On-line Covering the Unit Square with Squares

by

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Summary. The unit square can be on-line covered with any sequence of squares whose total area is not smaller than 4.

1. Introduction. Let $C, C_1, C_2, \ldots$ be planar convex bodies. We say that the sequence $(C_i)$ permits a covering [a translative covering] of $C$ if there exist rigid motions [translations, respectively] $\sigma_i$ such that $C \subseteq \bigcup \sigma_i C_i$. The on-line covering is a covering in which we are given $C_i$, where $i > 1$, only after the motion $\sigma_{i-1}$ has been provided; at the beginning we are given $C_1$. The on-line restriction means that each set $C_i$ must be assigned its place before the next one appears, and that the placement cannot be modified afterwards. The area of $C$ is denoted by $|C|$.

Denote by $v(C)$ the least number such that any (finite or infinite) sequence of positive homothetic copies of $C$ with total area greater than $v(C)|C|$ permits an on-line translative covering of $C$. Let

$$I = \{(x, y); 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$  

Moon and Moser showed in [8] that any sequence of homothetic copies of $I$ with total area not smaller than 3 permits a translative (off-line) covering of $I$ (see also the survey paper [1]). The bound of 3 cannot be reduced; three squares of side length smaller than 1, each with a side parallel to a side of $I$, cannot translatively cover $I$. Consequently, also $v(I) \geq 3$. Thus far it is still unknown whether or not $v(I) > 3$. The first upper bound presented by Kuperberg in [6] is that $v(I) \leq 16$. According to the method of the current bottom [3] we have $v(I) \leq 8$. By using the method of the $n$th segment [5] we obtain $v(I) \leq 389/60 \approx 6.483$. In [4] the method of the current bottom and

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top is presented. According to that method we have \( v(I) \leq \frac{7}{4} \sqrt[3]{9} + \frac{13}{8} \approx 5.265 \). The aim of this paper is to show that \( v(I) \leq 4 \). As a corollary, we show that \( v(C) \leq 14 \) for any planar convex body \( C \).

A survey of results concerning on-line packings and coverings is given in [7].

2. The method of the movable bottom. Let \( S_1, S_2, \ldots \) be a sequence of squares. Denote by \( s_i \) the side length of \( S_i \) and assume that a side of \( S_i \) is parallel to a side of \( I \) for \( i = 1, 2, \ldots \). We describe an on-line method of translatative covering of \( I \) with \( S_1, S_2, \ldots \). This method is a small modification of the method of the current bottom presented in [3].

If \( s_i < 1 \), then denote by \( d_i \) the number from the set \( \{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \} \) such that

\[
d_i \leq s_i < 2d_i.
\]

Let \( R_i \subset S_i \) be a rectangle of width \( d_i \) and height \( s_i \).

If \( s_1 \geq 1 \), then we cover \( I \) with \( S_1 \) and we stop the covering process. Otherwise, \( S_1 \supset R_1 \) is placed so that

\[
\sigma_1 R_1 = \{(x, y); 0 \leq x \leq d_1, 0 \leq y \leq s_1\}.
\]

By the part of \( S_1 \) used for the covering we mean \( Q_1 = \sigma_1 R_1 \).

Assume that \( i > 1 \) and that the translations \( \sigma_1, \ldots, \sigma_{i-1} \) have been provided. If \( s_i \geq 1 \), then we cover \( I \) with \( S_i \) and stop. Otherwise, we find the greatest number \( b_i \leq 1 \) such that each point of \( I \) whose \( y \)-coordinate is smaller than \( b_i \) has been covered with a rectangle \( \sigma_j R_j \) with \( j < i \). The set of points of \( I \) with \( y \)-coordinate \( b_i \) is called the \( i \)th bottom. A point of the \( i \)th bottom is a surface point if no point of \( I \) with the same \( x \)-coordinate and with a larger \( y \)-coordinate has been covered yet. We place \( S_i \supset R_i \) so that \( \sigma_i R_i \) contains a surface point and that \( \sigma_i R_i \) has the form

\[
\sigma_i R_i = \{(x, y); kd_i \leq x \leq (k + 1)d_i, b_i \leq y \leq b_i + s_i\},
\]

where \( k \in \{0, 1, \ldots, d_i^{-1} - 1\} \). The part of \( S_i \) used for the covering is defined as

\[
Q_i = \sigma_i R_i \setminus (\sigma_1 R_1 \cup \cdots \cup \sigma_{i-1} R_{i-1}).
\]

We stop when \( b_i = 1 \) for an integer \( i \); then \( I \) has been covered with \( \sigma_1 R_1, \ldots, \sigma_{i-1} R_{i-1} \). This method is called the method of the movable bottom.

