BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 58, No. 3, 2010

FUNCTIONAL ANALYSIS

Essentially Incomparable Banach Spaces of Continuous Functions

by

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Summary. We construct, under Axiom $\diamondsuit$, a family $(C(K_\xi))_{\xi<2^{(2^\omega)}}$ of indecomposable Banach spaces with few operators such that every operator from $C(K_\xi)$ into $C(K_\eta)$ is weakly compact, for all $\xi \neq \eta$. In particular, these spaces are pairwise essentially incomparable.

Assuming no additional set-theoretic axiom, we obtain this result with size 2^ω instead of $2^{(2^\omega)}$.

1. Introduction. An infinite-dimensional Banach space X is said to be *indecomposable* if there are no infinite-dimensional subspaces Y and Z of X such that $X = Y \oplus Z$. The first indecomposable Banach space was obtained by Gowers and Maurey, in [GM1]. In [Ko1], Koszmider constructed the first example of an indecomposable Banach space of the form $C(K)$ (the Banach space of continuous functions on a compact K, with the supremum norm).

Once we know that there exist indecomposable Banach spaces, we can ask how many of them there exist. In [Ga], Gasparis constructed a family of continuum many separable indecomposable Banach spaces which are pairwise totally incomparable. This means that no infinite-dimensional subspace of one space is isomorphic to a subspace of any other. The separability of the spaces implies that 2^ω is the largest possible cardinality of such a family.

The purpose of this paper is to study the analogous question for Banach spaces of continuous functions. Since any infinite-dimensional $C(K)$ contains a copy of c_0 as a subspace, two Banach spaces of continuous functions cannot be totally incomparable. Therefore we use a weaker notion of incomparabil-

2010 *Mathematics Subject Classification*: Primary 46E15; Secondary 46B03, 46B20.
Key words and phrases: Banach spaces, $C(K)$, indecomposable Banach spaces, few operators, essentially incomparable, diamond axiom.

DOI: 10.4064/ba58-3-7

ity, used by Aiena and González (see [AG]), called *essential incomparability*, as presented in Definition 1. On the other hand, since indecomposable Banach spaces of the form $C(K)$ built as in [Ko1] have density 2^ω (unlike Gowers and Maurey's space, which is separable) we may expect the existence of a family of size up to $2^{(2^\omega)}$ of non-isomorphic indecomposable Banach spaces of the form $C(K)$.

We say that a Banach space $C(K)$ *has few operators* if every operator on $C(K)$ has the form $gI + S$, where $g \in C(K)$, I is the identity operator and S is weakly compact. When K is connected, this implies that $C(K)$ is indecomposable, as shown in [Fa].

In this paper, we construct in ZFC a family $(K_\xi)_{\xi<2^\omega}$ of compact connected Hausdorff spaces such that $C(K_\xi)$ has few operators and every operator from $C(K_\xi)$ into $C(K_\eta)$ is weakly compact, for all $\xi \neq \eta$ in 2^ω. This implies that these spaces are essentially incomparable. Moreover, assuming $\diamondsuit$, a set-theoretic axiom which holds in Gödel's constructible universe and implies CH, we extend this result obtaining a family of size $2^{(2^\omega)}$. It remains open if the result holds in ZFC for some cardinal larger than continuum.

A similar result was obtained in Proposition 4.5 of [KMM]. But in that construction, each space K_ξ is totally disconnected, $C(K_\xi)$ has few operators in a slightly weaker sense (see [Ko1] and [Schl]), and moreover the family is countable.

The strategy of the proof is the following: for a cardinal κ we define a statement $A(\kappa)$ and we construct a family $\{K_\xi : \xi < \kappa\}$ of compact connected subspaces of $[0,1]^{2^\omega}$. During the construction, for each $\xi < \kappa$ we kill all operators on $C(K_\xi)$ which are not weak multipliers (see Definition 10), as in [Ko1]. Using $A(\kappa)$, for every $\xi \neq \eta$ in κ we kill all operators from $C(K_\xi)$ into $C(K_\eta)$ which are not weakly compact. Then we prove that $A(2^\omega)$ holds in ZFC and that $\diamondsuit$ implies $A(2^{(2^\omega)})$.

2. Essentially incomparable Banach spaces. We recall that an operator T on a Banach space X is said to be *Fredholm* if the dimension of its kernel and the codimension of its range are finite (see [DS]).

Definition 1. Two Banach spaces X and Y are to be *essentially incomparable* if for all operators $T : X \to Y$ and $S : Y \to X$ the operator $I - S \circ T : X \to X$ is a Fredholm operator.

