Omega$ satisfying the conclusions of the theorem with $B - A = \delta_{12}$ (Step 1). Then the mapping $u(x) = (T^{-1} \circ v \circ T)(x)$ defined on the simplex $\Omega_1 = T^{-1}\Omega$ satisfies our conclusions. Indeed, (1) and (2) follow directly from the definiton of u. Conditions (3) and (4) follow from the fact that the mapping $X \mapsto T^{-1}XT$ is linear, and hence continuous on $\mathbb{R}^{n\times n}$, and for each $t \in \mathbb{R}$ takes the matrix $\mathrm{Id} + t\delta_{12}$ to $\mathrm{Id} + t(B - A)$.

Step 3. Assume that F, A, B are arbitrary. In the case of Theorem 3.1 we have $\det(F + t(B - A)) = a$ for each $t \in \mathbb{R}$. Hence

$$\det(\mathrm{Id} + t(F^{-1}B - F^{-1}A)) = 1.$$

Therefore we may construct a mapping v like in Step 2 with $F^{-1}A$ and $F^{-1}B$ instead of A and B, respectively. Then the mapping $u(x) = (F \circ v)(x)$ satisfies our requirements.

In the case of Theorem 3.2 we have $\mathrm{tr}(F + t(B - A)) = n$ for each $t \in \mathbb{R}$. Since $\mathrm{tr}(\mathrm{Id}+t((B-F+\mathrm{Id})-(A-F+\mathrm{Id}))) = n$, we may construct a mapping v as in Step 2 with $A-F+\mathrm{Id}$ and $B-F+\mathrm{Id}$ instead of A and B, respectively. Then the mapping $u(x) = Fx - x + v(x)$ satisfies our requirements. ∎

Remarks. 1. Based on the mapping u defined on the square $A_0A_1A_2A_3$ (see Remark 2.2) instead of on the equilateral triangle $A_0A_1A_2$, one obtains a piecewise affine mapping having $5 \cdot 2^{n-2}$ gradients.

2. The piecewise affine mapping constructed in Step 1 of the above proof satisfies much more constraints than only $\det Du = 1$ or $\mathrm{div}\, u = 2$: at each point $x \in \Omega$ the matrix $Du(x)$ is *almost triangular*, i.e.

$$\frac{\partial u_i}{\partial x_j}(x) = 0 \quad \text{for } i > j \text{ with } (i,j) \neq (2,1) \quad \text{and} \quad \frac{\partial u_i}{\partial x_i}(x) = 1 \quad \text{for } i \geq 3.$$

In particular, at each $x \in \Omega$ the characteristic polynomial $p(\lambda)$ of $Du(x)$ is equal to

$$p(\lambda) = (\lambda^2 + a\lambda + 1)(1 - \lambda)^{n-2}$$

or

$$p(\lambda) = (\lambda^2 - 2\lambda + a)(1 - \lambda)^{n-2} \qquad (a \in \mathbb{R}).$$

(The first case corresponds to the constraint $\det Du = 1$ and the second one to $\mathrm{div}\, u = n$.) This property is still preserved in Step 2, but destroyed at the very last Step 3.

3. If $n = 2$, then by a linear change of variables we may transform the constraint $\mathrm{div}\, u = 0$ to $Du = (Du)^T$. Hence Theorem 3.2 holds also if $\mathrm{div}\, u = 0$ is replaced by $Du = (Du)^T$. However, for dimensions $n \geq 3$ our method of construction fails. In this case we refer the reader to [Ki02, Proposition 3.4], where the method uses the ideas of S. Müller and V. Šverák [MS99].

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