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ORDINARY DIFFERENTIAL EQUATIONS

Second-Order Viability Problem: A Baire Category Approach

by

Myelkebir AITALIOUBRAHIM and Said SAJID

Presented by Czesław OLECH

Summary. The paper deals with the existence of viable solutions to the differential inclusion

$$\ddot{x}(t) \in f(t, x(t)) + \operatorname{ext} F(t, x(t)),$$

where f is a single-valued map and $\operatorname{ext} F(t,x)$ stands for the extreme points of a continuous, convex and noncompact set-valued mapping F with nonempty interior.

1. Introduction. The aim of this work is to prove the existence of local viable solutions in a prescribed closed convex and bounded subset K of a separable Hilbert space H of the following Cauchy problem:

$$(1)\qquad \begin{cases} \ddot{x}(t) \in f(t, x(t)) + \operatorname{ext} F(t, x(t)) \quad \text{a.e. on } [0, T_0], \\ (x(0), \dot{x}(0)) = (x_0, v_0) \in K \times T_K(x_0), \\ \dot{x}(t) \in T_K(x(t)) \quad \text{a.e. on } [0, T_0], \end{cases}$$

where F is a Hausdorff continuous convex-valued map, f is a measurable function with respect to the first argument and Lipschitzean with respect to the second argument from $[0,T] \times H$, and $T_K(x)$ is the contingent cone to K at x. The proof is based on the Baire category approach developed by De Blasi and Pianigiani [5, 6, 7, 10] and the suitable use of the Choquet function. This result may be considered as an extension of our previous viability result for second order nonconvex differential inclusions without perturbation (i.e., with $f = 0$; see [9]).

Similar problems of first order have been studied by Sajid (see [11]), with the Baire category method. However, it is worth noting that the second-order

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Cauchy problem cannot be resolved via the classical transition to the first-order problem.

The second-order viability problems, considered first by Cornet and Haddad (see [4]), have been studied by several authors under various assumptions, the crucial ones being the tangential conditions (see [5, 6, 7, 10]). In all the above works, the second-order adjacent set, introduced by Ben-Tal (see [2]), is used in the proof. In this paper, we prove the existence of local solutions of problem (1) in a different way: we introduce a first-order tangential condition without the second-order adjacent set, but only involving the contingent cone. We give an example where this condition is satisfied.

2. Preliminaries and statement of the main result. Throughout this paper, H is a real separable Hilbert space with the inner product $\langle\, ,\, \rangle$ and the induced norm $\|\cdot\|$. Denote by $\mathcal{C}(H)$ the set of nonempty closed convex subsets of H, and by $\mathcal{B}(H)$ the set of nonempty closed convex subsets of H with nonempty interior. The space $\mathcal{B}(H)$ is equipped with the Hausdorff distance h. Let $\pi_K(x)$ be the projection of x onto K.

Let S be a nonempty metric space and A be a nonempty subset of S. Denote by $\operatorname{ext} A$, $\operatorname{int} A$, $\overline{\operatorname{co}}\, A$, $\chi_A(\cdot)$ and $d(x, A)$ the extreme points, the interior, the closed convex hull, the characteristic function of A and the distance from x to A respectively.

Let J be an interval of $\mathbb{R}$. Denote by $AC^2(J, H)$ the space of absolutely continuous functions $x(\cdot) : J \to H$ for which $\dot{x}(\cdot)$ is also absolutely continuous. On $AC^2(J, H)$ we consider a weaker topology than the natural one, namely we endow this space with the topology of uniform convergence inherited from the space of continuous functions on J, i.e. we consider the norm $\|x(\cdot)\| = \sup_{t\in J} \|x(t)\|$ for $x(\cdot) \in AC^2(J, H)$.

By a solution of (1) we mean a pair $(s, x(\cdot)) \in\,]0, T] \times AC^2([0, T], H)$ such that $\ddot{x}(t) \in f(t, x(t)) + \operatorname{ext} F(t, x(t))$ a.e. on $[0, s]$, $(x(0), \dot{x}(0)) = (x_0, v_0) \in K \times T_K(x_0)$, $x(t) \in K$ for all $t \in [0, s]$ and $\dot{x}(t) \in T_K(x(t))$ a.e. on $[0, s]$. We denote by $S^F_{[0,s]}$ the set of solutions of (1) on $[0, s]$.

Let $I = [0, T]$, $K \in \mathcal{C}(H)$ and consider a set-valued map $F : I \times K \to 2^H$ and a function $f : I \times K \to H$. Further we assume that for all $(t, x) \in I \times H$, $F(t, x) \in \mathcal{B}(H)$ and F is Hausdorff continuous. Moreover, we introduce the following hypotheses:

Hypothesis (Ha).

(A_1) For all $x \in K$, $t \mapsto f(t, x)$ is measurable.

(A_2) There exists an integrable function $k(\cdot) : I \to \mathbb{R}$ such that for all $(t, x, y) \in I \times K \times K$,

$$\|f(t, x) - f(t, y)\| \le k(t)\|x - y\|.$$

(A_3) $f(t,x) \in T_K(x)$ for all $(t,x) \in I \times K$.

(A_4) There exists $L > 0$ such that $\|f(t,x)\| \le L$ for all $(t,x) \in I \times K$.

HYPOTHESIS (Hb). There exists a convex compact subset D of H such that for all $(t,x) \in I \times K$,

$$\begin{cases} [\operatorname{int} F(t,x)] \cap T_K(x) \cap D \neq \emptyset, \\ \overline{\operatorname{co}}[\operatorname{ext} F(t,x) \cap T_K(x) \cap D] = F(t,x) \cap T_K(x) \cap D. \end{cases}$$

HYPOTHESIS (Hc).