Lemma 2. *Let X and Y be infinite-dimensional Banach spaces which are essentially incomparable. Then no infinite-dimensional complemented subspace of X is isomorphic to any complemented subspace of Y.*

Proof. Let X_1 and Y_1 be infinite-dimensional complemented subspaces of X and Y respectively, and suppose that X_1 and Y_1 are isomorphic. Let $P : X \to X$ and $Q : Y \to Y$ be projections whose ranges are X_1 and Y_1, respectively, and let ϕ be an isomorphism between X_1 and Y_1. Set $T = \phi \circ P$ and $S = \phi^{-1} \circ Q$. We have $\ker(I - S \circ T) = X_1$ and so $I - S \circ T$ is not Fredholm. ∎

LEMMA 3. *Let K and L be compact spaces such that every operator $T : C(K) \to C(L)$ is weakly compact. Then $C(K)$ and $C(L)$ are essentially incomparable.*

Proof. The composition of weakly compact operator with any bounded operator is weakly compact. By a result of [LT] a Fredholm operator plus a strictly singular operator is a Fredholm operator. By [Pe], weakly compact operators on $C(K)$ are strictly singular. Identity is clearly a Fredholm operator. So, for K and L as in the hypothesis, $I - S \circ T$ is a Fredholm operator on $C(K)$, for all $T : C(K) \to C(L)$ and $S : C(L) \to C(K)$. ∎

3. Strong extensions and few operators. In this section we cite some definitions and results of [Ko1] which will be used in this paper.

LEMMA 4 ([Ko1, 4.1]). *Let K be a compact space and let $(f_n)_{n\in\omega}$ be a pairwise disjoint sequence of functions from K into $[0,1]$ (i.e., $f_n \cdot f_m = 0$ for $n \neq m$). Then*

(a) *the set $D((f_n)_{n\in\omega}) = \{x \in K$: there is an open neighborhood U of x such that $U \cap \operatorname{supp}(f_n) = \emptyset$ for all but finitely many $n \in \omega\}$ is open and dense in K;*
(b) *$\sum_{n\in\omega} f_n$ is continuous in $D((f_n)_{n\in\omega})$.*

DEFINITION 5 ([Ko1, 4.2]). Let K be a compact space and $(f_n)_{n\in\omega}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$. We say that $L \subseteq K \times [0,1]$ is an *extension of K by $(f_n)_{n\in\omega}$* if L is the closure of the graph $\sum_{n\in\omega} f_n | D((f_n)_{n\in\omega})$. In that case we shall write $L = K((f_n)_{n\in\omega})$. Moreover we say that the extension is *strong* if it contains the graph of $\sum_{n\in\omega} f_n$.

LEMMA 6 ([Ko1, 4.3]). *Let K be a compact space and $(f_n)_{n\in\omega}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$. Let $L = K((f_n)_{n\in\omega})$. Denote by π the standard projection from L into K. Then*

(a) *if $M \subseteq K$ is nowhere dense in K, then $\pi^{-1}[M]$ is nowhere dense in L;*
(b) *$\sup\{f_n \circ \pi : n \in \omega\}$ exists in $C(L)$.*

LEMMA 7 ([Ko1, 4.4]). *Let K be a compact space and $(f_n)_{n\in\omega}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$. If K is connected, then the graph of $\sum_{n\in\omega} f_n$ is connected. In particular, a strong extension of a connected space is also connected.*

Lemma 8 ([Ko1, 4.5]). *Suppose that K is compact, and of topological weight $\kappa < 2^\omega$. Let X_1 and X_2 be two disjoint relatively discrete subsets of K such that $\overline{X}_1 \cap \overline{X}_2 \neq \emptyset$. Let $(f_n)_{n\in\omega}$ be a pairwise disjoint sequence of continuous functions from K into $[0,1]$ and $(N_\xi : \xi < 2^\omega)$ be a family of infinite subsets of ω such that $N_\xi \cap N_{\xi'}$ is finite for all $\xi \neq \xi'$. Then there exists $A \subseteq 2^\omega$ of cardinality not larger than κ such that, for all $\xi \in 2^\omega \setminus A$ and all infinite $b \subseteq N_\xi$, we have:*

(a) *$K((f_n)_{n\in b})$ is a strong extension;*
(b) *$\overline{\{(x, (\sum_{n\in b} f_n)(x)) : x \in X_1\}} \cap \overline{\{(x, (\sum_{n\in b} f_n)(x)) : x \in X_2\}} \neq \emptyset$ in $K((f_n)_{n\in b})$.*

