BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 58, No. 3, 2010

ORDINARY DIFFERENTIAL EQUATIONS

Practical Stability in Terms of Two Measures for Hybrid Dynamic Systems

by

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Presented by Czesław OLECH

Summary. We study hybrid dynamic systems on time scales. Using Lyapunov-like functions, we obtain sufficient conditions for practical stability and strict practical stability in terms of two measures for hybrid dynamic systems on time scales.

1. Introduction. In this paper, we present some sufficient conditions for practical stability and strict practical stability in terms of two measures for hybrid dynamic systems on time scales by using Lyapunov-like functions. The practical stability problem, introduced by LaSalle and Lefschetz [12], deals with the question of whether the system state evolves within certain subsets of the state-space. This is very useful in estimating the worst-case transient and steady-state responses and in verifying pointwise in time constraints imposed on the state trajectories. Thus, practical stability is concerned with quantitative analysis as opposed to Lyapunov analysis which is qualitative in nature. Lyapunov analysis may not always be useful in an engineering sense. For instance, an equilibrium point may not be stable in the sense of Lyapunov and yet the system response may be acceptable in the vicinity of this equilibrium; an example involving a van der Pol oscillator is given in [12]. On the other hand, an equilibrium point may be stable in the sense of Lyapunov, but the domain of stability (see [5] for its definition) could be so small as to render the system practically unstable. A striking example of this phenomenon is the Reynolds system used to study the flow of water along a tube of circular cross-section at various speeds [1]. Laminar

2010 *Mathematics Subject Classification*: 34N05, 34A38, 34D20.
Key words and phrases: hybrid systems on time scales, variational Lyapunov method, (h_0, h)-practical stability.

DOI: 10.4064/ba58-3-5

flow, i.e. smooth flow of water particles parallel to the axis, is stable at all speeds. However, the domain of stability at very high flowrates is so absurdly small as to render laminar flow practically unstable.

The study of hybrid systems [13] is caused by modeling, design and validation of interacting systems of continuous processes and computer programs. Therefore, the identifying characteristic of hybrid systems is that they incorporate both continuous components, usually called plants, which are governed by differential equations, and also digital components such as digital computers, sensors and actuators controlled by programs. Moreover, the growing demand for control systems that are capable of controlling complex nonlinear continuous plants with discrete intelligent controllers can be addressed by the method of hybrid systems [4].

In general, a hybrid dynamic system is a system with different kinds of time dynamics, e.g., continuous, discrete, or impulsive, in different interacting parts of the system. Recently, the theory of dynamic systems on time scales has gained impetus since it demonstrates the interplay of two different theories, namely, the theories of continuous and discrete dynamic systems [6, 2, 3, 9].

It is important to note that dynamic systems on time scales include hybrid systems in general. In the stability and practical stability aspect, Lakshmikantham and Vatsala [10] introduced hybrid dynamic systems on time scales and obtained the practical stability for such systems. P. G. Wang, Liu, Wu and Wu [16, 14, 15] studied the stability criteria in terms of two measures for discrete systems and the practical stability of impulsive hybrid differential systems in terms of two measures on time scales. Lakshmikantham and Mohapatra [8] advanced the concept of strict stability for differential systems, and Lakshmikantham and Zhang [11] developed the idea of strict practical stability of delay differential systems. For all kinds of practical stability for differential systems we refer to [7].

The main purpose of this paper is to establish criteria for practical stability and strict practical stability in terms of two measures for hybrid dynamic systems on time scales. Some ideas in this paper are motivated by [14–16], and some results are extensions of those in [16] for discrete hybrid systems.

2. Preliminaries. Let $\mathbb{T}$ be a *time scale* (a closed nonempty subset of $\mathbb{R}$). On $\mathbb{T}$ we define the *forward* and *backward jump operators* by

$$\sigma(t) := \inf\{s \in \mathbb{T} \mid s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} \mid s < t\}.$$

A point $t \in \mathbb{T}$ is said to be *left-dense* if $\rho(t) = t$, *right-dense* if $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$, and *right-scattered* if $\sigma(t) > t$. The *graininess* μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

For a function $f : \mathbb{T} \to \mathbb{R}$ the *delta derivative* $f^{\Delta}(t)$ at $t \in \mathbb{T}$ is defined to be the number (provided it exists) with the property that for every $\varepsilon > 0$, there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at each right-dense point and if there exists a finite left-sided limit at all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C_{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 2.1. For each $t \in \mathbb{T}$, let $\mathcal{N}$ be a neighborhood of t. Then we define the *lower right Dini derivative* $D_{+}u^{\Delta}(t)$ by the condition: for given $\varepsilon > 0$, there exists a right neighborhood $\mathcal{N}_{\varepsilon} \subset \mathcal{N}$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\mu^{*}(t, s)} > D_{+}u^{\Delta}(t) - \varepsilon \quad \text{for } s \in \mathcal{N}_{\varepsilon},\ s > t,$$

where $\mu^{*}(t, s) \equiv \sigma(t) - s$.

In case t is right-scattered and u is continuous at t, we have as in the case of the derivative

$$D_{+}u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)} = u^{\Delta}(t).$$

Definition 2.2 ([10]). For each $t \in \mathbb{T}$, let $\mathcal{N}$ be a neighborhood of t. Then we define the *upper right Dini derivative* $D^{+}u^{\Delta}(t)$ by the condition: for given $\varepsilon > 0$, there exists a right neighborhood $\mathcal{N}_{\varepsilon} \subset \mathcal{N}$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\mu^{*}(t, s)} < D^{+}u^{\Delta}(t) + \varepsilon \quad \text{for } s \in \mathcal{N}_{\varepsilon},\ s > t.$$

In case t is right-scattered and u is continuous at t, we have as in the case of the derivative

$$D^{+}u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\mu(t)} = u^{\Delta}(t).$$

Definition 2.3. A function $f(u, v)$ is said to be *quasimonotone nondecreasing* (respectively, *nonincreasing*) if for fixed u (or v), f is nondecreasing (respectively, nonincreasing) in v (or u).