(H_1) $K = H$.

(H_2) There is $M > 0$ such that $h(F(t,x),0) \le M$ for all $(t,x) \in I \times K$.

REMARK 1. The tangential condition (Hb) is weaker than the following:

$$\begin{cases} F(t,x(t)) \cap D \in T_K(x), \\ [\operatorname{int} F(t,x)] \cap D \neq \emptyset, \\ \overline{\operatorname{co}}[\operatorname{ext} F(t,x) \cap D] = F(t,x) \cap D. \end{cases}$$

Moreover, let K be a nonsingleton convex compact subset of H. Then the above condition is satisfied in the following case:

- $D := K - K$,
- $F(t,x) = \pi_D(x) + d(0, \partial_r D)\overline{B}$,

where $\partial_r D$ is the relative boundary of D and B is the unit ball of H.

We shall prove the following result:

THEOREM 1. *Suppose F and f are as above and that either* (Ha) *and* (Hb), *or* (Ha) *and* (Hc) *are satisfied. Then problem* (1) *has a solution.*

3. Preliminary results. For technical reasons, we consider the extensions of F and f to $I \times H$ defined by $G(t,x) = F(t, \pi_K(x))$ and $g(t,x) = f(t, \pi_K(x))$. Observe that G and g inherit all properties of F and f: G is Hausdorff continuous from $I \times H$ to $\mathcal{B}(H)$, g is continuous with respect to the first argument, $k(t)$-Lipschitzean with respect to the second argument and for all $(t,x) \in I \times H$,

$$\begin{cases} g(t,x(t)) \in T_K(\pi_K(x)), \\ [\operatorname{int} G(t,x)] \cap T_K(\pi_K(x)) \cap D \neq \emptyset, \\ \overline{\operatorname{co}}[\operatorname{ext} G(t,x) \cap T_K(\pi_K(x)) \cap D] = G(t,x) \cap T_K(\pi_K(x)) \cap D. \end{cases}$$

As in [3], the proof technique is based on the Baire category applied to the sets governed by upper semicontinuous functions, notably the Choquet function defined as follows. Let (e_n) be a dense sequence in the unit sphere of H and consider

$$h(t,x,v)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}\langle e_n,v\rangle^2/2^n & \text{if } v\in G(t,x),\\ \infty & \text{otherwise.}\end{cases}$$

Let $\mathcal{L}$ be the class of all affine functions $a(\cdot):H\to[0,\infty[$. We associate with h the function $\widehat{h}:I\times H\times H\to]-\infty,\infty[$ given by

$$\widehat{h}(t,x,v)=\inf\{a(v):a(\cdot)\in\mathcal{L},\ a(z)\geq h(t,x,z),\ \forall z\in G(t,x)\}.$$

Definition 1. The *Choquet function* $\phi:I\times H\times H\to[-\infty,\infty[$ associated to G is defined by

$$\phi(t,x,v)=\begin{cases}\widehat{h}(t,x,v)-h(t,x,v) & \text{if } v\in G(t,x),\\ -\infty & \text{otherwise.}\end{cases}$$

Some known properties of ϕ are collected in the following proposition:

Proposition 1. *The Choquet function has the following properties*:

(i) $0\leq\phi(t,x,v)\leq M^2$ *for all* $v\in G(t,x)$ *with* $\|v\|\leq M$.
(ii) $\phi(t,x,v)=0$ *if and only if* $v\in\operatorname{ext}G(t,x)$.
(iii) ϕ *is upper semicontinuous on* $I\times H\times H$.
(iv) *If* $(x_n(\cdot))_n\subset S_I^G$ *converges uniformly to* $x(\cdot)$ *and*

$$\{\ddot{x}_n(t)-g(t,x_n(t)):(t,n)\in I\times\mathbb{N}\}$$

is bounded, *then*

$$\limsup_{n\to\infty}\int_I\phi\big(t,x_n(t),\ddot{x}_n(t)-g(t,x_n(t))\big)\,dt\leq\int_I\phi\big(t,x(t),\ddot{x}(t)-g(t,x(t))\big)\,dt.$$

Lemma 1 ([1]). *Let* $x(\cdot):I\to H$ *be absolutely continuous. Then*

$$\frac{d}{dt}[d(x(t),K)]\leq d\big(\dot{x}(t),T_K(\pi_K(x(t)))\big)\qquad a.e.\ t\in I.$$

Lemma 2 ([1]). *The function* $(x,y)\mapsto d(y,T_K(x))$ *is upper semicontinuous.*

Lemma 3 ([8]). *For every* $(C,D)\in\mathcal{B}(H)$, $h(C,D)=h(\partial C,\partial D)$ *where* ∂C *denotes the boundary of* C.

This lemma will be used to prove the following proposition:

Proposition 2. *For every* $y\in H$, *the set* $\{(t,x)\in I\times H:y\in\operatorname{int}G(t,x)\}$ *is open in* $I\times H$.

For the proof, it suffices to remark, by using Lemma 3, that the set

$$\{(t,x) \in I \times H : y \in \partial G(t,x)\} \cup \{(t,x) \in I \times H : y \notin G(t,x)\}$$

is closed in $I \times H$.

4. Proof of the main result

LEMMA 4. *Under hypothesis* (Ha), *there exist* $T_0 \in \,]0,T]$ *and* $x_1(\cdot) \in AC^2([0,T_0],H)$ *such that*

(1) $\ddot{x}_1(\cdot) - g(\cdot, x_1(\cdot))$ *is constant on* $[0,T_0]$, $x_1(0) = x_0$, $\dot{x}_1(0) = v_0$,
(2) $\ddot{x}_1(t) - g(t,x_1(t)) \in [\operatorname{int} G(t,x_1(t))] \cap T_K(\pi_K(x_1(t))) \cap D$ *for* $t \in [0,T_0]$.