We recall the definition of inverse limits. Let $\prod_{\alpha<\kappa} X_\alpha$ be a product of topological spaces, where κ is a limit ordinal. Let Y_α be subspaces of $\prod_{\beta<\alpha} X_\beta$ such that $\pi_\beta[Y_\alpha] = Y_\beta$ when $\beta < \alpha$. We define the inverse limit of $(Y_\alpha)_{\alpha<\kappa}$ by

$$\varprojlim (Y_\alpha)_{\alpha<\kappa} = \Big\{(y_\alpha)_{\alpha<\kappa} \in \prod_{\alpha<\kappa} X_\alpha : \forall \alpha < \kappa \ ((y_\beta)_{\beta<\alpha} \in Y_\alpha)\Big\}.$$

Inverse limits preserve compactness (see [Eng, 2.5.1]).

Lemma 9 ([Ko1, 4.6]). *Let β be an ordinal and let $(K_\alpha)_{\alpha\leq\beta}$ be such that $K_2 = [0,1]^2$, $K_\alpha \subseteq [0,1]^\alpha$ is compact, K_α is the inverse limit of $(K_\gamma)_{\gamma<\alpha}$ when α is limit, and $K_{\alpha+1}$ is a strong extension of K_α by pairwise disjoint functions from K_α into $[0,1]$. Then*

(a) *if $f, f_n \in C(K_\alpha)$ for $n \in \omega$ and $\alpha \leq \beta$ are such that*
$$f = \sup\{f_n : n \in \omega\},$$
then
$$f \circ \pi_{\beta,\alpha} = \sup\{f_n \circ \pi_{\beta,\alpha} : n \in \omega\};$$
(b) *$K_\beta \setminus F$ is connected if $F \subseteq K_\beta$ is finite.*

Now we will give the main definitions and results related to Banach spaces of continuous functions with few operators.

Definition 10 ([Ko1, 2.1]). An operator $T : C(K) \to C(K)$ is called a *weak multiplier* if for every bounded sequence $(e_n : n \in \omega)$ of pairwise disjoint elements of $C(K)$ and any sequence $(x_n : n \in \omega) \subseteq K$ such that $e_n(x_n) = 0$ we have
$$\lim_{n\to\infty} T(e_n)(x_n) = 0.$$

Theorem 11 ([Ko1, 2.5]). *Suppose that all operators on $C(K)$ are weak multipliers and $K \setminus F$ is connected for all finite $F \subseteq K$. Then $C(K)$ is an indecomposable Banach space.*

Let us recall that $Y \subseteq X$ is C^*-*embedded* in X if every bounded continuous function on Y extends to a bounded continuous function on X.

Lemma 12 ([Ko1, 2.8]). *Suppose that K is a compact space with no disjoint open sets U_1 and U_2 such that $\overline{U}_1 \cap \overline{U}_2$ is a singleton. Then for every $x \in K$ the space $K \setminus \{x\}$ is C^*-embedded in K.*

Theorem 13 ([Ko1, 2.7]). *The following are equivalent for a compact space K:*

(a) *All operators $T : C(K) \to C(K)$ are of the form $gI + S$ where $g \in C(K)$ and S is weakly compact.*
(b) *All operators on $C(K)$ are weak multipliers and for every $x \in K$ the space $K \setminus \{x\}$ is C^*-embedded in K.*

4. Axiom $\diamondsuit$. We recall the definition of Axiom $\diamondsuit$. We know that Axiom $\diamondsuit$ holds in Gödel's Constructible Universe, and therefore it is relatively consistent with ZFC. It is easy to verify that $\diamondsuit$ implies CH. See [Ku] and [Ve] for references.

Definition 14. We say that a subset S of ω_1 is *stationary* if it intersects every closed unbounded subset of ω_1.

In particular, stationary subsets of ω_1 are unbounded, since $\{\alpha < \omega_1 : \alpha > \beta\}$ is closed unbounded for each $\beta < \omega_1$.

Axiom $\diamondsuit$. There exists a sequence $(A_\alpha : \alpha < \omega_1)$ such that $A_\alpha \subseteq \alpha$ and for every $A \subseteq \omega_1$ the set $\{\alpha < \omega_1 : A \cap \alpha = A_\alpha\}$ is stationary.

