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Addendum to "On Meager Additive and Null Additive Sets in the Cantor space 2^ω and in $\mathbb{R}$"

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by

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Summary. We prove in ZFC that there is a set $A \subseteq 2^\omega$ and a surjective function $H : A \to \langle 0,1\rangle$ such that for every null additive set $X \subseteq \langle 0,1)$, $H^{-1}(X)$ is null additive in 2^ω. This settles in the affirmative a question of T. Bartoszyński.

1. Introduction. Recall that by $(2^\omega, \oplus)$ we denote the Cantor space with modulo 2 coordinatewise addition, and $(\langle 0,1), +_1)$ is the unit interval with modulo 1 addition. For brevity, 2^ω (respectively, $\langle 0,1)$) stands for $(2^\omega, \oplus)$ (respectively, $(\langle 0,1), +_1)$).

We shall say that $X \subseteq 2^\omega$ is *null additive* if for every null set A, $X \oplus A = \{x \oplus a : x \in X,\ a \in A\}$ is null in 2^ω. By analogy, we define a null additive set in $\langle 0,1)$. In [4], it has been proven that if X is a null additive set in 2^ω, then $T(X)$ is null additive in $\langle 0,1\rangle$, where T is the Cantor–Lebesgue function that maps 2^ω into $\langle 0,1\rangle$. Thus the existence of an uncountable null additive set in 2^ω implies that there is an uncountable null additive set in $\mathbb{R}$. In this paper, we prove the converse implication which provides a complete answer to the measure version of T. Bartoszyński's question (see [4, p. 91]). To do this we show that there exists a set $A \subseteq 2^\omega$ and a surjective function $H : A \to \langle 0,1\rangle$ such that for every null additive set $X \subseteq \langle 0,1)$, $H^{-1}(X)$ is null additive in 2^ω.

2. Main theorems. In this paper, for $n \in \omega$, p_n denotes the nth prime number, and $Z_{p_n} = \{0,\dots,p_n - 1\}$ with modulo p_n addition. We define

$$C = Z_{p_0} \times Z_{p_1} \times \cdots$$

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and we assume that $\boxplus$ is coordinatewise addition in C or in any set of the form $Z_{p_r} \times \cdots \times Z_{p_s}$, where $r, s \in \omega$, and $r < s$. Let $f : C \to \langle 0, 1\rangle$ be the Cantor–Lebesgue function given by the formula

$$f(x) = \sum_{i=0}^{\infty} \frac{x(i)}{\prod_{j=0}^{i} p_j} \quad \text{for } x \in C,$$

where $x(i) \in \{0, \dots, p_i - 1\}$ for $i \in \omega$. It is not difficult to check that f is one-to-one except on a countable subset of C. Throughout the paper, x is often identified with $f(x)$.

Suppose that $X \subseteq \langle 0, 1)$ is a null additive set.

THEOREM 1. *Given a sufficiently fast increasing sequence $\{a_n\}_{n\in\omega}$ of positive integers, there is $\{\widetilde{K}_n\}_{n\in\omega}$, with $\widetilde{K}_n \subseteq Z_{p_{a_{2n}+1}} \times \cdots \times Z_{p_{a_{2n+1}}}$ and $|\widetilde{K}_n| \le 2^n$ for all $n \in \omega$, such that for every $\overline{x} \in X$,*

$$f^{-1}(\overline{x})\restriction Z_{p_{a_{2n}+1}} \times \cdots \times Z_{p_{a_{2n+1}}} \in \widetilde{K}_n$$

for almost every $n \in \omega$.

Proof. We will follow the notation, and we refine the proofs, of Theorem 2.7.18 in [1], and Lemma 0 and Claim ♠ in [2].

LEMMA 2. *For any non-negative integers k, l, m, with $k < l$, there is $n \in \omega$ and $T \subseteq Z_{p_k} \times \cdots \times Z_{p_l} \times \cdots \times Z_{p_n}$ such that $\mu(T) \sim 2^{-m}$, and for any $\langle \sigma_i, \tau_i\rangle \in Z_{p_k} \times \cdots \times Z_{p_n}$ $(i \in I)$, where σ_i $(i \in I)$ belong to $Z_{p_k} \times \cdots \times Z_{p_l}$ and are distinct, the sets $T \boxplus \langle \sigma_i, \tau_i\rangle$ $(i \in I)$ are stochastically independent.*