Proof. Let $w_0 \in [\operatorname{int} G(0,x_0)] \cap T_K(x_0) \cap D$ and denote by $x_1(\cdot)$ the solution on I of the Cauchy problem

$$\ddot{x}(t) = g(t,x(t)) + w_0, \qquad (x(0),\dot{x}(0)) = (x_0,v_0).$$

Since $x_1(\cdot)$ is continuous, by Proposition 2 there exists $T_0 \in \,]0,T]$ such that $w_0 \in \operatorname{int} G(t,x_1(t))$ for all $t \in [0,T_0]$. Hence the proof is complete.

Set $I_0 = [0,T_0]$ and denote by S the set of solutions on I_0 of the problem

$$\ddot{x}(t) \in g(t,x(t)) + G(t,x(t)), \qquad (x(0),\dot{x}(0)) = (x_0,v_0),$$

and S^* the subset of S such that for all $x(\cdot) \in S^*$ one has:

- $\ddot{x}(\cdot) - g(\cdot,x(\cdot))$ is constant on each interval $\operatorname{int} J_n$ where $(J_n)_{n\in\mathbb{N}}$ is a sequence of intervals such that $I_0 = \bigcup_{n\in\mathbb{N}} J_n$ and $\sup J_n = \inf J_{n+1}$, for all $n \in \mathbb{N}$,
- $\ddot{x}(t) - g(t,x(t)) \in [\operatorname{int} G(t,x)] \cap T_K(\pi_K(x)) \cap D$ a.e. on I_0.

The set S^* is nonempty because it contains the mapping $x_1(\cdot)$ given by Lemma 4. Since S is closed, $\overline{S^*}$ is a complete subset of S. For $\alpha > 0$, define

$$S_d^\alpha = \Big\{x(\cdot) \in \overline{S^*} : \int_0^{T_0} d\big(\dot{x}(t), T_K(\pi_K(x(t)))\big)\,dt < \alpha\Big\},$$

$$S_\phi^\alpha = \Big\{x(\cdot) \in \overline{S^*} : \int_0^{T_0} \phi\big(t,\dot{x}(t),\ddot{x}(t) - g(t,x(t))\big)\,dt < \alpha\Big\},$$

and consider the following subsets:

$$S^\alpha = S_d^\alpha \cap S_\phi^\alpha \quad \text{and} \quad R^n = S^{1/n}, \quad n \in \mathbb{N}.$$

To prove Theorem 1, it suffices to establish that $\bigcap_{n\in\mathbb{N}} R^n$ is nonempty. Indeed, every $x(\cdot) \in \bigcap_{n\in\mathbb{N}} R^n$ satisfies

$$\int_0^{T_0} \phi\big(t,\dot{x}(t),\ddot{x}(t) - g(t,x(t))\big)\,dt = \int_0^{T_0} d\big(\dot{x}(t), T_K(\pi_K(x(t)))\big)\,dt = 0.$$

Thus, by Proposition 1 and Lemma 1, it follows that

$$\begin{cases} \ddot{x}(t) \in g(t, x(t)) + \operatorname{ext} G(t, x(t)) & \text{a.e. on } I_0, \\ (x(0), \dot{x}(0)) = (x_0, v_0) \in K \times T_K(x_0), \\ (x(t), \dot{x}(t)) \in K \times T_K(x(t)) & \text{a.e. on } I_0. \end{cases}$$

Hence $(T_0, x(\cdot))$ is a solution to problem (1).

LEMMA 5. *For every $\alpha > 0$, the set S^α is open in $\overline{S^*}$.*

Proof. Let $(x_n(\cdot))_{n\in\mathbb{N}} \subset S^* \setminus S^\alpha$ be such that $x_n(\cdot)$ converges in $\overline{S^*}$, and let $x(\cdot)$ be its limit. By the definition of S^α, for any $n \in \mathbb{N}$,

$$\int_0^{T_0} \phi\big(t, \dot{x}_n(t), \ddot{x}_n(t) - g(t, x_n(t))\big)\, dt \geq \alpha$$

and

$$\int_0^{T_0} d\big(\dot{x}_n(t), T_K(\pi_K(x_n(t)))\big)\, dt \geq \alpha.$$

Hence by Proposition 1,

$$\int_0^{T_0} \phi\big(t, \dot{x}(t), \ddot{x}(t) - g(t, x(t))\big)\, dt \geq \limsup_{n\to\infty} \int_0^{T_0} \phi\big(t, \dot{x}_n(t), \ddot{x}_n(t) - g(t, x_n(t))\big)\, dt \geq \alpha,$$

thus $x(\cdot) \notin S_\phi^\alpha$ and consequently $x(\cdot) \notin S^\alpha$. This completes the proof.