The sequence $(A_\alpha : \alpha < \omega_1)$ as in Axiom $\diamondsuit$ is called a $\diamondsuit$-*sequence.*

We present a slight variation of Axiom $\diamondsuit$ which we will use in this paper. We use 2^α to denote the set of functions from α into $2 = \{0, 1\}$.

Lemma 15 ($\diamondsuit$). *Let X be a set of size ω_1. There exists a sequence $(t_\alpha, s_\alpha, x_\alpha)_{\alpha<\omega_1}$ such that $t_\alpha, s_\alpha \in 2^\alpha$, $x_\alpha \in X$ and for every $t, s \in 2^{\omega_1}$ and $x \in X$ the set $\{\alpha < \omega_1 : t|\alpha = t_\alpha,\ s|\alpha = s_\alpha$ and $x_\alpha = x\}$ is stationary.*

5. A large family of pairwise essentially incomparable indecomposable Banach spaces. We reduce the problem of constructing essentially incomparable Banach spaces to a purely combinatorial problem.

Statement $A(\kappa)$**.** Let κ be a cardinal such that $2^\omega \le \kappa \le 2^{(2^\omega)}$. We define *statement* $A(\kappa)$ as follows:

If X is a set of size 2^ω, then there exist functions $\varphi_1, \varphi_2 : 2^\omega \to \mathcal{P}(\kappa) \setminus \{\emptyset\}$ and $\psi : 2^\omega \to X$ such that:

1. For all $\beta \le \alpha < 2^\omega$ and $i, j \in \{1, 2\}$, we have either $\varphi_j(\alpha) \subseteq \varphi_i(\beta)$ or $\varphi_j(\alpha) \cap \varphi_i(\beta) = \emptyset$;

2. For all $\beta < 2^\omega$, $x \in X$ and $\xi, \eta < \kappa$ with $\xi \neq \eta$, there exists $\alpha > \beta$ such that $\psi(\alpha) = x$, $(\xi, \eta) \in \varphi_1(\alpha) \times \varphi_2(\alpha)$ and $\varphi_1(\alpha) \neq \varphi_2(\alpha)$.
3. For all $\beta < 2^\omega$, $x \in X$ and $\xi < \kappa$ there exists $\alpha > \beta$ such that $\psi(\alpha) = x$ and $\xi \in \varphi_1(\alpha) \cap \varphi_2(\alpha)$.

In the next few lines we explain how statement $A(\kappa)$ will be used to build a family of κ pairwise essentially incomparable indecomposable Banach spaces.

We will build a family $(K_\xi)_{\xi<\kappa}$ of compact spaces where each K_ξ is the inverse limit of $(K_{(\alpha,\xi)})_{\alpha<2^\omega}$. We will proceed by induction on $\alpha < 2^\omega$ to construct $K_{(\alpha,\xi)}$.

The elements of $\varphi_1(\alpha)$ indicate the compact spaces where we will add a supremum at stage α of the construction, and elements of $\varphi_2(\alpha)$ indicate the compact spaces where we will fix some coordinates that will be called "promises". These promises index some pairs of sets whose closures cannot be separated at further stages of the inductive construction. If $\xi \in \varphi_1(\alpha)$ and $\eta \in \varphi_2(\alpha)$, we kill an operator from $C(K_\xi)$ into $C(K_\eta)$ by adding a supremum of continuous functions in $C(K_\xi)$ and promises in K_η, at stage α. The function ψ will be used to list the sequence of functions whose supremum we want to add, at each stage.

From condition 1 we know that $\varphi_1(\alpha)$ and $\varphi_2(\alpha)$ are equal or disjoint. In the first case, we kill a non-weak multiplier on $C(K_\xi)$, for $\xi \in \varphi_1(\alpha)$, in a way very similar to the construction of [Ko1]. In the second case we kill a non-weakly compact operator from $C(K_\xi)$ into $C(K_\eta)$ for $\xi \in \varphi_1(\alpha)$ and $\eta \in \varphi_2(\alpha)$.

Condition 2 guarantees that we will kill all non-weakly compact operators from any $C(K_\xi)$ into any $C(K_\eta)$, and condition 3 guarantees that we kill all non-weak multipliers on any $C(K_\xi)$.

We could replace X by 2^ω in statement $A(\kappa)$. But we keep that notation because it is more convenient for further applications.

