BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 57, No. 1, 2009

CONVEX AND DISCRETE GEOMETRY

On-line Covering the Unit Square with Squares

by

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Summary. The unit square can be on-line covered with any sequence of squares whose total area is not smaller than 4.

1. Introduction. Let $C, C_1, C_2, \ldots$ be planar convex bodies. We say that the sequence (C_i) *permits a covering* [*a translative covering*] of C if there exist rigid motions [translations, respectively] σ_i such that $C \subseteq \bigcup \sigma_i C_i$. The *on-line* covering is a covering in which we are given C_i, where $i > 1$, only after the motion σ_{i-1} has been provided; at the beginning we are given C_1. The on-line restriction means that each set C_i must be assigned its place before the next one appears, and that the placement cannot be modified afterwards. The area of C is denoted by $|C|$.

Denote by $v(C)$ the least number such that any (finite or infinite) sequence of positive homothetic copies of C with total area greater than $v(C)|C|$ permits an on-line translative covering of C. Let

$$I = \{(x, y);\, 0 \le x \le 1,\, 0 \le y \le 1\}.$$

Moon and Moser showed in [8] that any sequence of homothetic copies of I with total area not smaller than 3 permits a translative (off-line) covering of I (see also the survey paper [1]). The bound of 3 cannot be reduced; three squares of side length smaller than 1, each with a side parallel to a side of I, cannot translatively cover I. Consequently, also $v(I) \ge 3$. Thus far it is still unknown whether or not $v(I) > 3$. The first upper bound presented by Kuperberg in [6] is that $v(I) \le 16$. According to the method of the current bottom [3] we have $v(I) \le 8$. By using the method of the nth segment [5] we obtain $v(I) \le 389/60 \approx 6.483$. In [4] the method of the current bottom and

2000 *Mathematics Subject Classification*: 52C15, 05B40.
Key words and phrases: on-line covering, square.

DOI: 10.4064/bp57-1-6

top is presented. According to that method we have $v(I) \le \frac{7}{4}\sqrt[3]{9}+\frac{13}{8} \approx 5.265$. The aim of this paper is to show that $v(I) \le 4$. As a corollary, we show that $v(C) \le 14$ for any planar convex body C.

A survey of results concerning on-line packings and coverings is given in [7].

2. The method of the movable bottom. Let $S_1, S_2, \ldots$ be a sequence of squares. Denote by s_i the side length of S_i and assume that a side of S_i is parallel to a side of I for $i = 1, 2, \ldots$. We describe an on-line method of translative covering of I with $S_1, S_2, \ldots$. This method is a small modification of the method of the current bottom presented in [3].

If $s_i < 1$, then denote by d_i the number from the set $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\right\}$ such that

$$d_i \le s_i < 2d_i.$$

Let $R_i \subset S_i$ be a rectangle of width d_i and height s_i.

If $s_1 \ge 1$, then we cover I with S_1 and we stop the covering process. Otherwise, $S_1 \supset R_1$ is placed so that

$$\sigma_1 R_1 = \{(x, y);\ 0 \le x \le d_1,\ 0 \le y \le s_1\}.$$

By the *part of S_1 used for the covering* we mean $Q_1 = \sigma_1 R_1$.

Assume that $i > 1$ and that the translations $\sigma_1, \ldots, \sigma_{i-1}$ have been provided. If $s_i \ge 1$, then we cover I with S_i and stop. Otherwise, we find the greatest number $b_i \le 1$ such that each point of I whose y-coordinate is smaller than b_i has been covered with a rectangle $\sigma_j R_j$ with $j < i$. The set of points of I with y-coordinate b_i is called the ith *bottom*. A point of the ith bottom is a *surface point* if no point of I with the same x-coordinate and with a larger y-coordinate has been covered yet. We place $S_i \supset R_i$ so that $\sigma_i R_i$ contains a surface point and that $\sigma_i R_i$ has the form

$$\sigma_i R_i = \{(x, y);\ k d_i \le x \le (k+1) d_i,\ b_i \le y \le b_i + s_i\},$$

where $k \in \{0, 1, \ldots, d_i^{-1} - 1\}$. The *part of S_i used for the covering* is defined as

$$Q_i = \sigma_i R_i \setminus (\sigma_1 R_1 \cup \cdots \cup \sigma_{i-1} R_{i-1}).$$

We stop when $b_i = 1$ for an integer i; then I has been covered with $\sigma_1 R_1, \ldots, \sigma_{i-1} R_{i-1}$. This method is called the *method of the movable bottom.*

