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Topological Pressure for One-Dimensional Holomorphic Dynamical Systems

by

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Summary. For a class of one-dimensional holomorphic maps f of the Riemann sphere we prove that for a wide class of potentials φ the topological pressure is entirely determined by the values of φ on the repelling periodic points of f. This is a version of a classical result of Bowen for hyperbolic diffeomorphisms in the holomorphic non-uniformly hyperbolic setting.

1. Introduction. In this paper we study the topological pressure $P_{\rm top}(\varphi) = P_{\rm top}(f,\varphi)$ of a continuous potential φ with respect to a one-dimensional holomorphic dynamical system f. To simplify the exposition we discuss in the introduction exclusively the case when f is a rational map of the Riemann sphere and present our more general results later on. Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational map of degree $d\geq 2$, and let J denote the Julia set of f, i.e. the closure of the repelling periodic points of f (see [2] for details). We are interested in the topological pressure with respect to the dynamical system $f|J$.

We denote by $\mathrm{Per}_n(f)$ the fixed points of f^n in J and by $\mathrm{Per}(f)=\bigcup_n \mathrm{Per}_n(f)$ the periodic points of f in J. Moreover, let $\mathrm{Per}_{\rm rep}(f)\subset \mathrm{Per}(f)$ denote the set of repelling periodic points of f. Given $\alpha>0$, $0<c\leq 1$, and $n\in\mathbb{N}$ we define

$$\mathrm{Per}_n(\alpha,c)=\{z\in \mathrm{Per}_n(f) : |(f^k)'(f^i(z))|\geq c\exp(k\alpha) \text{ for all } k\in\mathbb{N} \text{ and } 0\leq i\leq n-1\}. \tag{1}$$

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Thus, if $\alpha \geq \alpha'$, $c \geq c'$, then

$$\mathrm{Per}_n(\alpha, c) \subset \mathrm{Per}_n(\alpha', c') \tag{2}$$

and

$$\mathrm{Per}_{\mathrm{rep}}(f) = \bigcup_{\alpha>0} \bigcup_{c>0} \bigcup_{n=1}^{\infty} \mathrm{Per}_n(\alpha, c).$$

Let $\mathcal{M}$ denote the set of all f-invariant Borel probability measures on J endowed with the weak* topology. This makes $\mathcal{M}$ into a compact convex space. Moreover, let $\mathcal{M}_{\mathrm{E}} \subset \mathcal{M}$ be the subset of ergodic measures. For $\mu \in \mathcal{M}_{\mathrm{E}}$ we define the Lyapunov exponent of μ by

$$\chi(\mu) = \int \log |f'| \, d\mu. \tag{3}$$

It follows from Birkhoff's ergodic theorem that the *pointwise Lyapunov exponent* at z, which is defined by

$$\chi(z) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(z)|, \tag{4}$$

exists for μ-a.e. $z \in J$ and coincides (whenever it exists) with $\chi(\mu)$. We say that a measure μ is *hyperbolic* if $\chi(\mu) > 0$. We denote by $h_\mu(f)$ the measure-theoretic entropy of f with respect to μ (see for example [15] for the definition). Moreover, we denote by $P_{\mathrm{top}}(\varphi)$ the topological pressure of φ with respect to f (see Section 2.2 for the definition). For $\varphi \in C(J, \mathbb{R})$ we define

$$\alpha(\varphi) = \sup\{\chi(\mu) : \mu \in \mathcal{M}_{\mathrm{E}} \cap \mathrm{ES}(\varphi)\}, \tag{5}$$

where $\mathrm{ES}(\varphi)$ denotes the set of equilibrium states of φ, i.e. of measures $\mu \in \mathcal{M}$ satisfying $P_{\mathrm{top}}(\varphi) = h_\mu(f) + \int \varphi \, d\mu$ ([1]). We note that it follows from a general result of Newhouse [10] (or alternatively from a theorem of Lyubich [9] or from Freire *et al.* [5] in the case of rational maps) that for all $\varphi \in C(J, \mathbb{R})$ we have $\mathrm{ES}(\varphi) \cap \mathcal{M}_{\mathrm{E}} \neq \emptyset$. Our main goal in this paper is to prove the following result (for a more general case of not necessarily rational maps see Theorem 2 below):

THEOREM 1. *Let* $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *be a rational map of and let* $\varphi \in C(J, \mathbb{R})$ *be a Hölder continuous potential.*

(a) *If* $\alpha(\varphi) > 0$ *then for all* $0 < \alpha < \alpha(\varphi)$ *we have*

$$P_{\mathrm{top}}(\varphi) = \lim_{c\to 0} \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{z\in \mathrm{Per}_n(\alpha,c)} \exp\Big(\sum_{k=0}^{n-1} \varphi(f^k(z))\Big)\Big). \tag{6}$$

([1]) Note that the supremum in (5) is in fact a maximum. This follows from the fact that $\mathrm{ES}(\varphi)$ is a non-empty compact convex set whose extremal points are precisely the ergodic measures.