Lemma 16. *$A(2^\omega)$ holds in ZFC.*

Proof. Let $(\xi_\alpha, \eta_\alpha, x_\alpha)_{\alpha<2^\omega}$ be a sequence in $(2^\omega)^2 \times X$ such that, for all $\xi, \eta \in 2^\omega$, $\beta < 2^\omega$ and $x \in X$ there exists $\alpha > \beta$ such that $\xi_\alpha = \xi$, $\eta_\alpha = \eta$ and $x_\alpha = x$. Define $\varphi_1(\alpha) = \{\xi_\alpha\}$, $\varphi_2(\alpha) = \{\eta_\alpha\}$ and $\psi(\alpha) = x_\alpha$. ∎

Lemma 17. *$\diamondsuit$ implies $A(2^{(2^\omega)})$.*

Proof. We remark that $\diamondsuit$ implies CH, i.e., $\omega_1 = 2^\omega$.

Let X be a set of size $2^\omega = \omega_1$. Assuming $\diamondsuit$, by Lemma 15 there exists a sequence $(t_\alpha, s_\alpha, x_\alpha)_{\alpha<\omega_1}$ such that $t_\alpha, s_\alpha \in 2^\alpha$, $x_\alpha \in X$ and for all $t, s \in 2^{\omega_1}$ and all $x \in X$, the set

$$\{\alpha < \omega_1 : t|\alpha = t_\alpha,\, s|\alpha = s_\alpha,\, x_\alpha = x\}$$

is stationary.

Now, for $\alpha < \omega_1 = 2^\omega$, we define $\varphi_1(\alpha) = \{t \in 2^{\omega_1} : t|\alpha = t_\alpha\}$, $\varphi_2(\alpha) = \{s \in 2^{\omega_1} : s|\alpha = s_\alpha\}$ and $\psi(\alpha) = x_\alpha$. Conditions 2 and 3, as stationary sets in ω_1, are unbounded. To check condition 1 we note that if $\beta \le \alpha$ and $t_\alpha|\beta = t_\beta$, then $\varphi_1(\alpha) \subseteq \varphi_1(\beta)$, and if $t_\alpha|\beta \ne t_\beta$, then $\varphi_1(\alpha) \cap \varphi_1(\beta) = \emptyset$. The same argument can be repeated for other combinations of φ_1 and φ_2. ∎

Theorem 18. *$A(\kappa)$ implies that there exists a family $\{K_\xi : \xi < \kappa\}$ of compact connected spaces such that $C(K_\xi)$ is indecomposable for all $\xi < \kappa$, and every operator from $C(K_\xi)$ into $C(K_\eta)$ is weakly compact, for any different $\xi, \eta < \kappa$.*

Proof. Let φ_1, φ_2 and ψ be as in statement $A(\kappa)$, where X is the set of all triples $((f_n)_{n\in\omega}, (g_n)_{n\in\omega}, (l_n)_{n\in\omega})$ such that

- $((f_n))_{n\in\omega}$ is a pairwise disjoint sequence of continuous functions from $[0,1]^{2^\omega}$ into $[0,1]$;
- $((g_n))_{n\in\omega}$ is a pairwise disjoint sequence of continuous functions from $[0,1]^{2^\omega}$ into $\mathbb{R}$;
- $(l_n)_{n\in\omega}$ is a strictly increasing sequence of integers.

We denote $\psi(\alpha)$ by $((f_n(\alpha))_{n\in\omega}, (g_n(\alpha))_{n\in\omega}, (l_n(\alpha))_{n\in\omega})$.

We construct, by induction on α, connected compact spaces $K_{(\alpha,\xi)} \subseteq [0,1]^\alpha$ (for $\alpha > \omega$), sets $\{q_n(\alpha,\xi) : n \in \omega\}$ which are dense in $K_{(\alpha,\xi)}$, sets $b_\alpha \subseteq a_\alpha \subseteq \omega$ and sets $P(\alpha,\xi)$ of pairs of disjoint subsets of ω (which we call "promises").

In the process of induction we will have $|P(\alpha,\xi)| \le \alpha$ for all $\alpha < 2^\omega$ and $\xi < \kappa$. We assume by induction that whenever $\xi,\eta \in \varphi_1(\alpha)$ or $\xi,\eta \in \varphi_2(\alpha)$, we have $K_{(\alpha,\xi)} = K_{(\alpha,\eta)}$, $q_n(\alpha,\xi) = q_n(\alpha,\eta)$ and $P(\alpha,\xi) = P(\alpha,\eta)$. Therefore, for all $\xi \in \varphi_1(\alpha)$ we denote $K_{(\alpha,\xi)}$, $q_n(\alpha,\xi)$ and $P(\alpha,\xi)$ by K_α, $q_n(\alpha)$ and $P(\alpha)$ respectively, and for $\xi \in \varphi_2(\alpha)$ we denote $K_{(\alpha,\xi)}$, $q_n(\alpha,\xi)$ and $P(\alpha,\xi)$ by K'_α, $q'_n(\alpha)$ and $P'(\alpha)$ respectively.