Lemma 1. Let \( S_i \) be a square of side length smaller than 1 placed by the method of the movable bottom. The area of the part of \( S_i \) used for the covering exceeds \( \frac{1}{3} |S_i| \).

Proof. The part of \( \sigma_i R_i \) covered by \( \sigma_1 R_1, \ldots, \sigma_{i-1} R_{i-1} \) is of area smaller than

\[
\frac{1}{2} d_i \cdot d_i + \frac{1}{4} d_i \cdot \frac{1}{2} d_i + \frac{1}{8} d_i \cdot \frac{1}{4} d_i + \cdots = \frac{2}{3} d_i^2.
\]
As a consequence, $|Q_i| > |R_i| - \frac{2}{3} d_i^2 = s_i d_i - \frac{2}{3} d_i^2$. Since
\[
\frac{1}{3} d_i^2 - d_i d_i + \frac{2}{3} d_i^2 = 0,
\]
and $d_i \leq s_i < 2d_i$ it follows that
\[
\frac{1}{3} s_i^2 - s_i d_i + \frac{2}{3} d_i^2 \leq 0.
\]
Consequently,
\[
|Q_i| > s_i d_i - \frac{2}{3} d_i^2 \geq \frac{1}{3} s_i^2 = \frac{1}{3} |S_i|.
\]

**Theorem 1.** Any sequence of squares homothetic to $I$ with total area greater than or equal to 5 permits an on-line translative covering of $I$ by the method of the movable bottom.

**Proof.** Let $(S_i)$ be a sequence of homothetic copies of $I$ with total area not smaller than 5. Assume that $I$ cannot be covered with $S_1, S_2, \ldots$ by the method of the movable bottom. Obviously, the side length of each square is smaller than 1.

Since $s_i < 2d_i$ for each positive integer $i$ it follows that the area of $\bigcup \sigma_i R_i \setminus I$ is smaller than
\[
\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{4} + \cdots = \frac{2}{3}.
\]
The sets $Q_1, Q_2, \ldots$ are pairwise disjoint and contained in $\bigcup \sigma_i R_i$. This implies that $\sum |Q_i| < 1 + \frac{2}{3} = \frac{5}{3}$. By Lemma 1 we deduce that $\sum |S_i| < 3 \sum |Q_i| < 5$, which is a contradiction.

**3. The method of the movable bottom and immovable top.** Let $S_i$ be a homothetic copy of $I$ of side length $s_i$, for $i = 1, 2, \ldots$. The squares of side length smaller than 1 will be divided into two types: basic squares and special squares. Basic squares will be placed according to the method of the movable bottom. Special squares will be placed into $L = \{(x, y); x \geq 0, 0 \leq y \leq 1\}$, side by side, along the top of $L$.

If $s_1 \geq 1$, then we cover $I$ with $S_1$ and stop. Otherwise, the first square is basic. We place $S_1 \supset R_1$ so that
\[
\sigma_1 R_1 = \{(x, y); 0 \leq x \leq d_1, 0 \leq y \leq s_1\}.
\]
By the part of $S_1$ used for the covering we mean $Q_1 = \sigma_1 R_1$. Furthermore, we take $U_1 = \sigma_1 R_1$. 
Assume that $i > 1$ and that the translations $\sigma_1, \ldots, \sigma_{i-1}$ have been provided. Moreover, assume that $U_1, \ldots, U_{i-1}$ have been defined. If $s_i \geq 1$, then we cover $I$ with $S_i$ and stop. Assume that $s_i < 1$.

We find the greatest number $b_i \leq 1$ such that each point of $I$ whose $y$-coordinate is smaller than $b_i$ belongs to $\bigcup_{j=1}^{i-1} U_j$. The set of points of $I$ with $y$-coordinate $b_i$ is called the $i$th bottom. A point of the $i$th bottom is a surface point if no point of $I$ with the same $x$-coordinate and with a larger $y$-coordinate belongs to $\bigcup_{j=1}^{i-1} U_j$.