(b) *If* (6) *is true for some* $\alpha > 0$ *then there exists an ergodic equilibrium state* μ *of* φ *with* $\chi(\mu) \geq \alpha$.

We note that Theorem 1 generalizes a well-known result of Bowen for Axiom A diffeomorphisms to the case of holomorphic non-uniformly expanding dynamical systems. For a related result in the case of non-uniformly hyperbolic diffeomorphisms we refer to [6].

We briefly mention work where related assumptions on the potentials have been used. Note that the prerequisite of Theorem 1 is satisfied if $P_{\rm top}(\varphi) > \max_{z\in J}\varphi(z)$ and in particular if $\max_{z\in J}\varphi(z) - \min_{z\in J}\varphi(z) < h_{\rm top}(f|J)$ are satisfied, which are (much stronger than $\alpha(\varphi) > 0$) open conditions in the C^0 topology. The latter condition has been mentioned first in [7] in the context of piecewise monotonic maps of the unit interval and of a bounded variation potential φ to guarantee the existence and good ergodic properties of equilibrium states for φ, using a spectral gap approach. In [4], it is shown that for a rational map of degree ≥ 2 on the Riemann sphere for a Hölder continuous potential φ satisfying $P_{\rm top}(\varphi) > \sup\varphi$, there is a unique equilibrium state for φ. Analogous results are obtained for a class of non-uniformly expanding local diffeomorphisms and Hölder continuous potentials satisfying such a low oscillation condition (see [1] and references therein).

Przytycki *et al.* [12] consider a pressure of the potential $-t\log|f'|$ which is defined as in (6) except that they use *all* periodic points rather than only points in $\mathrm{Per}_n(\alpha, c)$. They prove the equality between this pressure and various other types of pressures in the case of rational maps satisfying an additional hypothesis that not too many periodic orbits with Lyapunov exponent close to 1 move close together (which is satisfied if f is a topological Collet–Eckmann map or, equivalently, if f is uniformly expanding on periodic orbits). It would be interesting to know under which conditions their pressure coincides with the pressure in (6).

This paper is organized as follows. In Section 2 we introduce a class of one-dimensional holomorphic (not necessarily rational) dynamical systems and discuss various notions of topological pressure. In Section 3 we prove our main result showing that for this class of systems the topological pressure is entirely determined by the values of the potential on the repelling periodic points.

2. Preliminaries

2.1. *A class of one-dimensional holomorphic dynamical systems.* Let $X \subset \overline{\mathbb{C}}$ be compact and let $f : X \to X$ be continuous. We say that $f \in \mathcal{A}(X)$ if there is an open neighborhood U of X such that f extends to a holomorphic map on U and for every $z \in U \setminus X$,

$$\text{(7)} \qquad \begin{aligned} &\text{either } z \text{ leaves } U \text{ under iteration of } f,\\ &\text{or } \liminf_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)| = 0. \end{aligned}$$

Without further mention we will always use a specific set U associated with X and f and we will also denote the extension of f to U by f. We note that in the particular case when f is a rational map on $\overline{\mathbb{C}}$ with Julia set J, a normal family argument shows that $f \in \mathcal{A}(J)$. For $f \in \mathcal{A}(X)$ we will continue to use the notation from Section 1 (e.g. $\mathrm{Per}(f)$, $\mathrm{Per}_{\mathrm{rep}}(f)$, $\mathrm{Per}_n(\alpha, c)$, $\mathcal{M}$, $\mathcal{M}_{\mathrm{E}}$, $\chi(\mu)$, $\chi(z)$, $\alpha(\varphi)$, etc.) for $f|X$.

Let now $U \subset \overline{\mathbb{C}}$ be open and $f : U \to \overline{\mathbb{C}}$ be holomorphic. We say that f is *expanding* on a compact f-invariant set $\Lambda \subset U$ if there exist constants $c > 0$ and $\beta > 1$ such that

$$|(f^n)'(z)| \geq c\beta^n$$

for all $n \in \mathbb{N}$ and all $z \in \Lambda$. We note that for $f \in \mathcal{A}(X)$ every invariant expanding set $\Lambda \subset U$ is contained in X. This follows from (7).

