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INTEGRAL EQUATIONS

Periodic and Almost Periodic Solutions of Integral Inclusions

by

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Summary. The existence of a continuous periodic and almost periodic solutions of the nonlinear integral inclusion is established by means of the generalized Schauder fixed point theorem.

In [7] O'Regan and Meehan obtained some results on the existence of periodic and almost periodic solutions of the nonlinear Fredholm integral equation. Their method was based on the nonlinear alternative. Employing similar approach we prove the existence of periodic and almost periodic solutions for the integral inclusion

$$y(t) \in h(t) + \int_I k(t,s)F(s,y(s))\,ds, \qquad t \in I, \tag{1}$$

where I is an interval of $\mathbb{R}$ (finite or infinite), $h\colon I \to \mathbb{R}^n$, $k\colon I \times I \to \mathbb{R}$ and $F\colon I \times \mathbb{R}^n \multimap \mathbb{R}^n$ is a multimap. Our conclusions rely on the following fixed point principle for so called admissible set-valued maps (see Corollary (41.13) in [6]).

THEOREM 1 (The Schauder fixed point theorem). *Let X be a convex subset of a normed space and let $\varphi\colon X \multimap X$ be a compact admissible multimap. Then there exists a point $x \in X$ such that $x \in \varphi(x)$.*

REMARK 1. For our purposes it is sufficient to know that any acyclic map $\varphi\colon X \multimap X$, i.e. any upper semicontinuous multimap with compact acyclic values is admissible.

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Remark 2. Observe that the composition of two acyclic mappings is an admissible map: in particular, the composition of two u.s.c. mappings with compact contractible values and the composition of two u.s.c. mappings with compact convex values are again admissible maps (see Section 40 in [6]).

Let us assume that $0 < T < \infty$ and I is an interval of $\mathbb{R}$ that contains at least one compact subinterval of length T, which is denoted by I_T. We denote by $P_T(I, \mathbb{R}^n)$ the space of T-periodic continuous mappings from I to $\mathbb{R}^n$. These mappings form a closed subspace of the space of bounded continuous functions $BC(I, \mathbb{R}^n)$ and for that reason they form a Banach space equipped with the usual supremum norm $\|\cdot\|_\infty$. $AP(\mathbb{R}^n)$ stands for the space of continuous almost periodic functions, in the Bohr sense, with values in $\mathbb{R}^n$. It is important for our considerations that continuous almost periodic functions are bounded and $AP(\mathbb{R}^n)$ endowed with the supremum norm $\|\cdot\|_\infty$ also forms a Banach space (see [4]). Finally we have the following compactness theorem, the proof of which is given in [4, Theorem 6.10].

Theorem 2. *A necessary and sufficient condition for a family $\mathcal{F}$ of functions from $AP(\mathbb{R}^n)$ to be relatively compact is that the following properties hold true:*

(i) *for any $t \in \mathbb{R}$, the set of values of functions from $\mathcal{F}$ is relatively compact in $\mathbb{R}^n$,*
(ii) *$\mathcal{F}$ is equi-continuous,*
(iii) *$\mathcal{F}$ is equi-almost periodic.*

We now formulate two existence theorems which follow from Theorem 1. Recall first the definition of an L^p-Carathéodory multifunction.

Definition 1. We say that a multimap $F\colon I \times \mathbb{R}^n \multimap \mathbb{R}^n$ is *L^p-Carathéodory* if the following conditions hold:

(F1) $F(\cdot, x)$ is measurable for every $x \in \mathbb{R}^n$,
(F2) $F(t, \cdot)$ is upper semicontinuous for almost every $t \in I$,
(F3) for any $M > 0$, there exists $\mu_M \in L^p(I, \mathbb{R}_+)$ such that $|x| \le M$ implies that $|F(t, x)| = \sup\{|y| \colon y \in F(t, x)\} \le \mu_M(t)$ for almost all $t \in I$.

Theorem 3. *Let $p \in [1, \infty)$, let q be such that $p^{-1} + q^{-1} = 1$, $T \in (0, \infty)$, and let I be an interval of $\mathbb{R}$ (finite in the case $p = 1$) that contains at least one compact subinterval I_T of length T. Assume that $F\colon I \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an L^p-Carathéodory multimap with nonempty compact convex values and $h\colon I \to \mathbb{R}^n$ is continuous and T-periodic. Let $k\colon I \times I \to \mathbb{R}$ be such that $k(t, \cdot) \in L^q(I)$ for every $t \in I$ and*

(2) *the map $I \ni t \mapsto k(t, \cdot) \in L^q(I)$ is continuous and T-periodic.*

In addition suppose that for some $M > 0$,

$$\|h\|_\infty + \sup_{t\in I} \|k(t,\cdot)\|_q \|\mu_M\|_p \le M. \tag{3}$$

Then the inclusion (1) *has a continuous T-periodic solution.*

Proof. First, observe that $I \ni t \mapsto \|k(t,\cdot)\|_q \in \mathbb{R}$ is T-periodic (cf. [7]). From (2) it follows that it is also continuous and thus

$$\sup_{t\in I} \|k(t,\cdot)\|_q = \sup_{t\in I_T} \|k(t,\cdot)\|_q < \infty.$$

This proves condition (3) makes sense.