To prove that S^α is dense in $\overline{S^*}$, we need the following approximation lemma:

LEMMA 6. *Let $x(\cdot) \in S^*$, $\alpha > 0$ and J_0 be an interval such that $\ddot{x}(\cdot) - g(\cdot, x(\cdot))$ is constant on $\operatorname{int} J_0 = \,]0, t_1[$. Then there exist two sequences $(y_n(\cdot))_{n\in\mathbb{N}}$ in $AC^2(J_0, H)$ and $(P_n)_{n\in\mathbb{N}}$ of families of intervals $(J_q^n)_{q\in\mathbb{N}}$ with $\sup J_q^n = \inf J_{q+1}^n$ for all $(q, n) \in \mathbb{N}^2$ such that*

(a) $(\ddot{y}_n(t) - g(t, y_n(t))) \in [\operatorname{int} G(t, y_n(t))] \cap T_K(\pi_K(y_n(t))) \cap D$ *for all* $t \in [0, t_1]$, $y_n(0) = x_0$ *and* $\dot{y}_n(0) = v_0$,

(b) $\ddot{y}_n(\cdot) - g(\cdot, y_n(\cdot))$ *is constant on each* $\operatorname{int} J_q^n$,

(c) $\displaystyle\int_0^{t_1} \phi\big(t, \dot{y}_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big)\, dt \leq \frac{\alpha t_1}{2T_0}$,

(d) $\displaystyle\int_0^{t_1} d\big(\dot{y}_n(t), T_K(\pi_K(y_n(t)))\big)\, dt \leq \frac{\alpha t_1}{2T_0}$,

(e) $\displaystyle\lim_{n\to\infty} \sup_{t\in J_0} \|y_n(t) - x(t)\| = 0.$

Proof. By Lemma 2, there exists $\delta > 0$ such that

$$d(y, T_K(\pi_K(x))) < \frac{\alpha}{2T_0}, \qquad \forall (x, y) \in B((x_0, v_0), \delta).$$

Put $M = \sup_{x \in D} \|x\|$, and denote by a the constant value of $\ddot{x}(\cdot) - g(\cdot, x(\cdot))$ on $]0, t_1[$. Assume that

$$t_1 < \min\left\{\frac{\delta}{2}, \frac{\delta}{2(\|v_0\| + M + L)}\right\}.$$

For any nonzero integer n, define

$$t_i^n = i t_1 / n, \qquad i = 0, \dots, n.$$

It is clear that

$$\bigcup_{i=0}^{n-1} [t_i^n, t_{i+1}^n] = \bar{J}_0.$$

Recall that, by hypothesis,

$$\overline{\text{co}}[\text{ext}\, G(0, x_0) \cap T_K(\pi_K(x_0)) \cap D] = G(0, x_0) \cap T_K(\pi_K(x_0)) \cap D.$$

Then for any n, there exist $\lambda_i^n > 0$ and $b_i^n \in [\text{ext}\, G(0, x_0) \cap T_K(\pi_K(x_0)) \cap D]$ for $i = 1, \dots, m_n$, $m_n \in \mathbb{N}$ and $\sum_{i=1}^{m_n} \lambda_i^n = 1$ such that

$$\left\| a - \sum_{i=1}^{m_n} \lambda_i^n b_i^n \right\| < \frac{1}{2^n}. \tag{2}$$

Let $\gamma \in]0, 1[$, $n \in \mathbb{N}$ and $i = 1, \dots, m_n$. Set $c_i^n(\gamma) = \gamma a + (1 - \gamma) b_i^n$. Observe that

$$c_i^n(\gamma) \in G(0, x_0) \cap T_K(\pi_K(x_0)) \cap D. \tag{3}$$

Moreover, by Proposition 1, we have

$$\phi(0, x_0, b_i^n) = 0, \tag{4}$$

hence, in view of Proposition 1, Lemma 2 and Proposition 2, for every $n \in \mathbb{N}$ there exist $\zeta_n \in]0, \delta[$ and $\gamma_0 > 0$ such that for all $i = 1, \dots, m_n$ and $(t, x) \in I \times H$ satisfying $\max(t, \|x - x_0\|) < \zeta_n$, one has

$$\begin{cases} c_i^n(\gamma_0) \in [\text{int}\, G(0, x)] \cap T_K(\pi_K(x)) \cap D, \\ \phi(t, x, c_i^n(\gamma_0)) < \dfrac{\alpha}{2T_0}, \\ d(c_i^n(\gamma_0), T_K(\pi_K(x))) < \dfrac{\alpha}{2T_0}. \end{cases} \tag{5}$$

Furthermore, by (2), we may assume that

$$\left\| a - \sum_{i=1}^{m_n} \lambda_i^n c_i^n(\gamma_0) \right\| < \frac{1}{2^n}. \tag{6}$$

For any positive integer n, $i = 0, \dots, n-1$ and $j = 0, \dots, m_n - 1$, define

$$\begin{cases} \tau_{i,0}^n = t_i^n = it_1/n, \\ \tau_{i,j+1}^n = \tau_{i,j}^n + \lambda_{j+1}^n (t_{i+1}^n - t_i^n), \end{cases}$$

and set $\Delta_{i,j}^n = [\tau_{i,j}^n, \tau_{i,j+1}^n]$. It is clear that

$$\bigcup_{j=0}^{m_n-1} \Delta_{i,j}^n = [t_i^n, t_{i+1}^n].$$

For any $n \in \mathbb{N}\setminus\{0\}$, let $y_{0,0}^n(\cdot)$ be the solution on $\Delta_{0,0}^n$ of the Cauchy problem

$$\ddot{x}(t) = g(t, x(t)) + c_1^n(\gamma_0), \qquad (x(0), \dot{x}(0)) = (x_0, v_0).$$

By induction, for $j = 1, \dots, m_n - 1$, denote by $y_{0,j}^n(\cdot)$ the solution on $\Delta_{0,j}^n$ of the Cauchy problem

$$\ddot{x}(t) = g(t, x(t)) + c_{j+1}^n(\gamma_0), \qquad (x(\tau_{0,j}^n), \dot{x}(\tau_{0,j}^n)) = (y_{0,j-1}^n(\tau_{0,j}^n), \dot{y}_{0,j-1}^n(\tau_{0,j}^n)).$$