We now state a result on existence of extremal solutions of the initial value problem for the dynamic system

$$u^{\Delta} = g(t, u), \quad u(t_0) = u_0, \tag{2.1}$$

where $g \in \mathrm{C_{rd}}[I_0 \times B, \mathbb{R}^n]$ is such that $\|g(t, u)\| \le M$, $(t, u) \in I_0 \times B$, $I_0 = [t_0 \le t \le t_0 + a] \cap \mathbb{T}$, $B = \{u \in \mathbb{R}^n : |u - u_0| \le b\}$, and M is a constant. $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

Lemma 2.1 ([9, Theorem 2.3.1]). *Assume that $g(t,u)$ is quasimonotone nondecreasing in u and for each i, $1 \le i \le n$, $g_i(t,u)\mu(t)+u_i$ is nondecreasing in u_i for each $t \in \mathbb{T}$. Then there exist minimal and maximal solutions of* (2.1) *on $[t_0, t_0+\eta] \cap \mathbb{T}$, where $\eta = \min(a, b/(2M+b))$.*

We now state a variational comparison result.

Lemma 2.2 ([9, Theorem 2.4.1]). *Let all the assumptions of Lemma* 2.1 *hold and let $m : I \equiv [t_0, t_0+a) \cap \mathbb{T} \to \mathbb{R}^n$ be differentiable for each $t \in I$ and satisfy $m^\Delta(t) \le g(t, m(t))$, $t \in I$. Then $m(t) \le \beta(t)$, $t \in I$, where $\beta(t)$ is the maximal solution of* (2.1).

Remark 2.1. Lemma 2.2 is also valid with $m^\Delta(t)$ replaced by $D^+m^\Delta(t)$.

Lemma 2.3. *Let all the assumptions of Lemma* 2.1 *hold and let $m : I \equiv [t_0, t_0+a) \cap \mathbb{T} \to \mathbb{R}^n$ be differentiable for each $t \in I$ and satisfy $m^\Delta(t) \ge g(t, m(t))$, $t \in I$. Then $m(t) \ge r(t)$, $t \in I$, where $r(t)$ is the minimal solution of* (2.1).

Proof. The proof is similar to that of Theorem 2.4.1 in [9], and so we omit it here. ■

Remark 2.2. Lemma 2.3 is also valid when $m^\Delta(t)$ is replaced by $D_+m^\Delta(t)$.

3. Comparison theorems. We consider the dynamic system

$$x^\Delta = f(t,x), \qquad x(t_0) = x_0, \tag{3.1}$$

on $\mathbb{T}$, where $f \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^n]$, $t_0 \in \mathbb{T}$, and $x_0 \in \mathbb{R}^n$. We assume, for convenience, that the solution $x(t) = x(t, t_0, x_0)$ of (3.1) exists and is unique for $t \ge t_0$ on $\mathbb{T}$.

Following Definition 2.1, we define $D_+V^\Delta(t, x(t))$ for $V \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$ by the condition: for given $\varepsilon > 0$, there exists a right neighborhood $\mathcal{N}_\varepsilon \subset \mathcal{N}$ of t such that

$$\frac{1}{\mu(t,s)}[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t,s) f(t, x(t))] > D_+V^\Delta(t, x(t)) - \varepsilon$$

for $s \in \mathcal{N}_\varepsilon$, $s > t$. As before, if t is right-scattered and $V(t, x(t))$ is continuous at t, this reduces to

$$D_+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu(t)} = V^\Delta(t, x(t)).$$

Following Definition 2.2, we define $D^+V^\Delta(t, x(t))$ for $V \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}+]$ by the condition: for given $\varepsilon > 0$, there exists a right neighborhood $\mathcal{N}_\varepsilon \subset \mathcal{N}$ of t such that

$$\frac{1}{\mu(t,s)}[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t))) - \mu(t,s) f(t, x(t))] < D^+V^\Delta(t, x(t)) + \varepsilon$$

for $s \in \mathcal{N}_\varepsilon$, $s > t$. As before, if t is right-scattered and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu(t)} = V^\Delta(t, x(t)).$$

From Lemmas 2.2 and 2.3, it is now easy to get necessary results in terms of Lyapunov-like functions.

LEMMA 3.1 ([9, Th. 3.1.2]). *Assume that the solution $x(t, t_0, x_0)$ of (3.1) is rd-continuous with respect to the initial data and $\|x(t, t_0, x_0)\|$ is locally Lipschitzian in x_0. Suppose that*

(i) *for $V \in \mathrm{C_{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, $V(t, x)$ is locally Lipschitzian in x for each right-dense point $t \in \mathbb{T}$, and for $t_0 < s \le t$, $z \in \mathbb{R}^n$,*

$$D^+V^\Delta(s, x(t, s, z)) \le G(s, V(s, x(t, s, z)));$$

(ii) *for $G \in \mathrm{C_{rd}}[\mathbb{T}^k \times \mathbb{R}^+, \mathbb{R}^+]$, $G(t, u)\mu(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and the maximal solution $r(t) = r(t, t_0, u_0)$ of $u^\Delta = G(t, u)$, $u(t_0) = u_0 \ge 0$, exists for $t \in \mathbb{T}$.*

Then if $y(t, t_0, x_0)$ is any solution of

$$y^\Delta = G(t, y), \qquad y(t_0) = x_0, \tag{3.2}$$

we have

$$V(t, y(t, t_0, x_0)) \le r(t, t_0, V(t_0, x(t, t_0, x_0))), \qquad t \in \mathbb{T},$$

provided $V(t_0, x(t, t_0, x_0)) \le u_0$.

LEMMA 3.2. *Assume that the solution $x(t, t_0, x_0)$ of (3.1) is rd-continuous with respect to the initial data and $\|x(t, t_0, x_0)\|$ is locally Lipschitzian in x_0. Suppose that*

(i) *for $V \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, $V(t, x)$ is locally Lipschitzian in x for each $t \in \mathbb{T}$ which is right-dense, and for $t_0 < s \le t$, $z \in \mathbb{R}^n$,*

$$D_+V^\Delta(s, x(t, s, z)) \ge G(s, V(s, x(t, s, z)));$$

(ii) *for $G \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^+, \mathbb{R}^+]$, $G(t, u)\mu(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and the minimal solution $r(t) = r(t, t_0, u_0)$ of $u^\Delta = G(t, u)$, $u(t_0) = u_0 \ge 0$, exists for $t \in \mathbb{T}$.*

Then if $y(t, t_0, x_0)$ is any solution of

$$y^\Delta(t) = G(t, y), \qquad y(t_0) = x_0, \tag{3.3}$$

we have

$$V(t, y(t, t_0, x_0)) \ge r(t, t_0, V(t_0, x(t, t_0, x_0))), \qquad t \in \mathbb{T},$$

provided $V(t_0, x(t, t_0, x_0)) \ge u_0$.