Proof. Assume that $\overline{m} = p_k \cdot \ldots \cdot p_l$. In $\{p_{l+1}, \dots, p_n\}$, where n is sufficiently large, find a family $\{A_j\}_{j<\overline{m}}$ of $\overline{m}$ disjoint sets, each of cardinality m. Fix $j < \overline{m}$, and for each $p_r \in A_j$, let $B_r \subseteq Z_{p_r}$ be such that $|B_r|/p_r \sim 1/2$. Put

$$T_j = \Big\{x \in Z_{p_{l+1}} \times \cdots \times Z_{p_n} : x\restriction A_j \in \prod_{p_r \in A_j} B_r\Big\}.$$

Define $T = \bigcup_{j<\overline{m}} \{\sigma_i\} \times T_j$, where $\{\sigma_j\}_{j<\overline{m}}$ is a bijective enumeration of $Z_{p_k} \times \cdots \times Z_{p_l}$, and then follow the proof of Lemma 0 in [2] to show that T is as required. ■

REMARK 3. Notice that for every $m \in \omega$, $m \ge 4$,

$$\left(\frac{1}{2}\right)^m \le \mu(T) \le \left(\frac{1}{2} + \frac{1}{m}\right)^m \le \left(\frac{3}{4}\right)^m.$$

LEMMA 4. *For any $r, s \in \omega$ with $r < s$, $Z_{p_r} \times \cdots \times Z_{p_s}$ is isomorphic to $Z_{p_r \cdot \ldots \cdot p_s}$.*

Proof. Put $q_i = \frac{p_r \cdot \ldots \cdot p_s}{p_i}$ for $r \le i \le s$, and define, for $(a_r, \ldots, a_s) \in Z_r \times \cdots \times Z_s$,

$$i_{r,s}(a_r, \ldots, a_s) = q_r \cdot a_r + q_{r+1} \cdot a_{r+1} + \cdots + q_s \cdot a_s \pmod{p_r \cdot \ldots \cdot p_s}.$$

It is well-known that $i_{r,s}$ is an isomorphism. ∎

Clearly.

$$i_{r,s}(a, b) = i_{r,r'}(a) + i_{r'+1,s}(b) \pmod{p_r \cdot \ldots \cdot p_s}$$

whenever $r < r' < r'+1 < s$ and $a \in Z_{p_r} \times \cdots \times Z_{p_{r'}}$, $b \in Z_{p_{r'+1}} \times \cdots \times Z_{p_s}$. Here $i_{r,r'}(a)$ is an element of the subgroup of $Z_{p_r \cdot \ldots \cdot p_s}$ that has order $p_r \cdot \ldots \cdot p_{r'}$, and $i_{r'+1,s}(b)$ belongs to the subgroup of $Z_{p_r \cdot \ldots \cdot p_s}$ of order $p_{r'+1} \cdot \ldots \cdot p_s$. Suppose that $\overline{x} \in Z_{p_0 \cdot \ldots \cdot p_n}$. From now on, depending on the context, we identify $\overline{x}$ with $\overline{x}/(p_0 \cdot \ldots \cdot p_n)$. Thus for every l with $0 < l < n$, $\overline{x}$ has the following (unique) form:

$$\overline{x} = \sum_{i=0}^{l} \frac{x(i)}{\prod_{j=0}^{i} p_j} + \sum_{i=l+1}^{n} \frac{x(i)}{\prod_{j=0}^{i} p_j}.$$

Let $\overline{x}\restriction[0, l]$ denote the first sum, and $\overline{x}\restriction[l+1, n]$ the second.

Lemma 5. *Let $x, y \in Z_{p_0} \times \cdots \times Z_{p_k} \times \cdots \times Z_{p_l} \times \cdots \times Z_{p_n}$, and suppose that*

$$x\restriction Z_{p_k} \times \cdots \times Z_{p_l} = y\restriction Z_{p_k} \times \cdots \times Z_{p_l}.$$