We can assign to each $1 \le p < \infty$ a set-valued Nemytskiĭ operator $N_F\colon P_T(I,\mathbb{R}^n) \multimap L^p(I,\mathbb{R}^n)$ by letting

$$N_F(y) = \{w \in L^p(I,\mathbb{R}^n)\colon w(s) \in F(s,y(s)) \text{ for a.a. } s \in I\}.$$

Next we define the integral operator $K\colon L^p(I,\mathbb{R}^n) \to P_T(I,\mathbb{R}^n)$ such that

$$K(w)(t) = h(t) + \int_I k(t,s)w(s)\,ds, \qquad t \in I.$$

In view of the continuity of h and $t \mapsto k(t,\cdot)$, Hölder's inequality implies that $K(w) \in C(I,\mathbb{R}^n)$. Obviously, $K(w)$ is also T-periodic, which proves the correctness of the definition of the operator K.

Now, the periodic solutions of the inclusion (1) are fixed points of the set-valued operator $G\colon P_T(I,\mathbb{R}^n) \multimap P_T(I,\mathbb{R}^n)$ defined by $G = K \circ N_F$. In fact, the inequality (3) and the property (F3) of the multimap F guarantee that $G\colon D(0,M) \multimap D(0,M)$, where $D(0,M)$ is the closed ball with radius M in the supremum norm of the space $P_T(I,\mathbb{R}^n)$. We will see that this operator is an admissible multimap in the sense of Theorem 1. To this end we prove that G is a compact multimap with nonempty convex values and closed graph.

G has nonempty values. Indeed, since F takes nonempty and compact values and satisfies (F1)–(F2), which means that F is upper Carathéodory, for every $y \in D(0,M)$ there exists a measurable selection w of $F(\cdot,y(\cdot))$ in view of Theorem 7, p. 124 in [1]. Then property (F3) implies that $w \in L^p(I,\mathbb{R}^n)$. Therefore $v = K(w) \in G(y)$.

$G(y)$ is convex for every y. This is true because the multimap F has convex values and the operator K is affine.

G is compact. Take any sequence $(v_n)_{n=1}^\infty$ in $G(D(0,M))$. Then $(v_n)_{n=1}^\infty$ is uniformly bounded on I and in particular on I_T. Let $v_n = K(w_n)$. From the inequality

$$|K(w_n)(t) - K(w_n)(\tau)| \le |h(t) - h(\tau)| + \|k(t,\cdot) - k(\tau,\cdot)\|_q \|\mu_M\|_p \tag{4}$$

for every $n \ge 1$, it follows that the sequence $(v_n)_{n=1}^\infty$ is also equi-continuous on I_T. Thus we can assume without loss of generality that $(v_n)_{n=1}^\infty$ converges uniformly on I_T to some v. Let $v(t+T)=v(t)$ for every t such that $t+T\in I$. Since

$$\|v_n-v\|_\infty=\sup_{t\in I}|v_n(t)-v(t)|=\sup_{t\in I_T}|v_n(t)-v(t)|\underset{n\to\infty}{\longrightarrow}0,$$

the sequence (v_n) tends to v in $P_T(I,\mathbb{R}^n)$ and $G(D(0,M))$ is relatively compact.

G has closed graph. To show this, take a sequence $(y_n,v_n)_{n=1}^\infty$ in the graph of G such that $\|v_n-v\|_\infty\to 0$ and $\|y_n-y\|_\infty\to 0$ as $n\to\infty$. Let $v_n=K(w_n)$, where $w_n\in L^p(I,\mathbb{R}^n)$ satisfies $w_n(s)\in F(s,y_n(s))$ a.e. in I. Assume $p=1$ and I is a finite interval. Then the set

$$\{f\in L^1(I,\mathbb{R}^n)\colon |f(t)|\le\mu_M(t)\text{ for a.a. }t\in I\}$$

is compact in the weak topology of $L^1(I,\mathbb{R}^n)$ (see [5] for the proof), which means that (w_n) has an L^1-weakly convergent subsequence. It is well known that $L^p(I,\mathbb{R}^n)$ is reflexive for $1<p<\infty$, whether or not I is bounded (i.e. I can be $\mathbb{R}$). Clearly the sequence $(w_n)_{n=1}^\infty$ is bounded in the L^p norm. So if $p\in(1,\infty)$ then in view of the Eberlein–Shmul'yan theorem (w_n) also has a convergent subsequence (again denoted by (w_n)). Let w be its limit. From Mazur's theorem it follows that w belongs to the strong closure of $\operatorname{co}\{w_n\colon n\ge l\}$ for every $l\ge 1$. Thus there is a sequence $(z_l)_{l=1}^\infty$ converging to w in the L^p norm such that $z_l\in\operatorname{co}\{w_n\colon n\ge l\}$ for every $l\ge 1$. Further, there is a subsequence (again denoted by) (z_l) converging to w a.e. in I. Let J be a set of full measure in the interval I satisfying:

