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PROBABILITY THEORY AND STOCHASTIC PROCESSES

Weak Type Inequality for the Square Function of a Nonnegative Submartingale

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Summary. Let f be a nonnegative submartingale and $S(f)$ denote its square function. We show that for any $\lambda > 0$,

$$\lambda\mathbb{P}(S(f) \geq \lambda) \leq \frac{\pi}{2}\, \|f\|_1,$$

and the constant $\pi/2$ is the best possible. The inequality is strict provided $\|f\|_1 \neq 0$.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n=0}^{\infty}$, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Assume $f = (f_n)_{n=0}^{\infty}$ is an adapted sequence of integrable real-valued random variables. The difference sequence $df = (df_n)_{n=0}^{\infty}$ of f is given by the equations $df_0 = f_0$ and $df_n = f_n - f_{n-1}$, $n = 1, 2, \ldots$. We define the *square function* of f by

$$S(f) = \Big(\sum_{k=0}^{\infty} |df_k|^2\Big)^{1/2}.$$

We will also use the notation

$$S_n(f) = \Big(\sum_{k=0}^{n} |df_k|^2\Big)^{1/2}$$

and write $\|f\|_p = \sup_n \|f_n\|_p$ for $p \geq 1$.

In the present paper we deal with weak type inequalities for the square function. As shown by Burkholder [2], if f is a martingale or nonnegative submartingale, then

$$\lambda\mathbb{P}(S(f) \geq \lambda) \leq 3\|f\|_1. \tag{1.1}$$

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Cox [5] showed that the best constant in the above inequality for real-valued martingales f equals $\sqrt{e}$ (it is worth mentioning that in the earlier paper [1] Bollobás conjectures that this is the right choice). The purpose of this note is to determine the optimal constant in (1.1) under the assumption that f is a nonnegative submartingale.

THEOREM 1. *If f is a nonnegative submartingale, then for any $\lambda > 0$,*

$$\lambda \mathbb{P}(S(f) \geq \lambda) \leq \frac{\pi}{2} \|f\|_1, \tag{1.2}$$

and the constant $\pi/2$ is the best possible. Furthermore, the inequality is strict unless $\|f\|_1 = 0$.

A few words about the organization of the paper. The proof of the inequality (1.2) is based on Burkholder's method, which translates the problem of proving a given (sub-)martingale inequality to the problem of finding a certain special function (for the description of the method, see e.g. [4] or [6]). We construct the function and thus establish (1.2) in Section 2. In the last section we show that the constant $\pi/2$ cannot be replaced by a smaller one and that (1.2) is strict in all nontrivial cases.

2. The proof of the inequality (1.2).

Let us start with the following auxiliary result.

LEMMA 1. *For any $x \in (0,1)$ and $d > -x$ such that $(x+d)^2 + d^2 < 1$ we have*

$$\frac{\sqrt{1-x^2} - \sqrt{1-(x+d)^2-d^2}}{x+d} + \arcsin x - \arcsin \frac{x+d}{\sqrt{1-d^2}} \leq 0. \tag{2.1}$$

Proof. Denote the left hand side of (2.1) by $F(x,d)$. If we fix d and differentiate with respect to x, we obtain

$$\begin{aligned} F_x(x,d)(x+d)^2 &= \sqrt{1-(x+d)^2-d^2} - \sqrt{1-x^2} + \frac{d(x+d)}{\sqrt{1-x^2}} \\ &= \sqrt{1-x^2-2d(x+d)} - \sqrt{1-x^2} - \frac{-2d(x+d)}{2\sqrt{1-x^2}}, \end{aligned}$$

which is nonnegative, due to the concavity of the function $t \mapsto \sqrt{t}$. Therefore the inequality $F(x,d) \leq 0$ will be established once we have shown that $F(-d+,d) < 0$ for $d < 0$ and $F(0+,d) \leq 0$ for $d \geq 0$. Suppose first that $d < 0$. Then

$$F(-d+,d) = \frac{d}{\sqrt{1-d^2}} + \arcsin(-d) = \int_0^{-d} \left(\frac{1}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-d^2}} \right) ds < 0.$$

If $d = 0$, then $F(x,d) = 0$ for any x. Finally, if $d > 0$, then

$$F(0+, d) = \frac{1-\sqrt{1-2d^2}}{d} - \arcsin\frac{d}{\sqrt{1-d^2}} \tag{2.2}$$
$$= \int_0^d \frac{\sqrt{1-2s^2}-1}{(1-s^2)(1+\sqrt{1-2s^2})}\, ds < 0.$$