If $i_{0,n}(x)\restriction[l+1, n]$ and $i_{0,n}(y)\restriction[l+1, n]$ belong to $Z_{p_{l+1} \cdot \ldots \cdot p_n}$, or more precisely, to the subgroup of $Z_{p_0} \times \cdots \times Z_{p_n}$ that has order $p_{l+1} \cdot \ldots \cdot p_n$, then

$$i_{0,n}(x)\restriction[k, l] = i_{0,n}(y)\restriction[k, l].$$

Proof. Assume that $i_{0,n}(x)\restriction[l+1, n] \in Z_{p_{l+1} \cdot \ldots \cdot p_n}$. Since $i_{0,n}$ is one-to-one, we have $i_{0,n}(x)\restriction[0, l] = i_{0,l}(x\restriction Z_{p_0} \times \cdots \times Z_{p_l})$. By the same argument, $i_{0,n}(y)\restriction[0, l] = i_{0,l}(y\restriction Z_{p_0} \times \cdots \times Z_{p_l})$. By the equality $x\restriction Z_{p_k} \times \cdots \times Z_{p_l} = y\restriction Z_{p_k} \times \cdots \times Z_{p_l}$, we have

$$i_{0,l}(x\restriction Z_{p_0} \times \cdots \times Z_{p_l})\restriction[k, l] = i_{0,l}(y\restriction Z_{p_0} \times \cdots \times Z_{p_l})\restriction[k, l].$$

Thus $i_{0,n}(x)\restriction[k, l] = i_{0,n}(y)\restriction[k, l]$. ∎

Corollary 6. *Let $x, y \in Z_{p_0} \times \cdots \times Z_{p_k} \times \cdots \times Z_{p_l} \times \cdots \times Z_{p_n}$. If $i_{0,n}(x)\restriction[l+1, n]$, $i_{0,n}(y)\restriction[l+1, n]$ belong to $Z_{p_{l+1} \cdot \ldots \cdot p_n}$, and $i_{0,n}(x)\restriction[k, l] \neq i_{0,n}(y)\restriction[k, l]$, then $x\restriction Z_{p_k} \times \cdots \times Z_{p_l}$ and $y\restriction Z_{p_k} \times \cdots \times Z_{p_l}$ are different as well.*

Proof. Follows from Lemma 5 above. ∎

Lemma 7. *Assume that $\overline{x} \in Z_{p_0 \cdot \ldots \cdot p_l \cdot \ldots \cdot p_n}$. Then there is $\overline{x}' \in Z_{p_0 \cdot \ldots \cdot p_l \cdot \ldots \cdot p_n}$, $\overline{x}' \le \overline{x}$, such that $\overline{x}\restriction[0, l] = \overline{x}'\restriction[0, l]$, $\overline{x}'\restriction[l+1, n] \in Z_{p_{l+1} \cdot \ldots \cdot p_n}$, and*

$$|\overline{x}\restriction[l+1, n] - \overline{x}'\restriction[l+1, n]| \le \frac{1}{p_{l+1} \cdot \ldots \cdot p_n}.$$

Proof. It is clear that

$$\overline{x}\restriction[l+1,n] < \frac{1}{p_0\cdot\ldots\cdot p_l}.$$

Also, the distance between consecutive elements of $Z_{p_{l+1}\cdot\ldots\cdot p_n}$ is equal to $\frac{1}{p_{l+1}\cdot\ldots\cdot p_n}$ Thus there exists $y < \frac{1}{p_0\cdot\ldots\cdot p_l}$, $y \in Z_{p_{l+1}\cdot\ldots\cdot p_n}$, with

$$|\overline{x}\restriction[l+1,n] - y| \le \frac{1}{p_{l+1}\cdot\ldots\cdot p_n}.$$

Then $\overline{x}' = \overline{x}\restriction[0,l] + y$ is as required. ∎

Let us notice that in many cases the fact that $\overline{x},\overline{y} \in Z_{p_0\cdot\ldots\cdot p_k\cdot\ldots\cdot p_l\cdot\ldots\cdot p_n}$ have different sums $\overline{x}\restriction[k,l]$ and $\overline{y}\restriction[k,l]$ does not imply that $i_{0,n}^{-1}(\overline{x})$, $i_{0,n}^{-1}(\overline{y})$ have different restrictions to $Z_{p_k}\times\cdots\times Z_{p_l}$. However, this holds true when we choose $\overline{x}',\overline{y}'$ as in Lemma 7, and moreover sufficiently "close" to $\overline{x}$ and $\overline{y}$.