LEMMA 1. *Let S_i be a square of side length smaller than* 1 *placed by the method of the movable bottom. The area of the part of S_i used for the covering exceeds $\frac{1}{3}|S_i|$.*

Proof. The part of $\sigma_i R_i$ covered by $\sigma_1 R_1, \ldots, \sigma_{i-1} R_{i-1}$ is of area smaller than

$$\frac{1}{2}\, d_i \cdot d_i + \frac{1}{4}\, d_i \cdot \frac{1}{2}\, d_i + \frac{1}{8}\, d_i \cdot \frac{1}{4}\, d_i + \cdots = \frac{2}{3}\, d_i^2.$$

As a consequence, $|Q_i| > |R_i| - \frac{2}{3}d_i^2 = s_i d_i - \frac{2}{3}d_i^2$. Since

$$\frac{1}{3}\,d_i^2 - d_i d_i + \frac{2}{3}\,d_i^2 = 0,$$

$$\frac{1}{3}\,(2d_i)^2 - 2d_i d_i + \frac{2}{3}\,d_i^2 = 0$$

and $d_i \le s_i < 2d_i$ it follows that

$$\frac{1}{3}\,s_i^2 - s_i d_i + \frac{2}{3}\,d_i^2 \le 0.$$

Consequently,

$$|Q_i| > s_i d_i - \frac{2}{3}\,d_i^2 \ge \frac{1}{3}\,s_i^2 = \frac{1}{3}\,|S_i|. \ \blacksquare$$

THEOREM 1. *Any sequence of squares homothetic to I with total area greater than or equal to 5 permits an on-line translative covering of I by the method of the movable bottom.*

Proof. Let (S_i) be a sequence of homothetic copies of I with total area not smaller than 5. Assume that I cannot be covered with $S_1, S_2, \ldots$ by the method of the movable bottom. Obviously, the side length of each square is smaller than 1.

Since $s_i < 2d_i$ for each positive integer i it follows that the area of $\bigcup \sigma_i R_i \setminus I$ is smaller than

$$\frac{1}{2}\cdot 1 + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{4} + \cdots = \frac{2}{3}.$$

The sets $Q_1, Q_2, \ldots$ are pairwise disjoint and contained in $\bigcup \sigma_i R_i$. This implies that $\sum |Q_i| < 1 + \frac{2}{3} = \frac{5}{3}$. By Lemma 1 we deduce that $\sum |S_i| < 3\sum |Q_i| < 5$, which is a contradiction. $\blacksquare$

3. The method of the movable bottom and immovable top. Let S_i be a homothetic copy of I of side length s_i, for $i = 1, 2, \ldots$. The squares of side length smaller than 1 will be divided into two types: basic squares and special squares. Basic squares will be placed according to the method of the movable bottom. Special squares will be placed into

$$L = \{(x, y);\ x \ge 0,\ 0 \le y \le 1\},$$

side by side, along the top of L.

If $s_1 \ge 1$, then we cover I with S_1 and stop. Otherwise, the first square is *basic*. We place $S_1 \supset R_1$ so that

$$\sigma_1 R_1 = \{(x, y);\ 0 \le x \le d_1,\ 0 \le y \le s_1\}.$$

By the *part of S_1 used for the covering* we mean $Q_1 = \sigma_1 R_1$. Furthermore, we take $U_1 = \sigma_1 R_1$.

Assume that $i > 1$ and that the translations $\sigma_1, \dots, \sigma_{i-1}$ have been provided. Moreover, assume that $U_1, \dots, U_{i-1}$ have been defined. If $s_i \geq 1$, then we cover I with S_i and stop. Assume that $s_i < 1$.

We find the greatest number $b_i \leq 1$ such that each point of I whose y-coordinate is smaller than b_i belongs to $\bigcup_{j=1}^{i-1} U_j$. The set of points of I with y-coordinate b_i is called the ith *bottom*. A point of the ith bottom is a *surface point* if no point of I with the same x-coordinate and with a larger y-coordinate belongs to $\bigcup_{j=1}^{i-1} U_j$.