Proof. The proof is similar to that of Theorem 3.1.2 in [9], and so we omit it here. ∎

We now consider the hybrid dynamic system

$$
\begin{aligned}
&x^{\Delta}(t) = f(t, x, \lambda_k(\tau_k, z_k)), \quad t \in [\tau_k, \tau_{k+1}],\\
&x(\tau_k) = z_k \in \mathbb{R}^n, \quad k = 0, 1, 2, \ldots,
\end{aligned}
\tag{3.4}
$$

and the perturbed hybrid dynamic system

$$
\begin{aligned}
&y^{\Delta} = g(t, y, \lambda_k(\tau_k, z_k)), \quad t \in [\tau_k, \tau_{k+1}],\\
&y(\tau_k) = z_k \in \mathbb{R}^n, \quad k = 0, 1, 2, \ldots,
\end{aligned}
\tag{3.5}
$$

on $\mathbb{T}$, where

(i) $\tau_k \in \mathbb{T}$ with $0 \le t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ and $\tau_k \to \infty$ as $k \to \infty$,
(ii) $f, g \in \mathrm{C_{rd}}[\mathbb{T} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$, $z_k \in \mathbb{R}^n$ and $\lambda_k : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$, $k = 0, 1, 2, \ldots$.

Remark 3.1. If $\mathbb{T} = \mathbb{N}$, then equations (3.4) and (3.5) are equations (2.1) and (2.2) of [16]. In the case of $\mathbb{T} = \mathbb{R}$ they coincide with equations (2.1.1) and (2.2.2) in [7].

Remark 3.2. The term “practical stability” is used to describe the ability of a time-delay system to be stable in the presence of small perturbations. Any equation governing the behavior of a perturbation is called a perturbed system. Sometimes, it may be convenient to involve perturbations in the definition of practical stability itself since one can then deal with perturbations directly as constraints [7].

By a solution $x(t) = x(t, t_0, x_0)$ of (3.4) we mean the following:

$$
x(t) = \begin{cases}
x_0(t), & t_0 \le t \le \tau_1,\\
x_1(t), & \tau_1 \le t \le \tau_2,\\
\vdots & \\
x_k(t), & \tau_k \le t \le \tau_{k+1},\\
\vdots &
\end{cases}
$$

where $x_k(t) = x_k(t, \tau_k, z_k)$ is the solution of the dynamic system

$$
x_k^{\Delta}(t) = f(t, x_k(t), \lambda_k(\tau_k, z_k)), \quad x_k(\tau_k) = z_k \in \mathbb{R}^n, \tag{3.6}
$$

for each $k = 0, 1, 2, \ldots$, and $\tau_k \le t \le \tau_{k+1}$. The description of solutions of system (3.5) can be related to above. We assume that solutions of (3.4) and (3.5) exist and are unique for $t \ge t_0$.

We also need the scalar comparison hybrid dynamic system

$$
\begin{aligned}
&u^{\Delta}(t) = G(t, u, \delta_k(u_k)), \quad t \in [\tau_k, \tau_{k+1}],\\
&u(\tau_k) = v_k \in \mathbb{R}^+,
\end{aligned}
\tag{3.7}
$$

where $G \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}]$ and $\delta_k \in C[\mathbb{R}^+, \mathbb{R}^+]$. By a solution $r(t) = r(t, t_0, u_0)$ of (3.7), we mean the following:

$$r(t) = \begin{cases} r_0(t), & t_0 \le t \le \tau_1, \\ r_1(t), & \tau_1 \le t \le \tau_2, \\ \vdots & \\ r_k(t), & \tau_k \le t \le \tau_{k+1}, \\ \vdots & \end{cases}$$

where $r_k(t) = r_k(t, \tau_k, u_k)$ is the solution of

$$u_k^\Delta = G(t, u_k(t), \delta_k(v_k)), \qquad u_k(\tau_k) = v_k \in \mathbb{R}^+, \tag{3.8}$$

for each $k = 0, 1, 2, \ldots$ and $\tau_k \le t \le \tau_{k+1}$.

We can now prove the required comparison result in terms of a Lyapunov-like function which is useful in discussing the qualitative behavior of solutions of (3.4) and (3.5).

Relating to system (3.1), we assume the following:

(H) The solution $x(t, t_0, x_0)$ of (3.1) is rd-continuous with respect to the initial data and $\|x(t, t_0, x_0)\|$ is locally Lipschitzian in x_0.

THEOREM 3.1. *Assume that condition* (H) *holds and*

(i) *for $V \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, $V(t, x)$ is locally Lipschitzian in x for each right-dense $t \in \mathbb{T}$, and for $\tau_j \le t \le \tau_{j+1}$, $j = 0, 1, 2, \ldots,$*

$$D_+ V^\Delta(s, x(t, s, x_s)) \ge G(s, V(s, x(t, s, x_s)), \delta_j(V(\tau_j, x_j))), \qquad \tau_j \le s \le t,$$

where $G \in C_{\mathrm{rd}}[\mathbb{T} \times \mathbb{R}^+, \mathbb{R}^+]$, $G(t, u, v)\mu(t) + u$ is nondecreasing in v for each (t, u) and in u for each (t, v), $x_s = x_j(s, \tau_j, x_j)$, and $\delta_j(v)$ is nondecreasing in v for all j,

(ii) *the minimal solution $r(t) = r(t, t_0, u_0)$ of* (3.7) *exists for $t \ge t_0$, $t \in \mathbb{T}$.*

Then for any solution $y(t, t_0, x_0)$ of (3.5) *we have*

$$V(t, y(t, t_0, x_0)) \ge r(t, t_0, V(t_0, x(t, t_0, x_0))), \qquad t \in \mathbb{T}, \tag{3.9}$$

provided $V(t_0, x(t, t_0, x_0)) \ge u_0$.

Proof. Let $y_j(t, \tau_j, x_j)$ be the solution of system (3.5) in the interval $[\tau_j, \tau_{j+1}]$ existing for $t \ge t_0$, $t \in \mathbb{T}$, where $x_j = y_{j-1}(\tau_j, \tau_{j-1}, x_{j-1})$, $j = 1, 2, \ldots$. Set

$$m(s) = V(s, x(t, s, x_s)), \qquad \tau_j \le s \le t \le \tau_{j+1},$$

where $x_s = x_j(s, \tau_j, x_j)$. Then by condition (i), it is easy to derive the differential inequality

$$D_+ m^\Delta(s) \ge G(s, m(s), \delta_r(m_r)), \qquad s \in [\tau_j, \tau_{j+1}],$$

where $m_j = V(\tau_j, x_j)$.