For $i = 1, \dots, n-1$ and $t \in [t_i^n, t_{i+1}^n]$, set

$$y_i^n(t) = \sum_{j=1}^{m_n} \chi_{\Delta_{i,j}^n}(t) y_{i,j}^n(t),$$

where $y_{i,j}^n(\cdot)$ stands for the solution on $[\tau_{i,j}^n, \tau_{i,j+1}^n[$ of the problem

$$\ddot{x}(t) = g(t, x(t)) + c_{j+1}^n(\gamma_0), \qquad (x(\tau_{i,j}^n), \dot{x}(\tau_{i,j}^n)) = (y_{i,j-1}^n(\tau_{i,j}^n), \dot{y}_{i,j-1}^n(\tau_{i,j}^n)).$$

Now, for all $n \in \mathbb{N}$, consider the function

$$y_n(t) = \sum_{i=0}^{n-1} \chi_{[t_i^n, t_{i+1}^n]}(t) y_i^n(t), \qquad t \in J_0.$$

Obviously, for all $n \in \mathbb{N}$ and $t \in [0, t_1[$,

$$\ddot{y}_n(t) - g(t, y_n(t)) \in \{c_j^n(\gamma_0) : j = 1, \dots, m_n\}. \tag{7}$$

Moreover, by the choice of t_1, for any $t \in [0, t_1]$ we have

$$\begin{aligned} \|\dot{y}_n(t) - v_0\| &\leq \int_0^{t_1} \|\ddot{y}_n(s)\| \, ds \\ &\leq \int_0^{t_1} \|\ddot{y}_n(s) - g(s, y_n(s))\| \, ds + \int_0^{t_1} \|g(s, y_n(s))\| \, ds \\ &\leq (M + L)t_1 < \delta/2 \end{aligned}$$

and

$$\|y_n(t) - x_0\| \leq \int_0^{t_1} \|\dot{y}_n(t)\| dt.$$

Since $\dot{y}_n(t) = v_0 + \int_0^{t_1} \ddot{y}_n(s)\,ds$ and $\|\ddot{y}_n(s)\| \le M + L$, we have

$$\|y_n(t) - x_0\| \le \|v_0\| t_1 + (M+L)(t_1)^2 < \frac{\delta}{2}.$$

Thus, combining (5) and (7), for any $t \in [0, t_1]$, one has

- $\ddot{y}_n(t) - g(t, y_n(t)) \in [\operatorname{int} G(t, y_n(t))] \cap T_K(\pi_K(y_n(t))) \cap D$,
- $\phi\big(t, y_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big) \le \dfrac{\alpha}{2T_0}$,
- $d\big(\dot{y}_n(t), T_K(\pi_K(y_n(t)))\big) \le \dfrac{\alpha}{2T_0}$.

Thus we have constructed a sequence $(y_n(\cdot))_n \in AC^2([0, t_1], H)$ such that for every positive integer n, we have

- $y_n(0) = x_0$, $\dot{y}_n(0) = v_0$ and $\ddot{y}_n(\cdot) - g(\cdot, y_n(\cdot))$ is constant on each $\operatorname{int} \Delta_{i,j}^n$, $i = 0, \dots, n-1$, $j = 0, \dots, m_n$,
- $\ddot{y}_n(t) - g(t, y_n(t)) \in [\operatorname{int} G(t, y_n(t))] \cap T_K(\pi_K(y_n(t))) \cap D$ for every $t \in [0, t_1]$,
- $\displaystyle\int_0^{t_1} \phi\big(t, y_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big)\,dt \le \frac{\alpha t_1}{2T_0}$,
- $\displaystyle\int_0^{t_1} d\big(\dot{y}_n(t), T_K(\pi_K(y_n(t)))\big)\,dt \le \frac{\alpha t_1}{2T_0}$,
- $\max\{\|y_n(t) - x_0\|, \|\dot{y}_n(t) - v_0\|\} < \dfrac{\delta}{2}$.

To complete the proof, it suffices to show that $\sup_{t \in J_0} \|y_n(t) - x(t)\|$ converges to 0. Indeed, for every positive integer n, we have

$$\begin{aligned}
\|\dot{y}_n(t_1^n) - \dot{x}(t_1^n)\| &= \left\| \int_0^{t_1^n} (\ddot{y}_n(t) - \ddot{x}(t))\,dt \right\| \\
&\le \left\| \int_0^{t_1^n} \big(\ddot{y}_n(t) - g(t, y_n(t)) - \big(\ddot{x}(t) - g(t, x(t))\big)\big)\,dt \right\| \\
&\quad + \left\| \int_0^{t_1^n} \big(g(t, y_n(t)) - g(t, x(t))\big)\,dt \right\|.
\end{aligned}$$

Since $\ddot{x}(t) - g(t, x(t))$ is equal to a, and $g(t, \cdot)$ is $k(t)$-Lipschitzean, by (7) it follows that

$$\|\dot{y}_n(t_1^n) - \dot{x}(t_1^n)\| \le \left\| \sum_{j=0}^{m_n - 1} \int_{\Delta_{0,j}^n} (a - c_{j+1}^n(\gamma_0))\,dt \right\| + \sup_{t \in J_0} \|y_n(t) - x(t)\| \int_0^{t_1^n} k(t)\,dt,$$

which in view of (6) implies

$$\|\dot{y}_n(t_1^n)-\dot{x}(t_1^n)\| \le t_1^n\Big\|a-\sum_{j=0}^{m_n-1}\lambda_j^n c_{j+1}^n(\gamma_0)\Big\| + \sup_{t\in J_0}\|y_n(t)-x(t)\|\int_0^{t_1^n} k(t)\,dt$$
$$\le \frac{t_1^n}{2^n} + \sup_{t\in J_0}\|y_n(t)-x(t)\|\int_0^{t_1^n} k(t)\,dt.$$