The proof is complete. ∎

The crucial role in this paper is played by the functions $U, V : [0,\infty) \times [0,\infty) \to \mathbb{R}$, given by

$$U(x,y) = \begin{cases} 1-\sqrt{1-x^2-y^2} - x\arcsin\dfrac{x}{\sqrt{1-y^2}} & \text{if } x^2+y^2 < 1,\\ 1-\pi x/2 & \text{if } x^2+y^2 \geq 1, \end{cases}$$

and $V(x,y) = I_{\{y \geq 1\}} - \pi x/2$.

The key properties of these functions are listed in the lemma below.

Lemma 2. *The functions U, V have the following properties.*

(i) *U is of class C^1 on $(0,\infty) \times (0,\infty)$.*

(ii) *For any $x, y \geq 0$, we have*

$$U_x(x,y) \leq 0 \tag{2.3}$$

(*if $x = 0$, then we understand $U_x(0,y)$ as the limit $U_x(0+,y)$*).

(iii) *For any $x, y \geq 0$,*

$$U(x,y) \geq V(x,y) \tag{2.4}$$

and

$$U(x,y) \leq 1-\pi x/2. \tag{2.5}$$

(iv) *For any $x, y \geq 0$ and $d \geq -x$ we have*

$$U(x+d, \sqrt{y^2+d^2}) \leq U(x,y) + U_x(x,y)d \tag{2.6}$$

(*again, if $x = 0$, then the partial derivative is understood as the limit*).

(v) *For any $x \geq 0$,*

$$U(x,x) \leq 0. \tag{2.7}$$

Furthermore, the inequality is strict if $x > 0$.

Proof. (i) A direct computation shows that

$$U_x(x,y) = \begin{cases} -\arcsin\dfrac{x}{\sqrt{1-y^2}} & \text{if } x^2+y^2 < 1,\\ -\pi/2 & \text{if } x^2+y^2 \geq 1, \end{cases} \tag{2.8}$$

and

$$U_y(x,y) = \begin{cases} \dfrac{y\sqrt{1-x^2-y^2}}{1-y^2} & \text{if } x^2+y^2<1, \\ 0 & \text{if } x^2+y^2 \geq 1. \end{cases}$$

Now it can be easily verified that both derivatives are continuous on $(0,\infty)\times(0,\infty)$.

(ii) This follows immediately from the formula for U_x above.

(iii) Clearly, it suffices to show the inequalities on the set $\{(x,y): x,y>0, x^2+y^2<1\}$. By (2.8) we have, for (x,y) in this set,

$$\frac{\partial}{\partial x}\left(U(x,y)+\frac{\pi}{2}x\right) = \frac{\pi}{2} - \arcsin\frac{x}{\sqrt{1-y^2}} \geq 0.$$

Hence

$$U(x,y)-V(x,y) \geq U(0,y)-V(0,y) = 1-\sqrt{1-y^2} \geq 0$$

and

$$U(x,y)+\frac{\pi}{2}x \leq U(\sqrt{1-y^2},y)+\frac{\pi}{2}\sqrt{1-y^2} = 1.$$

(iv) The inequality is easy if $x^2+y^2 \geq 1$: indeed, we have

$$U(x,y)+U_x(x,y)d = 1-\frac{\pi}{2}(x+d) \geq U(x+d,\sqrt{y^2+d^2}),$$

the latter estimate being a consequence of (2.5). Suppose then that $x^2+y^2<1$. If $(x+d)^2+(\sqrt{y^2+d^2})^2<1$, then the inequality (2.6) takes the form

$$-\sqrt{1-(x+d)^2-y^2-d^2}-(x+d)\arcsin\frac{x+d}{\sqrt{1-y^2-d^2}} \leq \sqrt{1-x^2-y^2}-(x+d)\arcsin\frac{x}{\sqrt{1-y^2}}.$$

The first observation is that we may assume that $y=0$: indeed, if this is not the case, divide both sides by $\sqrt{1-y^2}$ and substitute $x := x/\sqrt{1-y^2}$, $d := d/\sqrt{1-y^2}$. The second step is to note that, by continuity, we may assume $x+d>0$. Then the desired estimate is precisely (2.1). The only remaining case is that $x^2+y^2<1$ and $(x+d)^2+(\sqrt{y^2+d^2})^2 \geq 1$; then the inequality (2.6) is equivalent to

$$\sqrt{1-x^2-y^2}+(x+d)\left(\frac{\pi}{2}-\arcsin\frac{x}{\sqrt{1-y^2}}\right)-1 \geq 0.$$

It is clear that it suffices to prove it for the least possible d, i.e., satisfying $d \geq 0$ and $(x+d)^2+(\sqrt{y^2+d^2})^2=1$. However, then the estimate follows from continuity and the case $x^2+y^2<1$, $(x+d)^2+(\sqrt{y^2+d^2})^2<1$ already considered.