Consider the interval $[t_0, \tau_1]$, and set $m_0 = m(t_0) = V(t_0, x(t, t_0, x_0)) = u_0$. For $t_0 \le t \le \tau_1$, $t_0 \le s \le t$, we have

$$D_+ m^\Delta(s) \ge G(s, m(s), \delta_0(m_0)).$$

Hence, by Lemma 3.2, we get

$$V(t, y_0(t, t_0, x_0)) \ge r_0(t, t_0, V(t_0, x(t, t_0, x_0))),$$

where $r_0(t, t_0, u_0)$ is the minimal solution of

$$u_0^\Delta(t) = G(t, u_0(t), \delta_0(u_0)), \quad u_0(t_0) = u_0, \quad t \in [t_0, \tau_1],$$

and $y_0(t) = y_0(t, t_0, x_0)$ is the solution of

$$y_0^\Delta(t) = g(t, y_0(t), \lambda_0(x_0)), \quad y_0(t_0) = x_0, \quad t \in [t_0, \tau_1].$$

Next, consider the interval $[\tau_1, \tau_2]$. Choosing $u_1 = r_0(\tau_1, t_0, u_0)$ and $x_1 = y_0(\tau_1, t_0, x_0)$, we have

$$D_+ m^\Delta(s) \ge G(s, m(s), \delta_1(m_1)),$$

where $m_1 = V(\tau_1, y_0(\tau_1, t_0, x_0)) \ge r_0(\tau_1, t_0, u_0) = u_1$. In the interval $[\tau_1, \tau_2]$, using the monotonicity properties of $g(t, u, v)$ and $\delta_1(v)$ with respect to v and assumption (i), we have

$$D_+ m^\Delta(s) \ge G(s, m(s), \delta_1(u_1)).$$

Again, by Lemma 3.2 with $u_1 = V(\tau_1, x(t, \tau_1, x_1))$, we have

$$V(t, y_1(t, \tau_1, x_1)) \ge r_1(t, \tau_1, V(\tau_1, x(t, \tau_1, x_1))),$$

where $r_1(t, t_1, u_1)$ is the minimal solution of

$$u_1^\Delta(t) = G(t, u_1(t), \delta_1(u_1)), \quad u_1(\tau_1) = u_1, \quad t \in [\tau_1, \tau_2],$$

and $y_1(t) = y_1(t, \tau_1, x_1)$ is the solution of

$$y_1^\Delta(t) = g(t, y_1(t), \lambda_1(x_1)), \quad y_1(\tau_1) = x_1, \quad t \in [\tau_1, \tau_2].$$

We repeat the process using the special choice of $x_j = y_{j-1}(\tau_j, \tau_{j-1}, x_{j-1})$, $j = 1, 2, \ldots$, to get

$$V(t, x_j(t)) \ge r_j(t, \tau_j, V(\tau_j, x(t, \tau_j, x_j))),$$

where $r_j(t, \tau_j, u_j)$ is the minimal solution of

$$u_j^\Delta(t) = G(t, u_j(t), \delta_j(u_j)), \quad u_j(\tau_j) = u_j, \quad t \in [\tau_j, \tau_{j+1}],$$

and $y_j(t) = y_j(t, \tau_j, x_j)$ is the solution of

$$y_j^\Delta(t) = g(t, y_j(t), \lambda_j(x_j)), \quad y_j(\tau_r) = x_j, \quad t \in [\tau_j, \tau_{j+1}].$$

Defining for $t \in \mathbb{T}$, $t \geq t_0$,

$$V(t, y(t)) = \begin{cases} V(t, y_0(t)), & t_0 \leq t \leq \tau_1, \\ V(t, y_1(t)), & \tau_1 \leq t \leq \tau_2, \\ \vdots & \\ V(t, y_j(t)), & \tau_j \leq t \leq \tau_{j+1}, \\ \vdots & \end{cases}$$

with $V(t, x(t, t_0, x_0)) = u_0$, we have the desired estimate (3.9) for $t \geq t_0$ and the proof is complete. ∎

Estimate (3.9) shows a connection between solutions of systems (3.4) and (3.5) in terms of the minimal solution of (3.7). Similarly, we can give a connection between solutions of systems (3.4) and (3.5) in terms of the maximal solution of (3.7).

THEOREM 3.2. *Assume that condition* (H) *holds and*

(i) *for $V \in \mathrm{C_{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$, $V(t, x)$ is locally Lipschitzian in x for each right-dense $t \in \mathbb{T}$, and for $\tau_j \leq t \leq \tau_{j+1}$, $j = 0, 1, 2, \ldots$,*

$$D^+V^{\Delta}(s, x(t, s, x_k)) \leq G(s, V(s, x(t, s, x_s)), \delta_j(V(\tau_j, x_j))), \qquad \tau_j \leq s \leq t,$$

where $G \in \mathrm{C_{rd}}[\mathbb{T}^k \times \mathbb{R}^+, \mathbb{R}^+]$, $G(t, u, v)\mu(t) + u$ is nondecreasing in v for each (t, u) and in u for each (t, v), and $\delta_j(v)$ is nondecreasing in v for all j,

(ii) *the maximal solution $\beta(t) = \beta(t, t_0, u_0)$ of (3.7) exists for $t \geq t_0$, $t \in \mathbb{T}$.*

Then for any solution $y(t, t_0, x_0)$ of (3.5) we have

$$V(t, y(t, t_0, x_0)) \leq \beta(t, t_0, V(t_0, x(t, t_0, x_0))), \qquad t \in \mathbb{T}, \tag{3.10}$$

provided $V(t_0, x(t, t_0, x_0)) \leq u_0$.