$$z_l(s)\to w(s)\text{ as }l\to\infty\quad\text{for every }s\in J,\tag{5}$$

$$x\mapsto F(s,x)\text{ is u.s.c. for every }s\in J,\tag{6}$$

$$w_n(s)\in F(s,y_n(s))\quad\text{for every }s\in J,\ n\ge 1.\tag{7}$$

Take an arbitrary $t\in J$ and $\varepsilon>0$. From (6) we get $\delta>0$ such that $F(t,x)\subset B(F(t,y(t)),\varepsilon)$ for every $x\in B(y(t),\delta)$. Since $y_n(t)\to y(t)$ as $n\to\infty$, there is $N\in\mathbb{N}$ such that $w_n(t)\in B(F(t,y(t)),\varepsilon)$ for every $n\ge N$ (by (7)). Observe that $(z_l)_{l=N}^\infty\subset\operatorname{co}\{w_n\colon n\ge N\}$. By (5) it follows that $w(t)\in D(F(t,y(t)),\varepsilon)$. Since $\varepsilon>0$ and $t\in J$ was arbitrary we have $w(s)\in F(s,y(s))$ for almost all $s\in I$, i.e. $w\in N_F(y)$. Keeping in mind that $v_n=K(w_n)$ and $z_l\in\operatorname{co}\{w_n\colon n\ge l\}$ we find that $K(z_l)\in\operatorname{co}\{v_n\colon n\ge l\}$, since K is affine. Hence

$$\|K(z_l)-v\|_\infty\le\operatorname{diam}\operatorname{co}\overline{\{v_n\colon n\ge l\}}\tag{8}$$

for every $l\ge 1$. Notice that (v_n) is a Cauchy sequence. Therefore

$$\forall\varepsilon>0\ \exists l\in\mathbb{N}\ \forall n\ge l\quad v_n\in B(v_l,\varepsilon),$$

which implies

$$\forall\varepsilon>0\ \exists L\in\mathbb{N}\ \forall l\ge L\quad\operatorname{diam}\operatorname{co}\overline{\{v_n\colon n\ge l\}}\le 2\varepsilon.$$

Obviously $\operatorname{diam}\operatorname{co}\overline{\{v_n : n \geq l\}} \to 0$ as $l \to \infty$, which means that $K(z_l) \rightrightarrows v$ as $l \to \infty$, by (8). Applying again the Hölder inequality we see that

$$\begin{aligned}\|K(z_l) - K(w)\|_\infty &\leq \sup_{t\in I} \int_I |k(t,s)|\,|z_l(s) - w(s)|\,ds \\ &\leq \sup_{t\in I} \|k(t,\cdot)\|_q \|z_l - w\|_p.\end{aligned}$$

As (z_l) tends to w in the L^p norm we conclude $K(z_l) \rightrightarrows K(w)$. Thus $v = K(w)$ and $v \in G(y)$, i.e. the graph of G is closed.

According to Corollary 1 in [2, p. 42] the operator $G\colon D(0,M) \multimap D(0,M)$ is upper semicontinuous. Therefore it is an admissible multimap and from Theorem 1 it follows that G has a fixed point, i.e. there is y in $P_T(I,\mathbb{R}^n)$ with $\|y\|_\infty \leq M$ such that $y \in G(y)$. Hence the inclusion (1) has a continuous T-periodic solution. ∎

REMARK 3. If the multifunction F is L^p-integrably bounded, i.e. there exists $\mu \in L^p(I,\mathbb{R}_+)$ such that $|F(t,x)| \leq \mu(t)$ for all x and for almost all t, condition (3) is superfluous. In the proof of the theorem one can simply take $M = \|h\|_\infty + \sup_{t\in I} \|k(t,\cdot)\|_q \|\mu\|_p$.

Our second result establishes the existence of continuous almost periodic solutions of the integral inclusion (1).