By induction, we show that for all $i=1,\ldots,n$,

$$\|\dot{y}_n(t_i^n)-\dot{x}(t_i^n)\| \le \frac{t_i^n}{2^n} + \sup_{t\in J_0}\|y_n(t)-x(t)\|\int_0^{t_i^n} k(t)\,dt,$$

hence

$$\|\dot{y}_n(t_i^n)-\dot{x}(t_i^n)\| \le \frac{T_0}{2^n} + \sup_{t\in J_0}\|y_n(t)-x(t)\|\int_0^{T_0} k(t)\,dt. \tag{8}$$

Let $t\in[0,t_1]$. For any n, let i be the integer such that $t\in[t_i^n,t_{i+1}^n]$. Since

$$\|\dot{y}_n(t)-\dot{x}(t)\| \le \|\dot{y}_n(t)-\dot{y}_n(t_i^n)\| + \|\dot{y}_n(t_i^n)-\dot{x}(t_i^n)\| + \|\dot{x}(t_i^n)-\dot{x}(t)\|,$$

we have

$$\|\dot{y}_n(t)-\dot{x}(t)\| \le 2(M+L)(t-t_i^n) + \frac{T_0}{2^n} + \sup_{t\in J_0}\|y_n(t)-x(t)\|\int_0^{T_0} k(t)\,dt$$
$$\le 2(M+L)\frac{T_0}{n} + \frac{T_0}{2^n} + \sup_{t\in J_0}\|y_n(t)-x(t)\|\int_0^{T_0} k(t)\,dt.$$

Put $\rho=\int_0^{T_0} k(t)\,dt$. Then by the choice of T_0 and by an easy computation, the above inequality implies that

$$\|y_n(t)-x(t)\| \le \frac{1}{1-\rho T_0}\bigg(2(M+L)\frac{(T_0)^2}{n} + \frac{(T_0)^2}{2^n}\bigg),$$

so that $y_n(\cdot)$ converges uniformly to $x(\cdot)$ on J_0. Hence the proof of Lemma 6 is complete.

LEMMA 7. *For any $\alpha>0$, the set S^α is dense in $\overline{S^*}$.*

Proof. We shall use the Kuratowski–Zorn Lemma. Indeed, let $x(\cdot)\in S^*$, $\alpha>0$ and let Γ be the family of all $(s,(s_n)_{n\in\mathbb{N}},(y_n(\cdot))_{n\in\mathbb{N}})$ in the set $]0,T_0]\times\,]0,s]\times AC^2([0,s],H)$ with the following properties:

(C_1) $y_n(0)=x_0$, $\dot{y}_n(0)=v_0$ and $\ddot{y}_n(\cdot)-g(\cdot,y_n(\cdot))$ is constant on each $\operatorname{int} J_q^n$, where $(J_q^n)_{q\in\mathbb{N}}$ is a family of intervals satisfying $[0,s]=\bigcup_{q\in\mathbb{N}} J_q^n$ and $\sup J_q^n=\inf J_{q+1}^n$ for all $q\in\mathbb{N}$,

(C_2) $\|\ddot{y}_n(t)-g(t,y_n(t))\|\le M+L$,

(C_3) $(\ddot{y}_n(t)-g(t,y_n(t))) \in [\operatorname{int} G(t,y_n(t))]\cap T_K(\pi_K(y_n(t)))\cap D$ for every $t\in[0,s_n[$,

(C_4) $\displaystyle\int_0^{s_n} \phi\big(t,y_n(t),\ddot{y}_n(t)-g(t,y_n(t))\big)\,dt \le \frac{\alpha s_n}{2T_0}$,

(C_5) $\displaystyle\int_0^{s_n} d\big(\dot{y}_n(t),T_K(\pi_K(y_n(t)))\big)\,dt \le \frac{\alpha s_n}{2T_0}$,

(C_6) $\displaystyle\lim_{n\to\infty}\sup\|y_n(t)-x(t)\|=0$,

(C_7) $\displaystyle\max(\|y_n(t)-y_n(t_q^n)\|,\|\dot{y}_n(t)-\dot{y}_n(t_q^n)\|)\le\sum_{i=1}^{q+1}\delta/2^i$ for all $t\in(\operatorname{int} J_q^n)$ $\cap\,[0,s_n[$, where $t_q^n=\inf J_q^n$.

By Lemma 6, the family Γ is nonempty. On Γ we define the following order:

$$[(s^1,(s_n^1),(y_n^1(\cdot)))\le(s^2,(s_n^2),(y_n^2(\cdot)))]$$
$$\Updownarrow$$
$$[s^1\le s^2, s_n^1\le s_n^2 \text{ and } y_n^1(\cdot)=y_n^2(\cdot),\forall n\in\mathbb{N}].$$

Claim 1. *The above order is inductive.*

Proof. Let $(s^p,(s_n^p)_{n\in\mathbb{N}},(y_n^p(\cdot))_{n\in\mathbb{N}})_{p\in\Lambda}$ be a totally ordered subfamily of Γ. Without loss of generality, we may assume that $\Lambda=\mathbb{N}$ and

$$(s^p,(s_n^p),(y_n^p(\cdot)))\le(s^{p+1},(s_n^{p+1}),(y_n^{p+1}(\cdot)))\quad\text{for all } p\in\mathbb{N}.$$

Set $s=\sup s^p$ and $s_n=\sup s_n^p$. For all $n\in\mathbb{N}$, let us prove that $(y_n^p(s_n^p))_{p\in\mathbb{N}}$ is a Cauchy sequence. Indeed, for $p,q\in\mathbb{N}$, $p\ge q$, we have $y_n^q(s^q)=y_n^p(s^q)$. Then

$$\|\dot{y}_n^p(s^p)-\dot{y}_n^q(s^q)\|=\|\dot{y}_n^p(s^p)-\dot{y}_n^p(s^q)\|=\left\|\int_{s^q}^{s^p}\ddot{y}_n^p(t)\,dt\right\|\le(M+L)(s^p-s^q).$$