4. Practical stability in terms of two measures. Using the comparison Theorems 3.1 and 3.2, we prove some results on practical stability and strict practical stability of system (3.5) in terms of two measures when information about systems (3.4) and (3.7) is known. For the sake of convenience, we introduce the following function classes:

$$\mathcal{K} = \{\alpha \in \mathrm{C}[\mathbb{R}^+, \mathbb{R}^+] : \alpha \text{ is strictly increasing and } \alpha(0) = 0\},$$

$$\Gamma = \{h \in \mathrm{C_{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+] : \inf_{(t,x)} h(t, x) = 0\},$$

$$\Sigma = \{Q \in \mathrm{C}[\mathbb{R}^+, \mathbb{R}^+] : Q(0) = 0 \text{ and } Q \text{ is increasing}\},$$

$$S(h, \gamma) = \{(t, x) \in \mathbb{T} \times \mathbb{R}^n : h \in \Gamma \text{ and } h(t, x) < \gamma, \gamma > 0\}.$$

Definition 4.1. The hybrid dynamic system (3.4) is said to be

(i) *practically stable* if for given $0 < \lambda < A$, the condition $\|x_0\| < \lambda$ implies $\|x(t)\| < A$, $t \geq t_0$ for some $t_0 \in \mathbb{T}$, where $x(t) = x(t, t_0, x_0)$ is any solution of (3.4);

(ii) *practically quasi-stable* if for given $\lambda, B, T > 0$ and some $t_0 \in \mathbb{T}$ with $t_0 + T \in \mathbb{T}$, the condition $\|x_0\| < \lambda$ implies $\|x(t)\| < B$, $t \geq t_0 + T$, $t \in \mathbb{T}$;

(iii) *strongly practically stable* if (i) and (ii) hold simultaneously;

(iv) *practically asymptotically stable* if (i) holds and for any given $\varepsilon > 0$ there exists $T_0 > 0$ such that $t_0 + T_0 \in \mathbb{T}$ and $\|x_0\| < \lambda$ implies $\|x(t)\| < \varepsilon$, $t \geq t_0 + T_0$;

(v) *strictly practicallly stable* if there exist $0 < \lambda_1 \leq A_1$ such that $\|x_0\| < \lambda_1$ implies $\|x(t)\| < A_1$, $t \geq t_0$, for some $t_0 \in \mathbb{T}$, and for every $0 < \lambda_2 \leq \lambda_1$ there exists $A_2 \leq \lambda_2$ such that $\|x_0\| > \lambda_2$ implies $\|x(t)\| > A_2$, $t \geq t_0$.

Definition 4.2. Let $h, h_0 \in \Gamma$. The hybrid dynamic system (3.4) is said to be

(PS1) (h_0, h)-*practically stable* if for given $0 < \lambda < A$, the condition $h_0(t_0, x_0) < \lambda$ implies $h(t, x(t)) < A$, $t \geq t_0$, for some $t_0 \in \mathbb{T}$, where $x(t) = x(t, t_0, x_0)$ is any solution of (3.4);

(PS2) (h_0, h)-*practically quasi-stable* if for given $\lambda, B, T > 0$ and some $t_0 \in \mathbb{T}$ with $t_0 + T \in \mathbb{T}$, the condition $h_0(t_0, x_0) < \lambda$ implies $h(t, x(t)) < B$, $t \geq t_0 + T$, $t \in \mathbb{T}$;

(PS3) (h_0, h)-*strongly practically stable* if (PS1) and (PS2) hold simultaneously;

(PS4) (h_0, h)-*practically asymptotically stable* if (PS1) holds and for any given $\varepsilon > 0$ there exists $T_0 > 0$ such that $t_0 + T_0 \in \mathbb{T}$ and $h_0(t_0, x_0) < \lambda$ implies $h(t, x(t)) < \varepsilon$, $t \geq t_0 + T_0$;

(PS5) (h_0, h)-*strictly practically stable* if there exist $0 < \lambda_1 \leq A_1$ such that $h_0(t_0, x_0) < \lambda_1$ implies $h(t, x(t)) < A_1$, $t \geq t_0$, for some $t_0 \in \mathbb{T}$, and for every $0 < \lambda_2 \leq \lambda_1$ there exists $A_2 \leq \lambda_2$ such that $h_0(t_0, x_0) > \lambda_2$ implies $h(t, x(t)) > A_2$, $t \geq t_0$.

One can similarly define corresponding notions for the scalar comparison dynamic system (3.7).

Definition 4.3. Let $Q_0, Q \in \Sigma$. Then we say that the scalar comparison hybrid dynamic system (3.7) is (Q_0, Q)-*practically stable* if for given $0 < \lambda < A$, the condition $Q_0(t_0, u_0) < \lambda$ implies $Q(t, u(t)) < A$, $t \geq t_0$, $t \in \mathbb{T}$, where $u(t) = u(t, t_0, u_0)$ is any solution of (3.7).

Other practical stability notions can be defined similarly.

THEOREM 4.1. *Assume condition* (H) *holds and*

(A1) $0 < \lambda < A$;

(A2) $h_0, h^*, h \in \Gamma$, $h^*(t, x)$ *is nondecreasing in* t *and* h_0 *is uniformly finer than* h, *i.e.*,

$$h(t, x) \le \phi(h_0(t, x)),\ \phi \in \mathcal{K}, \quad \textit{whenever} \quad h_0(t, x) < \lambda;$$

(A3) *there exists* $V \in \mathrm{C_{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$ *such that* $V(t, u)$ *is locally Lipschitzian in* u *for each right-dense* $t \in \mathbb{T}$, *and for* $a, b \in \mathcal{K}$,

$$V(t, x) \ge b(h(t, x)) \qquad \textit{if } (t, x) \in S(h, A),$$
$$V(t, x) \le a(h^*(t, x)) \qquad \textit{if } (t, x) \in S(h^*, \lambda);$$

(A4) *for* $(t, x) \in S(h, A)$,

$$D^+V^{\Delta}(s, x(t, s, x_k)) \le G(s, V(s, x(t, s, x_k)), \delta_r(V(\tau_j, x_j))),$$

where $G \in \mathrm{C_{rd}}[\mathbb{T}^k \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$, $G(t, u, v)\mu(t) + u$ *is nondecreasing in* v *for each* (t, u) *and in* u *for each* (t, v), *and* $\delta_j(v)$ *is nondecreasing in* v *for all* j;

(A5) $\phi(\lambda) < A$ *and* $a(\lambda) < b(A)$;

(A6) *system* (3.4) *is* (h_0, h^*)*-practically stable with respect to* (λ, λ), *i.e.*,

$$h_0(t_0, x_0) < \lambda \quad \textit{implies} \quad h^*(t, x(t, t_0, x_0)) < \lambda, t \ge t_0. \tag{4.1}$$

Then practical stability properties of system (3.7) *imply the corresponding* (h_0, h)*-practical stability properties of the perturbed system* (3.5).