THEOREM 4. *Let $p \in (1,\infty)$, let q be such that $p^{-1} + q^{-1} = 1$, and $I = \mathbb{R}$. Assume that $F\colon I \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an L^p-Carathéodory multimap with nonempty compact convex values and $h\colon I \to \mathbb{R}^n$ is continuous almost periodic. Let $k\colon I \times I \to \mathbb{R}$ be such that $k(t,\cdot) \in L^q(I)$ for every $t \in I$ and*

(9) *the map $I \ni t \mapsto k(t,\cdot) \in L^q(I)$ is continuous almost periodic.*

In addition suppose that for some $M > 0$,

$$\|h\|_\infty + \sup_{t\in I} \|k(t,\cdot)\|_q \|\mu_M\|_p \leq M. \tag{10}$$

Then the inclusion (1) *has a continuous almost periodic solution.*

Proof. Take $\varepsilon > 0$. Then condition (9) implies that there exists an $l(\varepsilon) > 0$ such that any interval of length $l(\varepsilon)$ contains a number τ such that

$$\begin{aligned}|\,\|k(t+\tau,\cdot)\|_q - \|k(t,\cdot)\|_q| &\leq \|k(t+\tau,\cdot) - k(t,\cdot)\|_q \\ &= \Big(\int_I |k(t+\tau,s) - k(t,s)|^q\,ds\Big)^{1/q} < \varepsilon\end{aligned}$$

for all $t \in \mathbb{R}$. Hence the function $t \mapsto \|k(t,\cdot)\|_q$ is continuous almost periodic. Since almost periodic functions are bounded (see [4]), $\sup_{t\in I} \|k(t,\cdot)\|_q < \infty$ and the inequality (10) is justified.

Let $G\colon AP(\mathbb{R}^n) \multimap AP(\mathbb{R}^n)$ be as in the proof of the previous theorem, i.e. $G = K \circ N_F$. We have to show that the operator K acts from $L^p(I,\mathbb{R}^n)$

to $AP(\mathbb{R}^n)$. Indeed, let $\varepsilon > 0$ and $w \in L^p(I, \mathbb{R}^n)$ with $\|w\|_p > 0$. Then since $t \mapsto k(t, \cdot)$ and h are almost periodic, there exists $l > 0$ such that any interval of length l contains a τ such that

$$\|k(t+\tau, \cdot) - k(t, \cdot)\|_q < \frac{\varepsilon}{2\|w\|_p} \quad \text{and} \quad |h(t+\tau) - h(t)| < \frac{\varepsilon}{2}$$

for all $t \in \mathbb{R}$. Thus any interval of length l contains an element τ such that for all $t \in \mathbb{R}$,

$$\begin{aligned} |K(w)(t+\tau) - K(w)(t)| &\le |h(t+\tau) - h(t)| + \|k(t+\tau, \cdot) - k(t, \cdot)\|_q \|w\|_p \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2\|w\|_p}\|w\|_p = \varepsilon. \end{aligned}$$

Therefore $K(w) \in AP(\mathbb{R}^n)$ for every $w \in L^p(I, \mathbb{R}^n)$.

Using arguments similar to those in the proof of Theorem 3 we can show that $G\colon D(0,M) \multimap D(0,M)$ has nonempty convex values and closed graph. The only difference is that we must use the compactness criteria (see Theorem 2) for the space $AP(\mathbb{R}^n)$ to prove that G is a compact multimap.

Condition (i) is easily verified since $G(D(0,M))$ is uniformly bounded on $\mathbb{R}$ by M. Theorem 6.2 in [4] says that an almost periodic function with values in a Banach space is uniformly continuous on $\mathbb{R}$. Thus condition (ii) for the family $\mathcal{F} = G(D(0,M))$ follows from the uniform continuity of h and $t \mapsto k(t, \cdot)$ and inequality (4). Finally, the inequality

$$|K(w)(t+\tau) - K(w)(t)| \le |h(t+\tau) - h(t)| + \|k(t+\tau, \cdot) - k(t, \cdot)\|_q \|\mu_M\|_p$$

for every $w \in N_F(y)$ and $y \in D(0,M)$, together with the almost periodicity of h and $t \mapsto k(t, \cdot)$, implies condition (iii). Therefore the set $G(D(0,M))$ is relatively compact.

Concluding, the operator $G\colon D(0,M) \multimap D(0,M)$ is an admissible multimap and in view of Theorem 1 has a fixed point. This fixed point is a continuous almost periodic solution of the inclusion (1). ∎

Example 1. (a) If for $T = 2\pi$ and I a finite interval of $\mathbb{R}$ that contains at least one compact subinterval of length 2π,

$$k(t,s) = \sin(t+s), \qquad t, s \in I,$$

then k satisfies the assumptions in Theorem 3.

(b) If for some $0 < T < \infty$ and I an interval of $\mathbb{R}$ that contains at least one compact interval of length T,

$$k(t,s) = a(t)b(s), \qquad t, s \in I,$$

where $a \in P_T(I)$ and $b \in L^q(I)$ then k satisfies the assumption of Theorem 3. If $I = \mathbb{R}$, $a \in AP(\mathbb{R})$ and $b \in L^q(I)$ then k satisfies the assumption of Theorem 4.

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