Define $K_{(0,\xi)} = [0,1]^2$ and $P(0,\xi) = \emptyset$. Let $\{q_n(0,\xi) : n \in \omega\}$ be an enumeration of the pairs of rationals in $[0,1]^2$. If α is a limit ordinal, define $K_{(\alpha,\xi)}$ to be the inverse limit of $K_{(\beta,\xi)}$ for $\beta < \alpha$, $q_n(\alpha,\xi) = \bigcup\{q_n(\beta,\xi) : \beta < \alpha\}$ for all $n \in \omega$, and $P(\alpha,\xi) = \bigcup\{P(\beta,\xi) : \beta < \alpha\}$.

For $\xi \notin \varphi_1(\alpha)$, define $K_{(\alpha+1,\xi)} = K_{(\alpha,\xi)} \times \{0\}$ and $q_n(\alpha+1,\xi) = (q_n(\alpha,\xi), 0)$. For $\eta \notin \varphi_2(\alpha)$ define $P(\alpha+1,\eta) = P(\alpha,\eta)$.

It remains to define a_α, b_α, $K_{(\alpha+1,\xi)}$, $q_n(\alpha+1,\xi)$ and $P(\alpha+1,\eta)$ for $\xi \in \varphi_1(\alpha)$ and $\eta \in \varphi_2(\alpha)$. We say that α is a *non-trivial step* if:

1. There exist continuous functions $f'_n : K_\alpha \to [0,1]$ such that $f_n(\alpha)|_{\pi^{-1}[K_\alpha]} = f'_n \circ \pi$.

2. $\{q'_{l_n(\alpha)}(\alpha) : n \in \omega\}$ is relatively discrete in K'_α.
3. If $\varphi_1(\alpha) = \varphi_2(\alpha)$ then $f'_n(q_{l_m(\alpha)}(\alpha)) = 0$ for all $n, m \in \omega$.
4. If $\varphi_1(\alpha) \neq \varphi_2(\alpha)$ then there exist continuous functions $g'_n : K'_\alpha \to [0,1]$ such that $g_n(\alpha)|_{\pi^{-1}[K_\alpha]} = g'_n \circ \pi$.
5. If $\varphi_1(\alpha) \neq \varphi_2(\alpha)$ then there exists $\varepsilon > 0$ such that $g'_n(q'_{l_m(\alpha)}(\alpha)) > \varepsilon$.

Otherwise, we say that α is a *trivial step*. Suppose that α is a trivial step, $\xi \in \varphi_1(\alpha)$ and $\eta \in \varphi_2(\alpha)$. Then we define $a_\alpha = b_\alpha = \emptyset$, $K(\alpha+1,\xi) = K(\alpha,\xi) \times \{0\}$, $q_n(\alpha+1,\xi) = (q_n(\alpha,\xi),0)$ and $P(\alpha+1,\eta) = P(\alpha,\eta)$.

Let α be a non-trivial step. Using Lemma 8 we find infinite $b \subseteq a \subseteq \omega$ such that:

1. The extension of K_α by $(f'_n)_{n\in b}$ is strong.
2. $\overline{\{(q_n(\alpha),t_n) : n \in L\}} \cap \overline{\{(q_n(\alpha),t_n) : n \in R\}} \neq \emptyset$ in $K_\alpha((f'_n)_{n\in b})$, for all $(L,R) \in P(\alpha)$ and $t_n = \sum_{m\in b} f'_m(q_n(\alpha))$.
3. $\overline{\{(q'_{l_n(\alpha)}(\alpha),t_n) : n \in L\}} \cap \overline{\{(q'_{l_n(\alpha)}(\alpha),t_n) : n \in R\}} \neq \emptyset$ in K'_α.

Set $a_\alpha = a$ and $b_\alpha = b$. Finally, for $\xi \in \varphi_1(\alpha)$ and $\eta \in \varphi_2(\alpha)$, define $K(\alpha+1,\xi) = K_\alpha((f'_n)_{n\in b})$, $q_n(\alpha+1,\xi) = (q_n(\alpha,\xi),t_n)$ and $P(\alpha+1,\eta) = P(\alpha,\eta) \cup \{(\{l_n(\alpha) : n \in b_\alpha\}, \{l_n(\alpha) : n \in a_\alpha \setminus b_\alpha\})\}$.