Proof. Assume that (3.7) is practically stable. Then for given $(a(\lambda), b(A))$,

$$u_0 < a(\lambda) \quad \text{implies} \quad u(t, t_0, u_0) < b(A), \qquad t \ge t_0, \tag{4.2}$$

where $u(t, t_0, u_0)$ is any solution of (3.7). Since (3.4) is (h_0, h^*)-practically stable with respect to (λ, λ), (4.1) holds. Then by (A2) and (A5), it follows that

$$h(t_0, x_0) \le \phi(h_0(t_0, x_0)) < \phi(\lambda) < A.$$

We claim that $h(t, y(t)) < A$, $t \ge t_0$, where $y(t) = y(t, t_0, x_0)$ is any solution of (3.5). Indeed, if this were not true, there would exist a solution $y(t, t_0, x_0)$ of (3.5) with $h_0(t_0, x_0) < \lambda$ and a $t_1 > t_0$, $t_1 \in \mathbb{T}$, such that

$$h(t_1, y(t_1, t_0, x_0)) \ge A, \qquad h(t, y(t, t_0, x_0)) < A, \qquad t_0 \le t < t_1. \tag{4.3}$$

As $h_0(t_0, x_0) < \lambda$, Theorem 3.2 together with (A2), (A3) and (4.3) imply

$$V(t, y(t, t_0, x_0)) \le \beta(t, t_0, V(t_0, x(t, t_0, x_0))), \qquad t_0 \le t \le t_1, \tag{4.4}$$

and

$$V(t_0, x(t_1, t_0, x_0)) \le a(h^*(t_0, x(t_1, t_0, x_0))) \le a(h^*(t_1, x(t_1, t_0, x_0))) < a(\lambda).$$

Using (A3), (4.1), (4.3) and (4.4), we have

$$\begin{aligned} b(A) &\le b(h(t_1, y(t_1, t_0, x_0))) \le V(t_1, y(t_1, t_0, x_0)) \\ &\le \beta(t_1, t_0, V(t_0, x(t_1, t_0, x_0))) < b(A), \end{aligned}$$

and this contradiction proves that $h_0(t_0, x_0) < \lambda$ implies $h(t, y(t, t_0, x_0)) < A$, $t \ge t_0$.

Next, we prove that system (3.5) is (h_0, h)-strongly practically stable for given positive numbers λ, A, B, T. To do this, suppose that (3.7) is strongly practically stable for positive numbers $a(\lambda)$, $b(A)$, $b(B)$, T. This means we only need to prove (h_0, h)-practical quasi-stability of system (3.5). Practical quasi-stability of (3.7) means that

$$u_0 < a(\lambda) \text{ implies } u(t, t_0, u_0) < b(B),\ t \ge t_0 + T, \quad \text{with } t_0 + T \in \mathbb{T}.$$

From the foregoing argument, since $V(t_0, x(t, t_0, x_0)) < a(\lambda)$ if $h_0(t_0, x_0) < \lambda$, we have

$$b(h(t, y(t, t_0, x_0))) \le V(t, y(t, t_0, x_0)) \le u(t, t_0, V(t_0, x(t, t_0, x_0))) < b(B)$$

for all $t \ge t_0 + T$ if $h_0(t_0, x_0) < \lambda$. Thus we have $h(t, x(t)) < B$, $t \ge t_0 + T$, provided $h_0(t_0, x_0) < \lambda$. Hence system (3.5) is (h_0, h)-strongly practically stable.

Finally, we show that system (3.5) is (h_0, h)-practically asymptotically stable. Now, let us suppose that (3.7) is practically asymptotically stable. This implies we only need to prove that for any given $\varepsilon > 0$ there exists $T_0 > 0$ with $t_0 + T_0 \in \mathbb{T}$ such that $t_0 + T_0 \ge T$ and $h_0(t_0, x_0) < \lambda$ implies $h(t, x(t)) < B$, $t \ge t_0 + T_0$, for system (3.5). Practical asymptotic stability of (3.7) means that

$$u_0 < a(\lambda) \quad \text{implies} \quad u(t, t_0, u_0) < b(B),\ t \ge t_0 + T_0.$$

From the argument above, since $V(t_0, x(t, t_0, x_0)) < a(\lambda)$ whenever $h_0(t_0, x_0) < \lambda$, we obtain

$$b(h(t, y(t, t_0, x_0))) \le V(t, y(t, t_0, x_0)) \le u(t, t_0, V(t_0, x(t, t_0, x_0))) < b(B)$$

for all $t \ge t_0 + T_0$ if $h_0(t_0, x_0) < \lambda$. Thus we have $h(t, x(t)) < B$, $t \ge t_0 + T_0$ provided $h_0(t_0, x_0) < \lambda$. Hence system (3.5) is (h_0, h)-strongly practically stable, and the proof is complete. ∎

THEOREM 4.2. *Suppose that conditions of Theorem* 4.1 *are satisfied except that condition* (A3) *is replaced by*

(A7) *$Q_0, Q \in \Sigma$, and for $a, b \in \mathcal{K}$*

$$\begin{aligned} Q(V(t,x)) &\ge b(h(t,x)) \qquad \text{if } (t,x) \in S(h, A), \\ Q_0(V(t,x)) &\le a(h^*(t,x)) \qquad \text{if } (t,x) \in S(h^*, \lambda). \end{aligned}$$

Then (Q_0, Q)-practical stability properties of system (3.7) *imply the corresponding (h_0, h)-practical stability properties of the perturbed system* (3.5).

Proof. Assume that (3.7) is (Q_0, Q)-practically stable. Then we have $(a(\lambda), b(A))$ such that

$$Q_0(u_0) < a(\lambda) \quad \text{implies} \quad Q(u(t, t_0, u_0)) < b(A),\ t \geq t_0. \tag{4.5}$$

Suppose that the (h_0, h)-practical stability of (3.5) does not hold. Then arguing as in Theorem 4.1, by (A2), (A7) and (4.3) we can show that

$$\begin{aligned} Q_0(V(t_0, x(t_1, t_0, x_0))) &\leq a(h^*(t_0, x(t_1, t_0, x_0))) \\ &\leq a(h^*(t_1, x(t_1, t_0, x_0))) < a(\lambda). \end{aligned}$$

Then using (A7), (4.1), (4.3) and (4.4), we have

$$\begin{aligned} b(A) &\leq b(h(t_1, y(t_1, t_0, x_0))) \leq Q(V(t_1, y(t_1, t_0, x_0))) \\ &\leq Q(u(t_1, t_0, V(t_0, x(t_1, t_0, x_0)))) < b(A), \end{aligned}$$

which is a contradiction. The proof is complete. ∎

Remark 4.1. If $h_0 = h^* = h = \|x\|$, then we get the usual practical stability of system (3.5).