(v) This is a consequence of (iv): let $x = y = 0$ to obtain $U(d,d) \leq U(0,0) + U_x(0+,0)d = U(0,0) = 0$. Furthermore, for $d > 0$ the inequality is strict: this is precisely (2.2). ∎

Now we are ready to prove the main estimate of the paper.

Proof of (1.2). Let f be any nonnegative submartingale. By homogeneity, it suffices to show (1.2) for $\lambda = 1$ only. First we will show that the process $(U(f_n, S_n(f)))_{n=0}^{\infty}$ is a supermartingale. To this end, fix $n \geq 1$ and observe that, by (2.6),

$$\begin{aligned} U(f_n, S_n(f)) &= U(f_{n-1} + df_n, \sqrt{S_{n-1}(f) + |df_n|^2}) \\ &\leq U(f_{n-1}, S_{n-1}(f)) + U_x(f_{n-1}, S_{n-1}(f))df_n \end{aligned}$$

Both sides are integrable: indeed, one easily checks that $|U(x,y)| \leq K + \pi x/2$ for some absolute constant K; furthermore, $U_x(x,y)$ is bounded, in view of (2.8). Therefore, applying the conditional expectation with respect to $\mathcal{F}_{n-1}$ and using (2.3) together with the submartingale property yields

$$\begin{aligned} \mathbb{E}[U(f_n, S_n(f)) \,|\, \mathcal{F}_{n-1}] &\leq U(f_{n-1}, S_{n-1}(f)) + U_x(f_{n-1}, S_{n-1}(f))\mathbb{E}(df_n \,|\, \mathcal{F}_{n-1}) \\ &\leq U(f_{n-1}, S_{n-1}(f)). \end{aligned}$$

Combined with (2.4), this will imply the inequality (1.2) for the submartingales f of finite length (that is, satisfying $\mathbb{P}(df_n = df_{n+1} = \cdots = 0) = 1$ for some n). Namely, for any $n = 0, 1, 2, \ldots$, we write

$$\begin{aligned} \mathbb{P}(S_n(f) \geq 1) - \frac{\pi}{2}\mathbb{E}f_n &= \mathbb{E}V(f_n, S_n(f)) \\ &\leq \mathbb{E}U(f_n, S_n(f)) \leq \mathbb{E}U(f_0, S_0(f)) \leq 0, \end{aligned} \tag{2.9}$$

where in the last passage we have used the equality $f_0 = S_0(f)$ and the inequality (2.7). The final step is to let $n \to \infty$: for any $\varepsilon > 0$, we have, by (2.9) applied to the submartingale $f/(1-\varepsilon)$,

$$\begin{aligned} \mathbb{P}(S(f) \geq 1) &\leq \lim_{n\to\infty} \mathbb{P}(S_n(f) \geq 1 - \varepsilon) \\ &\leq \lim_{n\to\infty} \frac{\pi}{2(1-\varepsilon)}\mathbb{E}f_n \leq \frac{\pi}{2(1-\varepsilon)}\|f\|_1. \end{aligned} \tag{2.10}$$

Now let $\varepsilon \to 0$ to complete the proof. ∎

3. Strictness and sharpness

3.1. *Strictness.* Suppose $\|f\|_1 > 0$ and observe that if this is the case, then with no loss of generality we may assume that $\mathbb{P}(f_0 > 0) > 0$. Arguing as in (2.9) and (2.10), we obtain

$$\mathbb{P}(S(f) \geq 1) \leq \frac{\pi}{2}\|f\|_1 + \mathbb{E}U(f_0, S_0(f)).$$

It suffices to note that since $f_0 = S_0(f)$ almost surely, it follows that $\mathbb{E}U(f_0, S_0(f)) < 0$, by Lemma 2(v). This yields the claim.