2.2. *Various pressures.* We first recall the definition of the classical topological pressure. Let (X, d) be a compact metric space and let $f\colon X \to X$ be a continuous map. For $n \in \mathbb{N}$ we define a new metric d_n on X by $d_n(z, y) = \max_{k=0,\dots,n-1} d(f^k(z), f^k(y))$. A set $\{z_i\colon i \in I\} \subset X$ is called (n, ε)*-separated* (with respect to f) if $d_n(z_i, z_j) > \varepsilon$ for all z_i, z_j with $z_i \neq z_j$. For all $\varepsilon > 0$ and $n \in \mathbb{N}$ fix a maximal (with respect to the inclusion) (n, ε)-separated set $F_n(\varepsilon)$. The *topological pressure* (with respect to $f|X$) is a map $P_{\mathrm{top}}(f|X, \cdot)\colon C(X, \mathbb{R}) \to \mathbb{R}$ defined by

$$\text{(8)} \qquad P_{\mathrm{top}}(f|X, \varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{z\in F_n(\varepsilon)} \exp S_n\varphi(z)\Big),$$

where

$$\text{(9)} \qquad S_n\varphi(z) = \sum_{k=0}^{n-1} \varphi(f^k(z)).$$

The *topological entropy* of f is defined by $h_{\mathrm{top}}(f|X) = P_{\mathrm{top}}(f|X, 0)$. For simplicity we write $P_{\mathrm{top}}(\varphi)$ if there is no confusion about f and X. Note that the definition of $P_{\mathrm{top}}(\varphi)$ does not depend on the choice of the sets $F_n(\varepsilon)$ (see [15]). The topological pressure satisfies the following variational principle:

$$\text{(10)} \qquad P_{\mathrm{top}}(\varphi) = \sup_{\nu\in\mathcal{M}} \Big(h_\nu(f) + \int_\Lambda \varphi\, d\nu\Big).$$

Furthermore, the supremum in (10) can be taken only over all $\nu \in \mathcal{M}_{\mathrm{E}}$. We denote by $\mathrm{ES}(\varphi)$ the set of *equilibrium states* for φ, that is, of measures attaining the supremum in (10). We note that in general $\mathrm{ES}(\varphi)$ may be empty; however, if $f \in \mathcal{A}(X)$, then $\mathrm{ES}(\varphi)$ contains at least one (ergodic) measure. This follows from a result of Newhouse [10].

Next we introduce a pressure-like quantity by using the values of φ on the periodic points in X. Let $\varphi \in C(X,\mathbb{R})$ and let $0 < \alpha$, $0 < c \le 1$. We define

$$Q_{\mathrm{P}}(\varphi,\alpha,c,n) = \sum_{z\in \mathrm{Per}_n(\alpha,c)} \exp S_n\varphi(z)$$

if $\mathrm{Per}_n(\alpha,c) \neq \emptyset$, and

$$Q_{\mathrm{P}}(\varphi,\alpha,c,n) = \exp(n \min_{z\in X} \varphi(z))$$

otherwise. Furthermore, we define

$$P_{\mathrm{P}}(\varphi,\alpha,c) = \limsup_{n\to\infty} \frac{1}{n} \log Q_{\mathrm{P}}(\varphi,\alpha,c,n).$$

It follows from the definition that if $\mathrm{Per}_n(\alpha,c) \neq \emptyset$ for some $n \in \mathbb{N}$ then this is true for infinitely many $n \in \mathbb{N}$. Therefore, in the case when $\mathrm{Per}_n(\alpha,c) \neq \emptyset$ for some $n \in \mathbb{N}$ then $P_{\mathrm{P}}(\varphi,\alpha,c)$ is entirely determined by the values of φ on $\bigcup_{n\in\mathbb{N}} \mathrm{Per}_n(\alpha,c)$.

3. Pressure equals periodic point pressure. In this section we show for $f \in \mathcal{A}(X)$ and a rather general class of potentials that the topological pressure is entirely determined by the values of the potential on the repelling periodic points. More precisely, we prove the following theorem.

THEOREM 2. *Let $f \in \mathcal{A}(X)$ and let $\varphi \in C(X,\mathbb{R})$ be a Hölder continuous potential with $\alpha(\varphi) > 0$.*

(a) *If $\alpha(\varphi) > 0$ then for all $0 < \alpha < \alpha(\varphi)$ we have*

$$P_{\mathrm{top}}(\varphi) = \lim_{c\to 0} \limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{z\in \mathrm{Per}_n(\alpha,c)} \exp\Big(\sum_{k=0}^{n-1} \varphi(f^k(z))\Big)\Big). \tag{11}$$

(b) *If (11) is true for some $\alpha > 0$ then there exists an ergodic equilibrium state μ of φ with $\chi(\mu) \ge \alpha$.*

RREMARK. Note that Theorem 2 immediately implies Theorem 1.