Remark 4.2. If $Q_0(u) = Q(u) = u$, we deduce Theorem 4.1.

The following result concerns (h_0, h)-strict practical stability of (3.5).

Theorem 4.3. *Assume that condition* (H) *holds and*

(B1) $h_0, h^*, h \in \Gamma$, $h^*(t, x)$ *is nondecreasing in* t *and* h_0 *is uniformly finer than* h, *i.e.*, $h(t, x) \leq \phi(h_0(t, x))$, $\phi \in \mathcal{K}$, *whenever* $h_0(t, x) < \lambda$, *and* $h(t, x) \geq \phi(h_0(t, x))$, $\phi \in \mathcal{K}$, *whenever* $h_0(t, x) > \lambda$;

(B2) *for each* $0 < \eta \leq A_i$, *there exist* $V_i \in \mathrm{C_{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$ *such that* $V_i(t, x)$ *is locally Lipschitzian in* x *for each right-dense* $t \in \mathbb{T}$, *and there exist* $a_i, b_i \in \mathcal{K}$, $i = 1, 2$, *such that*

$$\begin{aligned} V_1(t, x) &\geq b_1(h(t, x)) && \text{if } h(t, x) \leq \eta, \\ V_1(t, x) &\leq a_1(h^*(t, x)) && \text{if } h^*(t, x) \leq \eta, \end{aligned}$$

and

$$\begin{aligned} V_2(t, x) &\leq a_2(h(t, x)) && \text{if } h(t, x) \geq \eta, \\ V_2(t, x) &\geq b_2(h^*(t, x)) && \text{if } h^*(t, x) \geq \eta; \end{aligned}$$

(B3)

$$D^+ V_1^{\Delta}(s, x(t, s, x_k)) \leq G(s, V(s, x(t, s, x_k)), \delta_r(V(\tau_r, x_r))) \qquad \textit{for } h(t, x) \leq A_1,$$

$$D_+ V_2^{\Delta}(s, x(t, s, x_k)) \geq G(s, V(s, x(t, s, x_k)), \delta_r(V(\tau_r, x_r))) \qquad \textit{for } h(t, x) \geq A_2,$$

where $G \in C_{\mathrm{rd}}[\mathbb{T}^k \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+]$, $G(t,u,v)\mu(t) + u$ is nondecreasing in v for each (t,u) and in u for each (t,v), and $\delta_r(v)$ is nondecreasing in v for all r;

(B4) *$\phi(\lambda_1) < A_1$, $\phi(\lambda_2) > A_2$, $a_1(\lambda_1) < b_1(A_1)$ and $a_2(A_2) < b_2(\lambda_2)$;*

(B5) *system (3.4) is (h_0, h^*)-strictly practically stable with respect to (λ_1, λ_1) and (λ_2, λ_2), i.e., $h_0(t_0, x_0) < \lambda_1$ implies $h^*(t, x(t, t_0, x_0)) < \lambda_1$ and $h_0(t_0, x_0) > \lambda_2$ implies $h^*(t, x(t, t_0, x_0)) > \lambda_2$, $t \geq t_0$.*

Then strict practical stability properties of system (3.7) imply the corresponding strict (h_0, h)-practical stability properties of the perturbed system (3.5).

Proof. Assume that (3.7) is strictly practically stable. Then there exist $0 < \lambda_1 \leq A_1$ such that $a_1(\lambda_1) < b_1(A_1)$ and $u_0 < a_1(\lambda_1)$ implies $u(t) < b_1(A_1)$, $t \geq t_0$, for some $t_0 \in \mathbb{T}$, and for every $0 < \lambda_2 \leq \lambda_1$ there exists $A_2 \leq \lambda_2$ such that $v_0 > b_2(\lambda_2)$ implies $v(t) > a_2(A_2)$, $t \geq t_0$, where $u(t) = u(t, t_0, u_0)$ and $v(t) = v(t, t_0, u_0)$ are maximal and minimal solutions of (3.7), respectively.

Since (3.4) is (h_0, h^*)-strictly practically stable with respect to (λ_1, λ_1) and (λ_2, λ_2), it follows that

$$h_0(t_0, x_0) < \lambda_1 \quad \text{implies} \quad h^*(t, x(t, t_0, x_0)) < \lambda_1, \qquad t \geq t_0, \tag{4.6}$$

$$h_0(t_0, x_0) > \lambda_2 \quad \text{implies} \quad h^*(t, x(t, t_0, x_0)) > \lambda_2, \qquad t \geq t_0. \tag{4.7}$$

Then by (B1) and (B4),

$$\begin{aligned} h(t_0, x_0) &\leq \phi(h_0(t_0, x_0)) < \phi(\lambda_1) < A_1 \quad \text{whenever} \quad h_0(t_0, x_0) < \lambda_1, \\ h(t_0, x_0) &\geq \phi(h_0(t_0, x_0)) > \phi(\lambda_2) > A_2 \quad \text{whenever} \quad h_0(t_0, x_0) > \lambda_2. \end{aligned}$$

Now we claim that $h_0(t_0, x_0) < \lambda_1$ implies $h(t, y(t)) < A_1$, $t \geq t_0$, where $y(t) = y(t, t_0, x_0)$ is any solution of (3.5). If this were not true, there would exist a solution $y(t, t_0, x_0)$ of (3.5) with $h_0(t_0, x_0) < \lambda_1$ and a $t_1 \in \mathbb{T}$, $t_1 > t_0$, such that

$$h(t_1, y(t_1, t_0, x_0)) \geq A_1, \qquad h(t, y(t, t_0, x_0)) < A_1, \qquad t_0 \leq t < t_1. \tag{4.8}$$

Then by Theorem 3.2, we obtain

$$V_1(t, y(t, t_0, x_0)) \leq u_1(t, t_0, V_1(t_0, x(t, t_0, x_0))), \qquad t_0 \leq t \leq t_1. \tag{4.9}$$

Because of (B1), (B2) and (4.6), we have

$$\begin{aligned} V_1(t_0, x(t_1, t_0, x_0)) &\leq a_1(h^*(t_0, x(t_1, t_0, x_0))) \\ &\leq a_1(h^*(t_1, x(t_1, t_0, x_0))) < a_1(\lambda_1) \end{aligned}$$

for $h_0(t_0, x_0) < \lambda_1$. Using (B2), (4.8) and (4.9), we have

$$\begin{aligned} b_1(A_1) &\leq b_1(h(t_1, y(t_1, t_0, x_0))) \leq V_1(t_1, y(t_1, t_0, x_0)) \\ &\leq u_1(t_1, t_0, V_1(t_0, x(t_1, t_0, x_0))) < b(A_1), \end{aligned}$$

and this contradiction proves that $h_0(t_0, x_0) < \lambda_1$ implies $h(t, y(t, t_0, x_0)) < A_1$, $t \geq t_0$.