Suppose now that $\{a_n\}_{n\in\omega}$ is a given increasing sequence of positive integers. By taking a subsequence, we may assume that the triples $a_0 < a_1 < a_2$, $a_2 < a_3 < a_4$, etc. correspond to $k < l < n$ as in Lemma 2 above. For $n \in \omega$, let $\overline{T}_n$ be equal to $i_{0,a_{2n+2}}$ $(Z_{p_0}\times\cdots\times Z_{p_{a_{2n}}}\times T_n)$, where T_n included in $Z_{p_{a_{2n+1}}}\times\cdots\times Z_{p_{a_{2n+2}}}$ has the same property as T in Lemma 2 above. Also, by the preceding remarks, $\overline{T}_n$ can be viewed as a family of intervals of equal length $1/(p_0\cdot\ldots\cdot p_{a_{2n+2}})$ contained in $\langle 0,1)$ with the group operation being modulo 1 addition.

LEMMA 8. *For every $n\in\omega$, and each set $T\subseteq Z_{p_0}\times\cdots\times Z_{p_{a_{2n+2}}}$, the sets $T\boxplus x_j$ $(j\in J)$ are stochastically independent iff $i_{0,a_{2n+2}}(T) + i_{0,a_{2n+2}}(x_j)$ $(j\in J)$ are stochastically independent in $Z_{p_0\cdot\ldots\cdot p_{a_{2n+2}}}$ (respectively, in $\langle 0,1)$).*

Proof. Follows immediately from the fact that $i_{0,a_{2n+2}}$ (respectively, $i_{0,a_{2n+2}}/(p_0\cdot\ldots\cdot p_{a_{2n+2}})$) is an isomorphism. ∎

Assume that for $n\in\omega$, $\widetilde{T}_n$ is obtained from $\overline{T}_n$ by adding to each interval $t\in\overline{T}_n$ its translations of the form

$$t -_1 \frac{i}{p_0\cdot\ldots\cdot p_{a_{2n+2}}},\ t +_1 \frac{1}{p_0\cdot\ldots\cdot p_{a_{2n+2}}} \quad \text{where} \quad i \le p_0\cdot\ldots\cdot p_{a_{2n+1}}.$$

Notice that for fixed $n\in\omega$,

$$\mu(\widetilde{T}_n) = (p_0\cdot\ldots\cdot p_{a_{2n+1}} + 1)\cdot\mu(\overline{T}_n).$$

Thus, by making $p_{a_{2n+2}}$ sufficiently large, we can have

$$\left(\frac{1}{2}\right)^n \le \mu(\widetilde{T}_n) \le \left(\frac{3}{4}\right)^n \quad \text{for almost every } n\in\omega$$

(see Lemma 2 and Remark 3 above). The advantage of using a larger set $\widetilde{T}_n$ instead of $\overline{T}_n$ is that if $(\widetilde{T}_n +_1 \overline{x}) \cap F = \emptyset$ for some $\overline{x} \in Z_{p_0 \cdot \ldots \cdot p_{a_{2n+2}}}$ and a closed set $F \subseteq \langle 0,1)$ then $(\overline{T}_n +_1 \overline{x}') \cap F = \emptyset$, where $\overline{x}'$ is an in Lemma 7.

Assume that X is a null additive set in $\langle 0,1)$. Let G be an open set with $\mu(G) < 1$ such that for every basic closed set $\tau \not\subseteq G$, we have $\mu(\tau \setminus G) > 0$ and

$$\bigcap_{m\in\omega} \bigcup_{n\geq m} \widetilde{T}_n +_1 X \subseteq G.$$

As in the proof of Claim ♠ in [2], we define, for each basic set τ and $n \in \omega$,

$$K_{\tau,n} = \{x \restriction Z_{p_{a_{2n}+1}} \times \cdots \times Z_{p_{a_{2n+1}}} : x \in Z_{p_0} \times \cdots \times Z_{p_{a_{2n+2}}}, \\ (\overline{T}_n +_1 i_{0,a_{2n+2}}(x)) \cap (\tau \setminus G) = \emptyset\}.$$

Suppose that $\overline{x} \in X$. Clearly, for some $m_0 \in \omega$ and some basic interval τ,

$$\Big(\bigcup_{n\geq m_0} \widetilde{T}_n +_1 \overline{x}\Big) \cap (\tau \setminus G) = \emptyset.$$