3.2. *Sharpness.* Throughout this subsection we assume that the underlying probability space is the interval $[0,1]$ equipped with its Borel subsets and Lebesgue's measure. We will show that the constant is optimal even if we restrict ourselves to the submartingales f satisfying $S(f)\geq 1$ almost surely. One could show this by giving appropriate examples; however, we take the opportunity here to provide a different proof.

Recall that the process f is called *simple* if it is of finite length (hence its limit f_∞ exists almost surely) and for any n the variable f_n takes only a finite number of values. For any (x,y), let $Z(x,y)$ be the class of all nonnegative simple submartingales f for which $f_0=x$ and $y^2-x^2+S^2(f)\geq 1$ almost surely. Here the filtration is no longer fixed—it may be different for different submartingales.

LEMMA 3. *Let the function* $W:[0,\infty)\times[0,\infty)\to\mathbb{R}$ *be given by*

$$W(x,y)=\inf_{f\in Z(x,y)}\mathbb{E}f_\infty.$$

The function W *has the following properties*:

(i) *For all* $x\geq 0$, $y\in[0,1)$,

$$W(x,y)=\sqrt{1-y^2}\,W(x/\sqrt{1-y^2},0). \tag{3.1}$$

(ii) *For all* $x,y,d\geq 0$,

$$W(x+d,\sqrt{y^2+d^2})\geq W(x,y). \tag{3.2}$$

(iii) *For all* $x,y\geq 0$, $\alpha\in(0,1)$ *and any* $d_1,d_2\geq -x$ *satisfying* $\alpha d_1+(1-\alpha)d_2=0$,

$$\alpha W\Big(x+d_1,\sqrt{y^2+d_1^2}\Big)+(1-\alpha)W\Big(x+d_2,\sqrt{y^2+d_2^2}\Big)\geq W(x,y). \tag{3.3}$$

Proof. (i) Suppose f is a simple nonnegative submartingale. Then f lies in $Z(x,y)$ if and only if $f'=f/\sqrt{1-y^2}$ belongs to the class $Z(x/\sqrt{1-y^2},0)$; indeed, $f_0=x$ is equivalent to $f_0'=x/\sqrt{1-y^2}$, and furthermore

$$y^2-x^2+S^2(f)\geq 1$$

is equivalent to

$$-\frac{x^2}{1-y^2}+S^2(f')\geq 1.$$

This implies

$$\begin{aligned}W(x,y)=\inf_{f\in Z(x,y)}\mathbb{E}f_\infty&=\inf_{f'\in Z(x/\sqrt{1-y^2},0)}\mathbb{E}\sqrt{1-y^2}f'_\infty\\&=\sqrt{1-y^2}\,W(x/\sqrt{1-y^2},0).\end{aligned}$$

(ii) Suppose $f \in Z(x+d, \sqrt{y^2+d^2})$ and consider a sequence f' such that, with probability 1, $f'_0 = x$, $df'_1 = d$ and $df'_{n+1} = df_n$ for $n = 1, 2, \ldots$. Since $d \geq 0$, f' is a simple submartingale (with respect to its natural filtration) and

$$y^2 - x^2 + S^2(f') = y^2 + d^2 + \sum_{n=2}^{\infty} |df'_n|^2 = y^2 + d^2 - (x+d)^2 + S^2(f) \geq 1.$$

Hence $f' \in Z(x, y)$ and since $f'_n = f_{n-1}$ for $n = 1, 2, \ldots$, we have

$$W(x,y) \leq \mathbb{E} f'_\infty = \mathbb{E} f_\infty.$$

As $f \in Z(x+d, \sqrt{y^2+d^2})$ was arbitrary, (3.2) follows.

(iii) We will use the so-called "splicing" argument; see e.g. [3] for details. Let $f^{(1)}$, $f^{(2)}$ be two submartingales belonging to $Z(x+d_1, \sqrt{y^2+d_1^2})$ and $Z(x+d_2, \sqrt{y^2+d_2^2})$, respectively. Consider the process f such that (recall that $\Omega = [0,1]$)

$$f_0 = xI_{[0,1]}, \qquad df_1 = d_1 I_{[0,\alpha]} + d_2 I_{(\alpha,1]}$$

and, for $\omega \in \Omega$,

$$df_n(\omega) = df^{(1)}_{n-1}(\omega/\alpha) I_{[0,\alpha]}(\omega) + df^{(2)}_{n-1}((\omega-\alpha)/(1-\alpha)) I_{(\alpha,1]}(\omega)$$