On the other hand, for every $0 < \lambda_2 \leq \lambda_1$, we claim that $h_0(t_0, x_0) > \lambda_2$ implies $h(t, y(t)) > A_2$, $t \geq t_0$. If this were not true, there would exist a solution $y(t, t_0, x_0)$ of (3.5) with $h_0(t_0, x_0) > \lambda_2$ and a $t_1 \in \mathbb{T}$ with $t_1 > t_0$ such that

$$\begin{aligned} &h(t, y(t)) \leq A_2 && \text{for } t \geq t_1, \\ &A_2 \leq h(t, y(t)) \leq A_1 && \text{for } t_0 \leq t < t_1. \end{aligned} \tag{4.10}$$

Then by Theorem 3.1, we get

$$V_2(t, y(t, t_0, x_0)) \geq u_2(t, t_0, V_2(t_0, x(t, t_0, x_0))), \qquad t_0 \leq t \leq t_1. \tag{4.11}$$

By (B1), (B2) and (3.7), we have

$$V_2(t_0, x(t_1, t_0, x_0)) \geq b_2(h^*(t_0, x(t_1, t_0, x_0))) > b_2(\lambda_2)$$

for $h_0(t_0, x_0) > \lambda_2$. Using (B2), (4.10) and (4.11), we have

$$\begin{aligned} a_2(A_2) &\geq a_2(h(t_1, y(t_1, t_0, x_0))) \geq V_2(t_1, y(t_1, t_0, x_0)) \\ &\geq u_2(t_1, t_0, V_2(t_0, x(t_1, t_0, x_0))) > a_2(A_2), \end{aligned}$$

which is a contradiction. Hence system (3.5) is (h_0, h)-strictly practically stable, and the proof is complete. ∎

Theorem 4.4. *Suppose that the conditions of Theorem* 4.3 *are satisfied except that the condition* (B2) *is replaced by*

(B6) *for any $0 < \eta \leq A_i$, there exist $V_i \in \mathrm{C_{rd}}[\mathbb{T} \times \mathbb{R}^n, \mathbb{R}^+]$ such that $V_i(t, x)$ is locally Lipschitzian in x for each right-dense $t \in \mathbb{T}$, and there exist $Q_0, Q \in \Sigma$ and $a_i, b_i \in \mathcal{K}$, $i = 1, 2$, such that*

$$\begin{aligned} Q(V_1(t, x)) &\geq b_1(h(t, x)) && \text{if } h(t, x) \leq \eta, \\ Q_0(V_1(t, x)) &\leq a_1(h^*(t, x)) && \text{if } h^*(t, x) \leq \eta \end{aligned}$$

and

$$\begin{aligned} Q(V_2(t, x)) &\leq a_2(h(t, x)) && \text{if } h(t, x) \geq \eta, \\ Q_0(V_2(t, x)) &\geq b_2(h^*(t, x)) && \text{if } h^*(t, x) \geq \eta. \end{aligned}$$

Then (Q_0, Q)-strict practical stability properties of system (3.7) *imply the corresponding (h_0, h)-strict practical stability properties of the perturbed system* (3.5).

The proof of Theorem 4.4 is similar to the proof of Theorem 4.2 and hence is omitted.

Remark 4.3. If $\mathbb{T} = \mathbb{R}$, then Theorems 3.2, 4.1 and 4.2 of this paper coincide with Theorems 2.1, 3.1 and 3.2 in [16].

Acknowledgements. The authors wish to thank the referees for their valuable suggestions and useful comments.

This research is supported by the Natural Science Foundation of China (60774004, 60904024), China Postdoctoral Science Foundation Funded Project (20080441126, 200902564), Shandong Postdoctoral Foundation (200802018), the Natural Science Foundation of Shandong (Y2008A28, ZR2009AL003), and the Fund of Doctoral Program Research of the University of Jinan (XBS0843).

References

[1] D. Acheson, *From Calculus to Chaos: An Introduction to Dynamics*, Oxford Univ. Press, Oxford, 1997.

[2] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, 2001.

[3] —, —, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[4] F. H. Clark, Yu. S. Ledyaev, R. I. Stern and P. R. Woleski, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.

[5] L. T. Grujić, *Exact determination of a Lyapunov function and the asymptotic stability domain*, Int. J. Systems Sci. 23 (1992), 1871–1888.

[6] S. Hilger, *Analysis on measure chains—a unified approach to continuous and discrete calculus*, Results Math. 18 (1990), 18–56.

[7] V. Lakshmikantham, S. Leela and A. A. Martynyuk, *Practical Stability of Nonlinear Systems*, World Sci., Singapore, 1990.

[8] V. Lakshmikantham and R. N. Mohapatra, *Strict stability of differential equations*, Nonlinear Anal. 46 (2001), 915–921.

[9] V. Lakshmikantham, S. Sivasundaram and B. Kaymakcalan, *Dynamic Systems on Measure Chains*, Kluwer, Dordrecht, 1996.

[10] V. Lakshmikantham and A. S. Vatsala, *Hybrid systems on time scales*, J. Comput. Appl. Math. 141 (2002), 227–235.

[11] V. Lakshmikantham and Y. Zhang, *Strict practical stability of delay differential equation*, Appl. Math. Comput. 118 (2001), 275–285.

[12] J. LaSalle and S. Lefschetz, *Stability by Liapunov's Direct Method*, Academic Press, New York, 1961.

[13] A. M. Nerode and W. Kohn, *Models for Hybrid Systems*, Lecture Notes in Comput. Sci. 36, Springer, Berlin, 1993.

[14] P. G. Wang and X. Liu, *New comparison principle and stability criteria for impulsive hybrid systems on time scales*, Nonlinear Anal. Real World Appl. 7 (2006), 1096–1103.

[15] —, —, *Practical stability of impulsive hybrid differential systems in terms of two measures on time scales*, Nonlinear Anal. 65 (2006), 2035–2042.

[16] P. G. Wang, M. Wu and Y. H. Wu, *Practical stability in terms of two measures for discrete hybrid systems*, Nonlinear Anal. Hybrid Systems 2 (2008), 58–64.

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Received June 6, 2010;
received in final form December 9, 2010 (7770)