Since

$$\sum_{i>a_{2n+2}} \frac{\overline{x}(i)}{\prod_{j=0}^{i} p_j} \leq \frac{1}{p_0 \cdot \ldots \cdot p_{a_{2n+2}}} = \mathrm{diam}(t),$$

for every $n \in \omega$ and each interval $t \in \overline{T}_n$, we have

$$\left(\overline{T}_n +_1 \sum_{i\leq a_{2n+2}} \frac{\overline{x}(i)}{\prod_{j=0}^{i} p_j}\right) \cap (\tau \setminus G) = \emptyset$$

for $n \geq m_0$. By Lemma 7 above, for $n \geq m_0$, there is $x' \in Z_{p_0} \times \cdots \times Z_{p_{a_{2n+2}}}$ such that

$$\overline{x} \restriction [0, a_{2n+1}] = i_{0,a_{2n+2}}(x') \restriction [0, a_{2n+1}],$$

and $i_{0,a_{2n+2}}(x') \restriction [a_{2n+1}+1, a_{2n+2}]$ is sufficiently "close" to $\overline{x} \restriction [a_{2n+1}+1, a_{2n+2}]$. Hence, by the construction of $\widetilde{T}_n$,

$$(\overline{T}_n +_1 i_{0,a_{2n+2}}(x')) \cap (\tau \setminus G) = \emptyset.$$

This implies (see Corollary 6 above) that the cardinality of the set

$$\{\overline{x} \restriction [a_{2n}+1, a_{2n+1}] : \overline{x} \in X \text{ and } (\overline{T}_n +_1 \overline{x} \restriction [0, a_{2n+2}]) \cap (\tau \setminus G) = \emptyset\}$$

is at most $|K_{\tau,n}|$, for $n \geq m_0$. Using Lemma 8 above, we now proceed exactly as in the proof of Claim ♠ in [2] to show that $|K_{\tau,n}| \leq 2^n$ for almost every $n \in \omega$.

LEMMA 9. *For almost every $n \in \omega$, $|K_{\tau,n}| \leq 2^n$.*

Proof. As in the proof of Claim ♠ in [2], let $k_n = |K_{\tau,n}|$ for $n \in \omega$, and suppose that $x_1^n, \ldots, x_{k_n}^n$ are elements of $Z_{p_0} \times \cdots \times Z_{p_{a_{2n+2}}}$ whose restrictions to $Z_{p_{a_{2n}+1}} \times \cdots \times Z_{p_{a_{2n+1}}}$ are different, and exhaust the whole $K_{\tau,n}$. We have

$$\mu\Big(\bigcap_{j \le k_n} i_{0,a_{2n+2}}(Z_{p_0} \times \cdots \times Z_{p_{a_{2n+2}}} \setminus ((Z_{p_0} \times \cdots \times Z_{p_{a_{2n}}} \times T_n) \boxplus x_j^n))\Big)$$
$$= \mu\Big(\bigcap_{j \le k_n} (\langle 0,1) \setminus (\overline{T}_n +_1 i_{0,a_{2n+2}}(x_j^n))\Big).$$

By independence (see Lemma 8), the latter number is not greater than $(1 - 1/2^n)^{k_n}$. Now, let

$$B_n = \bigcap_{j \le k_n} (\langle 0,1) \setminus (\overline{T}_n + i_{0,a_{2n+2}}(x_j^n))) \quad \text{for } n \in \omega.$$

Claim 10. *For every $m \in \omega$, $\mu(B_0 \cap \cdots \cap B_m) = \mu(B_0) \cdot \ldots \cdot \mu(B_m)$.*

Proof. It suffices to prove Claim 10 for $m = 1$. Consider the sets B_0, B_1. We may assume without loss of generality that both are included in $Z_{p_0 \cdot \ldots \cdot p_{a_4}}$. Then, by symmetry of B_0 and B_1 (recall the definition of $\overline{T}_n$), we have $\mu(B_0 \cap B_1) = \mu(B_0) \cdot \mu(B_1)$. ∎