By property 1 of ϕ_1 and ϕ_2 we know that $\phi_1(\alpha)$ and $\phi_2(\alpha)$ are either equal or disjoint. Note that the construction at stage α is the same for ξ and η when for each i either both belong to $\phi_i(\alpha)$, or neither does. Hence, the inductive assumption that $K_{(\alpha,\xi)} = K_{(\alpha,\eta)}$, $q_n(\alpha,\xi) = q_n(\alpha,\eta)$ and $P(\alpha,\xi) = P(\alpha,\eta)$, for $\xi,\eta \in \phi_1(\alpha)$ or $\xi,\eta \in \phi_2(\alpha)$, is preserved at the next stage.

For each $\xi < \kappa$ define K_ξ to be the inverse limit of $K(\alpha,\xi)$ for $\alpha < 2^\omega$.

Claim 1. *For all $\xi,\eta \in \kappa$ with $\xi \neq \eta$, every operator from $C(K_\xi)$ into $C(K_\eta)$ is weakly compact.*

The proof is similar to that of Lemma 5.2 of [Ko1]. We will omit some details which are exactly as in [Ko1].

Suppose that for some $\xi \neq \eta$ there exists $T : C(K_\xi) \to C(K_\eta)$ which is not weakly compact. Then there exist a bounded sequence $(f_n)_{n\in\omega}$ in $C(K)$ and $\varepsilon > 0$ such that $\|T(f_n)\| > \varepsilon$ for all n (see [DU, VI, Cor. 17]). Taking multiples of $\max(f_n,0)$ and $-\min(f_n,0)$ we may assume without loss of generality that f_n has its range included in $[0,1]$.

For each n we may choose $x_n \in K_\eta$ such that $|T(f_n)(x_n)| > \varepsilon$. Since $\{q^n_\eta : n \in \omega\}$ is dense in K_η we may assume that $x_n = q^{l_n}_\xi$ for some integer l_n. Note that l_n cannot be constant for infinitely many n's, because this would contradict the boundedness of T. Thus, passing to a subsequence, we may assume that $(l_n : n \in \omega)$ is a strictly increasing sequence and $\{q^{l_n}_\xi : n \in \omega\}$ is relatively discrete.

Let $\mu_n = T^*(\delta_{x_n})$. We have

$$|T(f_n)(x_n)| = \left|\int f_n \, d\mu_n\right| > \varepsilon.$$

Applying a lemma of Rosenthal (see [Di, p. 82]), we find an infinite $N' \subseteq \omega$ such that

$$\sum \left\{ \left|\int f_m \, d\mu_n\right| : n \neq m,\, m \in N' \right\} < \frac{\varepsilon}{3}.$$

As is shown in [Ko1, Lemma 5.2], there exists $N'' \subseteq N'$ such that, for all $b \subseteq N''$ and for all $n \in N''$,

$$\int \sup\{f_m : m \in b\} \, d\mu_n = \int \sum_{m \in b} f_m \, d\mu_n,$$

whenever the supremum exists in $C(K)$.

Passing again to a subsequence, we assume that $N'' = \omega$.

Let $g_n = T(f_n)$ for $n \in \omega$. By a theorem of Mibu [Mi], applied to extensions of f_n and g_n to the entire $[0,1]^{2^\omega}$, there exist $\alpha' < 2^\omega$ and functions $f'_n : K_{\alpha'} \to [0,1]$ and $g'_n : K_{\alpha'} \to \mathbb{R}$ such that $f_n = f'_n \circ \pi$ and $g_n = g'_n \circ \pi$. The existence of such functions still holds for any $\beta > \alpha'$, since we can take $f'_n \circ \pi_{K_\beta, K_{\alpha'}}$ instead of f'_n.

Let $\tilde{f}_n : [0,1]^{2^\omega} \to [0,1]$ and $\tilde{g}_n : [0,1]^{2^\omega} \to [0,1]$ be continuous extensions of f_n and g_n, respectively. By condition 2 of $A(\kappa)$ we can take $\alpha > \alpha'$ such that $\varphi_1(\alpha) \neq \varphi_2(\alpha)$, $(\xi, \eta) \in \varphi_1(\alpha) \times \varphi_2(\alpha)$, $f_n(\alpha) = \tilde{f}_n$, $g_n(\alpha) = \tilde{g}_n$ and $l_n(\alpha) = l_n$. Clearly α is a non-trivial step.