If $b_i < 1 - s_i$, then $S_i$ is basic. We place $S_i \supset R_i$ so that $\sigma_i R_i$ contains a surface point and that $\sigma_i R_i$ has the form

$$\sigma_i R_i = \{(x, y); kd_i \leq x \leq (k + 1)d_i, b_i \leq y \leq b_i + s_i\},$$

where $k \in \{0, 1, \ldots, d_i^{-1} - 1\}$. Obviously, $\sigma_i R_i \subset I$. We take $U_i = \sigma_i R_i$. The part of $S_i$ used for the covering is defined as

$$Q_i = \sigma_i R_i \setminus \bigcup_{j=1}^{i-1} U_j.$$

If $b_i \geq 1 - s_i$, then $S_i$ is special. We take $U_i = \emptyset$. If $S_i$ is the first special square in the sequence, then

$$\sigma_i S_i = \{(x, y); 0 \leq x \leq s_i, 1 - s_i \leq y \leq 1\}.$$  

If there is a special square $S_j$ with $j < i$, then denote by $k$ the greatest integer smaller than $i$ such that $S_k$ is special. Let $t_k$ be the number such that

$$\sigma_k S_k = \{(x, y); t_k - s_k \leq x \leq t_k, 1 - s_k \leq y \leq 1\}.$$  

We place $S_i$ so that

$$\sigma_i S_i = \{(x, y); t_k \leq x \leq t_k + s_i, 1 - s_i \leq y \leq 1\}.$$  

Obviously,

$$\sigma_i S_i \supset \{(x, y); t_k \leq x \leq t_k + s_i, b_i \leq y \leq 1\}.$$  

This method is called the method of the movable bottom and immovable top. It is easy to see that if the point $(1, 1)$ has been covered (with a special square), then $I$ has been covered.

**Theorem 2.** The unit square $I$ can be on-line translatively covered with any sequence of homothetic copies of $I$ whose total area is greater than or equal to 4.

**Proof.** Let $(S_i)$ be a sequence of homothetic copies of $I$ with total area not smaller than 4. Assume that $I$ cannot be covered with $S_1, S_2, \ldots$ by the method of the movable bottom and immovable top. Obviously, the side length of each square is smaller than 1. The parts of the basic squares used for the covering are pairwise disjoint and they are contained in $I$. By Lemma 1
we deduce that the total area of the basic squares is smaller than $3|I| = 3$. The total area of the special squares is smaller than 1 (even if the sequence of squares is infinite). Consequently, $\sum |S_i| < 3 + 1 = 4$, which is a contradiction. 

4. Covering a convex body. We will cover $I$ with squares of side length smaller than $1/2$ in the proof of Theorem 3 below.

**Lemma 2.** Any sequence of squares homothetic to $I$ of side length smaller than or equal to $s$, where $0 < s < 1$, with total area not smaller than $3 + s$ permits an on-line translative covering of $I$.

**Proof.** Let $S_i$ be a homothetic copy of $I$ with a positive homothety ratio not greater than $s$, for $i = 1, 2, \ldots$. Moreover, let $\sum |S_i| \geq 3 + s$. Assume that $I$ cannot be covered with $S_1, S_2, \ldots$ by the method of the movable bottom and immovable top. The total area of the basic squares is smaller than 3. The total area of the special squares is smaller than $s$. As a consequence, $\sum |S_i| < 3 + s$, which is a contradiction. 

**Theorem 3.** Let $C$ be a planar convex body. Any sequence of positive homothetic copies of $C$ with total area not smaller than $14|C|$ permits an on-line translative covering of $C$.

**Proof.** Let $C_i$ be a homothetic copy of $C$ with a ratio $\lambda_i \geq 0$, for $i = 1, 2, \ldots$. Assume that $\sum |C_i| \geq 14|C|$. Obviously, if $\lambda_i \geq 1$ for some $i$, then $C$ can be covered with $C_i$. Assume that $\lambda_i < 1$ for $i = 1, 2, \ldots$. 

Let $P$ and $T$ be homothetic parallelograms with the homothety ratio 2 such that $P \subset C \subset T$ (see [1]), and let $A$ be an affine transformation of the plane such that $A(T) = I$. Denote by $P_i$ a homothetic copy of $P$ with the ratio $\lambda_i$, for $i = 1, 2, \ldots$. Each $A(P_i)$ is a square of side length smaller than $1/2$. Furthermore,

$$\sum |A(P_i)| \geq 14|A(P)| = 14 \cdot \frac{1}{4} |A(T)| = 3 + \frac{1}{2}.$$ 

By Lemma 2, the sequence $(A(P_i))$ permits an on-line translative covering of $I$. Since $A(P_i) \subset A(C_i)$ it follows that $(A(C_i))$ permits an on-line translative covering of $I = A(T) \supset A(C)$. Consequently, $(C_i)$ permits an on-line translative covering of $C$. 

**References**


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