To finish the proof of Lemma 9, notice that for every $m \in \omega$, $B_0 \cap \cdots \cap B_m$ contains $\tau \setminus G$. Hence for each $m \in \omega$,

$$\prod_{n=0}^{m} \left(1 - \frac{1}{2^n}\right)^{k_n} \ge \lambda > 0,$$

where $\lambda = \mu(\tau \setminus G)$. This implies that

$$\sum_{n \in \omega} k_n \cdot 2^{-n}$$

is convergent. ∎

Since there are countably many basic sets τ in $\langle 0,1)$, we easily find a sequence $\{\widetilde{K}_n\}_{n \in \omega}$, with $\widetilde{K}_n \subseteq Z_{p_{a_{2n}+1}} \times \cdots \times Z_{p_{a_{2n+1}}}$ and $|\widetilde{K}_n| \le 2^n$ for $n \in \omega$, such that

$$f^{-1}(\overline{x}) \restriction Z_{p_{a_{2n}+1}} \times \cdots \times Z_{p_{a_{2n+1}}} \in \widetilde{K}_n$$

for almost every $n \in \omega$ whenever $\overline{x} \in X$. This finishes the proof of Theorem 1. ∎

Let X be a null additive set in $\langle 0,1)$.

COROLLARY 11. *Given a sufficiently fast increasing sequence* $\{a_n\}_{n\in\omega}$ *of positive integers, there is* $\{\widetilde{K}_n\}_{n\in\omega}$ *with* $\widetilde{K}_n \subseteq Z_{p_{a_n}} \times \cdots \times Z_{p_{a_{n+1}-1}}$ *and* $|\widetilde{K}_n| \leq 2^n$ *for all* $n \in \omega$ *so that for every* $\overline{x} \in X$,

$$f^{-1}(\overline{x}) \restriction Z_{p_{a_n}} \times \cdots \times Z_{p_{a_{n+1}-1}} \in \widetilde{K}_n$$

for almost every $n \in \omega$.

Proof. We follow the proof of Theorem 1 to calculate the cardinalities of the sets $\{f^{-1}(\overline{x}) \restriction Z_{p_{a_{2n}}} \times \cdots \times Z_{p_{a_{2n+1}-1}} : \overline{x} \in X\}$ and $\{f^{-1}(\overline{x}) \restriction Z_{p_{a_{2n+1}}} \times \cdots \times Z_{p_{a_{2n+2}-1}} : \overline{x} \in X\}$ for $n \in \omega$. ∎

Next we define a one-to-one correspondence between C and a subset of the Cantor space 2^ω, denoted by A. Let $n_{-1} = 0$, $n_0 = 1$, and for $k \in \omega$, $k \geq 1$, put $n_k = \min\{l : 2^{l-n_{k-1}} \geq p_k\}$. Fix p_k leftmost nodes in $2^{[n_{k-1}, n_k)}$ for $k \in \omega$, and denote them by $\{s_i^k\}_{i<p_k}$. Define a one-to-one function $g : C \to 2^\omega$ as follows: if $x \in C$, then $g(x) \restriction [n_{k-1}, n_k) = s_i^k$ iff $x(k) = i$, for $k \in \omega$ and $i < p_k$.

Put $A = \mathrm{range}(g)$, and let $H : A \to \langle 0,1 \rangle$ be the composition $f \circ g^{-1}$.

THEOREM 12. *Assume that* $X \subseteq \langle 0,1)$ *is a null additive set. Then* $Y = H^{-1}(X)$ *is null additive in* 2^ω.

Proof. Let G be a measure zero subset of 2^ω. We can assume without loss of generality that $G \subseteq \bigcap_{m\in\omega} \bigcup_{n\geq m} G_n$, where for $n \in \omega$,

$$G_n = \{x : x \restriction [a_n, a_{n+1}) \in G'_n\} \quad \text{with} \quad \frac{|G'_n|}{2^{a_{n+1}-a_n}} \leq \frac{1}{2^{2n}},$$

and $\{a_n\}_{n\in\omega}$ is a sufficiently fast increasing sequence of positive integers. Also we may require that $\{a_n\}_{n\in\omega}$ is a subsequence of the sequence $\{n_k\}_{k\in\omega}$ defined above. By Corollary 11, there is a sequence $\{\widetilde{K}_n\}_{n\in\omega}$, with $\widetilde{K}_n \subseteq 2^{[a_n, a_{n+1})}$ and $|\widetilde{K}_n| \leq 2^n$ for $n \in \omega$, such that $\forall y \in Y$, $\forall_n^\infty\ y \restriction [a_n, a_{n+1}) \in \widetilde{K}_n$. Clearly, this suffices to prove that $Y \oplus G$ is null (cf. [4, Theorem 13]). ∎

Now we can provide a complete solution of Problem 2.4 from [3].