Since $(s^p)_{p\in\mathbb{N}}$ is a Cauchy sequence, we obtain the desired property. For any $n\in\mathbb{N}$, denote by l_n the limit of the sequence $(y_n^p(s_n^p))_{p\in\mathbb{N}}$ and define

$$y_n(s)=l_n\quad\text{and}\quad y_n(t)=\sum_{p=1}^{\infty}\chi_{[s^{p-1},s^p]}(t)y_n^p(t)\quad\text{if } t\in[0,s[.$$

It is clear that

$$(s^p,(s_n^p),(y_n^p(\cdot)))\le(s,(s_n),(y_n(\cdot)))\quad\forall p\in\mathbb{N}.$$

So to complete the proof, it suffices to prove that $(s,(s_n),(y_n(\cdot)))$ satisfies conditions $(C_1),\dots,(C_7)$. To begin, put $s^{-1}=q_{-1}=0$ and for any $p\in\mathbb{N}$, define

$$q_p=\min\{q: s^p\in\overline{J_q^{n,p+1}}\}\quad\text{and}\quad L_i^{n,p}=J_q^{n,p+1}\cap[s^{p-1},s^p],\ i=q_{p-1},\dots,q_p,$$

where $(J_i^{n,p})_i$ is the family of subintervals of $[0, s^p]$ associated to the function $(y_n^p(\cdot))$ in the definition of Γ. By construction of the family $(L_i^{n,p})$ we prove easily that $(s, (s_n), (y_n(\cdot)))$ satisfies (C_1) and (C_2). Furthermore, it is clear that (C_3) is satisfied. Now, to complete the proof of the claim, we will show that conditions (C_4), (C_5), (C_6) and (C_7) are satisfied.

Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Since

$$\lim_{p\to\infty} \int_0^{s_n^p} \phi\big(t, y_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big)\, dt = \int_0^{s_n} \phi\big(t, y_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big)\, dt,$$

there exists p_0 such that, for any $p \geq p_0$, we have

$$\begin{aligned}\int_0^{s_n} \phi\big(t, y_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big)\, dt &\leq \int_0^{s_n^p} \phi\big(t, y_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big)\, dt + \varepsilon \\ &\leq \frac{\alpha s_n^p}{2T_0} + \varepsilon \leq \frac{\alpha s_n}{2T_0} + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\int_0^{s_n} \phi\big(t, y_n(t), \ddot{y}_n(t) - g(t, y_n(t))\big)\, dt \leq \frac{\alpha s_n}{2T_0}.$$

This gives (C_4). Similarly, we prove (C_5). Finally, let us prove (C_6) and (C_7). Since (s^p) converges to s, there exists p_0 such that, for all $p \geq p_0$,

$$s - s^p \leq \frac{\varepsilon}{8(M+L)}. \tag{9}$$

On the other hand, since $\lim_{n\to\infty} \sup_{t\in[0,s^{p_0}]} \|y_n^{p_0}(t) - x(t)\| = 0$, there exists n_0 such that for all $n \geq n_0$, we have

$$\sup_{t\in[0,s^{p_0}]} \|\dot{y}_n^{p_0}(t) - \dot{x}(t)\| \leq \frac{\varepsilon}{4}. \tag{10}$$

In addition, since

$$\dot{y}_n^{p_0}(t) - \dot{x}(t) = \dot{y}_n^{p_0}(s^{p_0}) - \dot{x}(s^{p_0}) + \int_{s^{p_0}}^{t} (\ddot{y}_n^{p_0}(s) - \ddot{x}(s))\, ds,$$

we have

$$\begin{aligned}\sup_{t\in[s^{p_0},s]} \|\dot{y}_n^{p_0}(t) - \dot{x}(t)\| &\leq \Big\|\dot{y}_n^{p_0}(s^{p_0}) - \dot{x}(s^{p_0}) + \int_{s^{p_0}}^{s} (\ddot{y}_n^{p_0}(t) - \ddot{x}(t))\, dt\Big\| \\ &\leq \|\dot{y}_n^{p_0}(s^{p_0}) - \dot{x}(s^{p_0})\| + \Big\|\int_{s^{p_0}}^{s} (\ddot{y}_n^{p_0}(t) - \ddot{x}(t))\, dt\Big\| \\ &\leq \|\dot{y}_n^{p_0}(s^{p_0}) - \dot{x}(s^{p_0})\| + 2(M+L)(s - s^{p_0}).\end{aligned}$$

Hence by (9) and (10), we obtain

$$\sup_{t\in[s^{p_0},s]} \|\dot{y}_n^{p_0}(t)-\dot{x}(t)\| \le \frac{\varepsilon}{2}. \tag{11}$$

Thus combining (10) and (11), it follows that

$$\sup_{t\in[0,s]} \|\dot{y}_n^{p_0}(t)-\dot{x}(t)\| \le \frac{\varepsilon}{4}, \quad \forall n \ge n_0,$$

and therefore $\lim_{n\to\infty}\sup_{t\in[0,s]}\|y_n^{p_0}(t)-x(t)\| = 0$. Thus, it is clear that (C_7) is satisfied. Hence the claim is proved.

We conclude, by the Kuratowski–Zorn Lemma, that Γ admits a maximal element, say $(r,(r_n),(u_n))$.