By construction and Lemma 6, $\{f_n : n \in b_\alpha\}$ has supremum in $C(K_\xi)$ and

$$\overline{\{q_\eta^{l_n} : n \in L\}} \cap \overline{\{q_\eta^{l_n} : n \in R\}} \neq \emptyset$$

for all $(L, R) \in P(\alpha, \eta)$, with the closures taken in K_η.

Take $f = \sup\{f_n : n \in b_\alpha\}$. Repeating the arguments of Lemma 5.2 of [Ko1] we conclude that

$$|T(f)(q_\eta^{l_n})| \geq 2\varepsilon/3$$

if $n \in b_\alpha$, and

$$|T(f)(q_\eta^{l_n})| \leq \varepsilon/3$$

if $n \in a_\alpha \setminus b_\alpha$, contradicting that $T(f)$ is continuous and $\overline{\{q_\eta^{l_n} : n \in b_\alpha\}} \cap \overline{\{q_\eta^{l_n} : n \in a_\alpha \setminus b_\alpha\}} \neq \emptyset$.

Claim 2. *For all $\xi \in \kappa$, $C(K_\xi)$ is indecomposable.*

The proof of this claim is analogous to the proof of Claim 1 and exactly like Lemma 5.2 of [Ko1]. We show that all opearators on $C(K_\xi)$ are weak multipliers. From Lemma 9 and Theorem 11 it follows that $C(K_\xi)$ is indecomposable. ∎

6. Final remarks. This paper provides the analogue of the result of Gasparis [Ga] for Banach $C(K)$ spaces. While Gasparis constructed totally incomparable Banach spaces, we constructed essentially incomparable Banach spaces of the form $C(K)$, which seems to be the natural adaptation of the problem, since $C(K)$ spaces necessarily have c_0 as a subspace, and thus they are never totally incomparable. While the spaces constructed by Gasparis are hereditarily indecomposable, we work with indecomposable Banach spaces, since c_0 is a decomposable subspace of $C(K)$. Since the first example of a hereditarily indecomposable Banach space, due to [GM1], is separable, Gasparis got the largest possible family of non-isomorphic separable hereditarily indecomposable Banach spaces. On the other hand, the first indecomposable $C(K)$, due to [Ko1], has density 2^ω, and so $2^{(2^\omega)}$ is the maximum possible cardinality for a family of non-isomorphic Banach spaces of that kind.

However, Gasparis obtained his construction entirely in ZFC, while this paper provides only a consistent result about the existence of $2^{(2^\omega)}$ essentially incomparable indecomposable Banach $C(K)$ spaces. Therefore the main question which arises from this paper is the following:

Problem 1. *Can it be proved in ZFC that there exist $2^{(2^\omega)}$ essentially incomparable Banach $C(K)$ spaces with few operators (or indecomposable)?*

A weaker version of this problem is also important.

Problem 2. *Can it be proved in ZFC that there exist more than continuum many essentially incomparable Banach $C(K)$ spaces with few operators* (*or indecomposable*)*?*

We may ask similar questions for the combinatorial statement $A(\kappa)$. See [Ku, Chapter II] for references about Martin's Axiom (MA).

Problem 3. *Does $A(2^{(2^\omega)})$ hold in ZFC? Does MA+¬CH imply the negation of $A(2^{(2^\omega)})$? Is it consistent that $A(\kappa)$ holds for no $\kappa > 2^\omega$?*

In [Ko2] it was proved by forcing that there exists consistently a Banach $C(K)$ space of density larger than continuum which has few operators. It is still open if such a space can be produced in ZFC. A notable open problem in this field is whether there exists (consistently or in ZFC) an indecomposable Banach space of density larger than continuum (the problem is open for $C(K)$ spaces as well as general Banach spaces). For hereditarily indecomposable Banach spaces it is proved in [PY] that they cannot have density larger than continuum.

These observations lead us to our next question.

Problem 4. *Is it consistent (or, can it be proved in ZFC) that there exist more than $2^{(2^\omega)}$ essentially incomparable Banach $C(K)$ spaces with few operators (or indecomposable)?*

Acknowledgements. This research was supported by a grant from FAPESP (Fundação de Amparo à Pesquisa do Estado de São Paulo), process number 07/54661-7. The author thanks Professor Ricardo Bianconi for his assistance and supervision.

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Received May 24, 2010;
received in final form October 30, 2010 (7761)