THEOREM 13. *Suppose that* X *and* Y *are null additive sets in* $\langle 0,1)$ (*respectively,* $\mathbb{R}$). *Then* $X \times Y$ *is null additive in* $\langle 0,1) \times \langle 0,1)$ (*respectively,* $\mathbb{R} \times \mathbb{R}$).

Proof. According to the introductory remarks we identify an infinite series $x \in \langle 0,1)$ with $f^{-1}(x) \in C$. Proceeding as in the proof of Theorem 2.5.7 in [1], we show that every null set $G \subseteq \langle 0,1) \times \langle 0,1)$ is included in the union of two sets of the form

$$\{(x,y) \in \langle 0,1) \times \langle 0,1) : \exists_n^\infty (x,y) \restriction (Z_{p_{a_n}} \times \cdots \times Z_{p_{a_{n+1}-1}})^2 \in K_n\},$$

where $\{a_n\}_{n\in\omega}$ is a certain increasing sequence of positive integers, $K_n \subseteq (Z_{p_{a_n}} \times \cdots \times Z_{p_{a_{n+1}-1}})^2$ for $n \in \omega$, and

$$\sum_{n\in\omega} \frac{|K_n|}{(p_{a_n} \cdot \ldots \cdot p_{a_{n+1}-1})^2} < \infty.$$

Assume that X and Y are null additive in $\langle 0,1)$. Using Corollary 11 and the above characterization of null sets in $\langle 0,1) \times \langle 0,1)$, we can follow the proof of Theorem 13 in [4] to show that both sets $X \times \{0\}, \{0\} \times Y$ are null additive in $\langle 0,1) \times \langle 0,1)$ with modulo 1 coordinatewise addition. Applying the same argument as in [4, Corollary 3] (see also [4, Remark 11]) completes the proof. ∎

Finally, we prove for sets included in $\langle 0,1)$ a version of the influential theorem of Shelah (see [1, Theorem 2.7.20]) which can be stated as "every null additive subset of 2^ω is meager additive".

We define a *meager additive* set in 2^ω (or in $\langle 0,1)$) analogously to null additive by replacing "null" with "meager". Suppose that $X \subseteq 2^\omega$ is meager additive in 2^ω. Then (see [1, Theorem 2.7.17]) X can be characterized by the following property due to Bartoszyński, Judah and Shelah. For every $\tilde{f} \in \omega^{\omega\uparrow}$, there are $\tilde{g} \in \omega^{\omega\uparrow}$ and $y \in 2^\omega$ such that

$$\forall x \in X,\ \forall_n^\infty\ \exists k\ \tilde{g}(n) \le \tilde{f}(k) < \tilde{f}(k+1) < \tilde{g}(n+1),$$

and

$$x{\restriction}[\tilde{f}(k), \tilde{f}(k+1)) = y{\restriction}[\tilde{f}(k), \tilde{f}(k+1)).$$

THEOREM 14. *Every null additive set $X \subseteq \langle 0,1)$ is meager additive.*

Proof. Suppose $\tilde{f} \in \omega^{\omega\uparrow}$ is a function with $\mathrm{range}(\tilde{f}) \subseteq \mathrm{range}(\{n_k\})_{k\in\omega}$, where $\{n_k\}_{k\in\omega}$ is as in the definition of the set A. Since X is null additive, $H^{-1}(X)$ is null additive in 2^ω (by Theorem 12), and it is meager additive by an argument of Shelah. From this we derive that $H^{-1}(X)$ satisfies the Bartoszyński–Judah–Shelah characterization for the function $\tilde{f}$. Hence $X = H(H^{-1}(X))$ satisfies a condition which is similar to the above characterization, and this suffices to show that X is meager additive in $\langle 0,1)$ (see [4, proof of Theorem 1]). ∎

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References

[1] T. Bartoszyński and H. Judah, *Set Theory. On the Structure of the Real Line*, A K Peters, Wellesley, MA, 1995.

[2] J. Pawlikowski, *A characterization of strong measure zero sets*, Israel J. Math. 93 (1996), 171–183.

[3] B. Tsaban and T. Weiss, *Products of special sets of real numbers*, Real Anal. Exchange 30 (2004/2005), 819–835.
[4] T. Weiss, *On meager additive and null additive sets in the Cantor space* 2^ω *and in* $\mathbb{R}$, Bull. Polish Acad. Sci. Math. 57 (2009), 91–99.

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