CLAIM 2. *For any $n\in\mathbb{N}$, $r_n = T_0$.*

Proof. Assume, to the contrary, that there exists p such that $r_p < T_0$. Let

$$b \in \operatorname{ext} G(r_p,u_p(r_p)) \cap T_K(\pi_K(u_p(r_p))) \cap D$$

and

$$v \in \operatorname{int} G(r_p,u_p(r_p)) \cap T_K(\pi_K(u_p(r_p))) \cap D.$$

Let $\delta > 0$ be as in the proof of Lemma 6. By Proposition 1 there exists $\eta \in \,]0,\delta]$ such that for all $(t,x,y)\in I_0\times H\times H$, we have

$$\max\{|t-r_p|, \|x-u_p(r_p)\|, \|y-b\|\} < \eta \;\Rightarrow\; \phi(t,x,y) < \frac{\alpha}{2T_0}. \tag{12}$$

Choose $s\in\,]r_p,r]$ such that

$$s-r_p < \min\left\{\eta, \frac{\eta}{\|v_0\|+(M+L)T_0}, \frac{\delta}{2^{\varphi_p(r_p)}}\right\}, \tag{13}$$

where $\varphi_p(r_p)$ is the order of the p-th partition to which r_p belongs. Put

$$c = \frac{\eta}{2M}v + \left(1-\frac{\eta}{2M}\right)b.$$

Let $z(\cdot)$ be the solution on $[r_p,r]$ of the problem

$$\ddot{x}(t) = g(t,x(t)) + c, \qquad (x(r_p),\dot{x}(r_p)) = (u_p(r_p),\dot{u}(r_p)),$$

and define on $[0,r]$ the function

$$y(\cdot) = u_p(\cdot)\chi_{[0,r_p[}(\cdot) + z(\cdot)\chi_{[r_p,r[}(\cdot).$$

By the choice of b and v, it is clear that $c\in\operatorname{int} G(r_p,u_p(r_p))$. Moreover, according to Proposition 2, we can assume that

$$\ddot{y}(t) - g(t,y(t)) = c \in \operatorname{int} G(t,y(t)), \qquad \forall t\in[r_p,r]. \tag{14}$$

Since $\sup_{t\in[0,r]} \|\dot{y}(t)\| \le \|v_0\| + (M+L)T_0$, for all $t \in [r_p, s]$, by (13), the choice of c and (14), we have

$$\max\{|t-r_p|, \|y(t)-y(r_p)\|, \|\ddot{y}(t)-g(t,y(t))-b\|\} < \eta$$

which, in view of (12), implies that for all $t \in [r_p, s]$,

$$\phi\big(t, y(t), \ddot{y}(t) - g(t,y(t))\big) < \frac{\alpha}{2T_0}.$$

Thus

$$\int_{r_p}^{s} \phi\big(t, y(t), \ddot{y}(t) - g(t,y(t))\big)\,dt < \frac{\alpha(s-r_p)}{2T_0},$$

so that

$$\int_{0}^{s} \phi\big(t, y(t), \ddot{y}(t) - g(t,y(t))\big)\,dt < \frac{\alpha s}{2T_0}. \tag{15}$$

Moreover, since (u_n) satisfies (C_7) of Lemma 7, and since $\eta < \delta$, from (13) it follows that

$$\max\{\|z(t)-x_0\|, \|\dot{z}(t)-v_0\|\} \le \sum_{i=1}^{\theta(t)+1} \frac{\delta}{2^i}, \quad \forall t \in [0,s],$$

where $\theta(t)$ is the order of the p-th partition to which t belongs. Thus

$$\int_{0}^{s} d\big(\dot{z}(t), y(t), T_K(\pi_K(z(t))\big)\,dt < \frac{\alpha s}{2T_0}. \tag{16}$$

Define $(k, (k_n)_n, (v_n)_n)$ as follows:

$$k = r,$$
$$k_n = \begin{cases} s & \text{if } n = p, \\ r_n & \text{otherwise,} \end{cases}$$
$$v_n = \begin{cases} y(\cdot) & \text{if } n = p, \\ u_n & \text{otherwise.} \end{cases}$$

Obviously, by the construction of $(k, (k_n)_n, (v_n)_n)$ together with (13)–(16), we deduce that $(k, (k_n)_n, (v_n)_n) \in \Gamma$. Furthermore, it is clear that

$$(r, (r_n)_n, (u_n)_n) < (k, (k_n)_n, (v_n)_n),$$

a contradiction, and so Claim 2 is valid.

Consequently, by the Baire category theorem, the set $\bigcap_{n\ge1} R^n$ is nonempty. Let $x(\cdot) \in \bigcap_{n\ge1} R^n$. Then

$$\begin{cases} \ddot{x}(t) - g(t,x(t)) \in \operatorname{ext} F(t,x(t)) \quad \text{a.e. on } [0,T_0], \\ (x(0), \dot{x}(0)) = (x_0, v_0) \in K \times T_K(x_0), \\ \dot{x}(t) - g(t,x(t)) \in T_K(\pi_K(x(t))), \forall t \in [0,T_0]. \end{cases}$$

Moreover, at the beginning of Section 3, it has been mentioned that $g(t, x(t))$ belongs to the cone $T_K(\pi_K(x(t)))$, hence $\dot{x}(t) \in T_K(\pi_K(x(t)))$ for all $t \in [0, T_0]$. By Lemma 1, it follows that $x(t) \in K$. This completes the proof of Theorem 1.

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Myelkebir Aitalioubrahim and Said Sajid
U.F.R. Mathematics and Applications, F.S.T.
University Hassan II–Mohammedia
BP 146, Mohammedia, Morocco
E-mail: aitalifr@yahoo.fr
saidsajid@hotmail.com

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