If $b_i < 1 - s_i$, then S_i is *basic*. We place $S_i \supset R_i$ so that $\sigma_i R_i$ contains a surface point and that $\sigma_i R_i$ has the form

$$\sigma_i R_i = \{(x,y);\ kd_i \leq x \leq (k+1)d_i,\ b_i \leq y \leq b_i + s_i\},$$

where $k \in \{0, 1, \dots d_i^{-1} - 1\}$. Obviously, $\sigma_i R_i \subset I$. We take $U_i = \sigma_i R_i$. The *part of S_i used for the covering* is defined as

$$Q_i = \sigma_i R_i \setminus \bigcup_{j=1}^{i-1} U_j.$$

If $b_i \geq 1 - s_i$, then S_i is *special*. We take $U_i = \emptyset$. If S_i is the first special square in the sequence, then

$$\sigma_i S_i = \{(x,y);\ 0 \leq x \leq s_i,\ 1 - s_i \leq y \leq 1\}.$$

If there is a special square S_j with $j < i$, then denote by k the greatest integer smaller than i such that S_k is special. Let t_k be the number such that

$$\sigma_k S_k = \{(x,y);\ t_k - s_k \leq x \leq t_k,\ 1 - s_k \leq y \leq 1\}.$$

We place S_i so that

$$\sigma_i S_i = \{(x,y);\ t_k \leq x \leq t_k + s_i,\ 1 - s_i \leq y \leq 1\}.$$

Obviously,

$$\sigma_i S_i \supseteq \{(x,y);\ t_k \leq x \leq t_k + s_i,\ b_i \leq y \leq 1\}.$$

This method is called the *method of the movable bottom and immovable top*. It is easy to see that if the point $(1,1)$ has been covered (with a special square), then I has been covered.

THEOREM 2. *The unit square I can be on-line translatively covered with any sequence of homothetic copies of I whose total area is greater than or equal to* 4.

Proof. Let (S_i) be a sequence of homothetic copies of I with total area not smaller than 4. Assume that I cannot be covered with $S_1, S_2, \dots$ by the method of the movable bottom and immovable top. Obviously, the side length of each square is smaller than 1. The parts of the basic squares used for the covering are pairwise disjoint and they are contained in I. By Lemma 1 we deduce that the total area of the basic squares is smaller than $3|I| = 3$. The total area of the special squares is smaller than 1 (even if the sequence of squares is infinite). Consequently, $\sum |S_i| < 3 + 1 = 4$, which is a contradiction. ∎

4. Covering a convex body. We will cover I with squares of side length smaller than $1/2$ in the proof of Theorem 3 below.

Lemma 2. *Any sequence of squares homothetic to I of side length smaller than or equal to s, where $0 < s < 1$, with total area not smaller than $3 + s$ permits an on-line translative covering of I.*

Proof. Let S_i be a homothetic copy of I with a positive homothety ratio not greater that s, for $i = 1, 2, \dots$. Moreover, let $\sum |S_i| \geq 3+s$. Assume that I cannot be covered with $S_1, S_2, \dots$ by the method of the movable bottom and immovable top. The total area of the basic squares is smaller than 3. The total area of the special squares is smaller than s. As a consequence, $\sum |S_i| < 3 + s$, which is a contradiction. ∎

Theorem 3. *Let C be a planar convex body. Any sequence of positive homothetic copies of C with total area not smaller than $14|C|$ permits an on-line translative covering of C.*

Proof. Let C_i be a homothetic copy of C with a ratio $\lambda_i \geq 0$, for $i = 1, 2, \dots$. Assume that $\sum |C_i| \geq 14|C|$. Obviously, if $\lambda_i \geq 1$ for some i, then C can be covered with C_i. Assume that $\lambda_i < 1$ for $i = 1, 2, \dots$.

Let P and T be homothetic parallelograms with the homothety ratio 2 such that $P \subset C \subset T$ (see [1]), and let $\mathcal{A}$ be an affine transformation of the plane such that $\mathcal{A}(T) = I$. Denote by P_i a homothetic copy of P with the ratio λ_i, for $i = 1, 2, \dots$. Each $\mathcal{A}(P_i)$ is a square of side length smaller than $1/2$. Furthermore,

$$\sum |\mathcal{A}(P_i)| \geq 14|\mathcal{A}(P)| = 14 \cdot \frac{1}{4}\, |\mathcal{A}(T)| = 3 + \frac{1}{2}.$$

By Lemma 2, the sequence $(\mathcal{A}(P_i))$ permits an on-line translative covering of I. Since $\mathcal{A}(P_i) \subset \mathcal{A}(C_i)$ it follows that $(\mathcal{A}(C_i))$ permits an on-line translative covering of $I = \mathcal{A}(T) \supset \mathcal{A}(C)$. Consequently, (C_i) permits an on-line translative covering of C. ∎

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Received December 19, 2008;
received in final form March 24, 2009 (7694)