for $n = 2, 3, \ldots$. It can be verified easily that f is a simple nonnegative submartingale such that $y^2 - x^2 + S^2(f)(\omega)$ equals

$$\begin{aligned}&[y^2 + d_1^2 - (x+d_1)^2 + S^2(f^{(1)})(\omega/\alpha)] I_{[0,\alpha]}(\omega)\\ &\qquad + [y^2 + d_2^2 - (x+d_2)^2 + S^2(f^{(2)})((\omega-\alpha)/(1-\alpha))] I_{(\alpha,1]}(\omega) \geq 1.\end{aligned}$$

Thus $f \in Z(x,y)$. Moreover, by the construction, we have

$$f_\infty(\omega) = f^{(1)}_\infty(\omega/\alpha) + f^{(2)}_\infty((\omega-\alpha)/(1-\alpha)),$$

so

$$W(x,y) \leq \mathbb{E} f_\infty = \alpha \mathbb{E} f^{(1)}_\infty + (1-\alpha)\mathbb{E} f^{(2)}_\infty,$$

and since $f^{(1)}$, $f^{(2)}$ were arbitrary, the inequality (3.3) is satisfied. ∎

The lemma above is the tool to show that $\pi/2$ in (1.2) is the best possible.

Sharpness of (1.2). In terms of the function W, the proof will be complete if we show that $W(0,0) \leq 2/\pi$. Let N be a fixed (large) integer and $\delta = 1/(N+1)$. By (3.2), applied to $x = y = 0$ and $d = \delta$, we have

$$W(0,0) \leq W(\delta, \delta). \tag{3.4}$$

Now, for $n \in \{1, \ldots, N\}$, use (3.3) with $x = n\delta$, $y = \sqrt{n}\,\delta$, $d_1 = -n\delta$, $d_2 = \delta$

and $\alpha = 1/(n+1)$ to obtain

$$W(n\delta, \sqrt{n}\,\delta) \leq \frac{W(0, \sqrt{n\delta^2 + n^2\delta^2})}{n+1} + \frac{nW((n+1)\delta, \sqrt{n+1}\,\delta)}{n+1}$$
$$= \frac{\sqrt{1 - n\delta^2 - n^2\delta^2}}{n+1} W(0,0) + \frac{nW((n+1)\delta, \sqrt{n+1}\,\delta)}{n+1},$$

where in the last passage we have exploited (3.1). This inequality yields

$$\frac{W(n\delta, \sqrt{n}\,\delta)}{n} - \frac{W((n+1)\delta, \sqrt{n+1}\,\delta)}{n+1} \leq \frac{\sqrt{1 - n^2\delta^2}}{n(n+1)} W(0,0),$$

and combining this with (3.4), we get

$$W(0,0) \leq \frac{W((N+1)\delta, \sqrt{N+1}\,\delta)}{N+1} + W(0,0) \sum_{n=1}^{N} \frac{\sqrt{1 - n^2\delta^2}}{n(n+1)}. \tag{3.5}$$

Now we make two observations. First, we have $W((N+1)\delta, \sqrt{N+1}\,\delta) = W(1, \sqrt{\delta}) = 1$. To see this, note that for any submartingale $f \in Z(1, \sqrt{\delta})$ we have $\mathbb{E}f_\infty \geq \mathbb{E}f_0 = 1$, so $W(1, \sqrt{\delta}) \geq 1$. On the other hand, the martingale f starting from 1 such that $df_1 = -I_{[0,1/2)} + I_{[1/2,1]}$ and $df_n = 0$ for $n \geq 2$, belongs to $Z(1, \sqrt{\delta})$ and satisfies $\mathbb{E}f_\infty = \mathbb{E}f_0 = 1$. The second observation is that

$$\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}.$$

Therefore, (3.5) can be rewritten in the form

$$W(0,0) \leq 1 + W(0,0) \sum_{n=1}^{N} \delta\, \frac{\sqrt{1 - n^2\delta^2} - 1}{n\delta(n+1)\delta}.$$

Now if we let $N \to \infty$ (so $\delta \to 0$), then the sum above converges to $\int_0^1 (\sqrt{1-x^2} - 1)x^{-2}\,dx = 1 - \pi/2$ and then the inequality becomes $W(0,0) \leq 2/\pi$. This completes the proof. ∎

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