We delay the proof of Theorem 2 for a while and first prove some preliminary results.

LEMMA 1. *Let $f \in \mathcal{A}(X)$ and let Λ be an invariant set on which f is expanding. Let $\varphi \in C(\Lambda,\mathbb{R})$ be a Hölder continuous potential. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log\Big(\sum_{z\in \mathrm{Per}_n(f|\Lambda)} \exp S_n\varphi(z)\Big) \le P_{\mathrm{top}}(f|\Lambda,\varphi). \tag{12}$$

In particular, if $f|\Lambda$ is topologically mixing, then we have equality in (12), *and the limit superior is in fact a limit.*

Proof. Since $f|\Lambda$ is expanding it is expansive. Given any expansivity constant δ, for every $n \in \mathbb{N}$ and every $0 < \varepsilon \leq \delta$ the set $\mathrm{Per}_n(f|\Lambda)$ is (n,ε)-separated. Now, inequality (12) follows from the fact that the definition (8) can be replaced by the supremum taken over all (n,ε)-separated sets (see [15]). For the proof of the second statement we refer to [14, Chapter 7]. ∎

REMARK 1. The identity in (12) holds in the more general case of topologically mixing expanding maps (see [14, Chapter 7]). In particular, if Λ is a repeller of a differentiable map f such that $f|\Lambda$ is conjugate to a (one-sided) irreducible aperiodic subshift of finite type then (12) is an identity.

PROPOSITION 1. *Let $f \in \mathcal{A}(X)$ and let $\varphi \in C(X,\mathbb{R})$ be a continuous potential. Then for all $\mu \in \mathcal{M}_{\mathrm{E}}$ with $\chi(\mu) > 0$ and for all $0 < \alpha < \chi(\mu)$ we have*

$$h_\mu(f) + \int \varphi\, d\mu \leq \lim_{c\to 0} P_{\mathrm{P}}(\varphi,\alpha,c). \tag{13}$$

Proof. Consider $\mu \in \mathcal{M}_{\mathrm{E}}$ with $\chi(\mu) > 0$, and fix $0 < \alpha < \chi(\mu)$. Since $\chi(\mu) > 0$, condition (7) implies that $\mathrm{supp}(\mu) \subset X$ and thus the left hand side of (13) is well-defined. It is a consequence of Katok's theory [8] in its version for holomorphic endomorphisms developed by Przytycki and Urbański ([13, Chapter 9], see also [11]) that there exists a sequence of measures $\mu_n \in \mathcal{M}_{\mathrm{E}}$ supported on expanding sets $X_n \subset X$ such that

$$h_\mu(f) + \int \varphi\, d\mu \leq \liminf_{n\to\infty} P_{\mathrm{top}}(f|X_n,\varphi) \tag{14}$$

and $\mu_n \to \mu$ with respect to the weak* topology. Moreover, for each $n \in \mathbb{N}$ there exist $m = m(n) \in \mathbb{N}$ and $s = s(n) \in \mathbb{N}$ such that $f^m|X_n$ is conjugate to the full shift on s symbols. For every $0 < \varepsilon < \chi(\mu) - \alpha$ there is a number $n = n(\varepsilon) \in \mathbb{N}$ such that

$$h_\mu(f) + \int \varphi\, d\mu - \varepsilon \leq P_{\mathrm{top}}(f|X_n,\varphi). \tag{15}$$

Moreover, there exists a number $c_0 = c_0(n,\varepsilon)$ with $0 < c_0(n) \leq 1$ such that for every periodic point $z \in X_n$ and every $k \in \mathbb{N}$ we have

$$c_0^{-1} e^{k(\chi(\mu)-\varepsilon)} \leq |(f^k)'(z)| \leq c_0 e^{k(\chi(\mu)+\varepsilon)}. \tag{16}$$

Note that (16) is a consequence of the construction of the sets X_n in [13, Chapter 9.6]. This implies that

$$\mathrm{Per}_k(f) \cap X_n \subset \mathrm{Per}_k(\alpha, c_0) \tag{17}$$

for all $k \in \mathbb{N}$. Let $m, s \in \mathbb{N}$ be such that $f^m|X_n$ is topologically conjugate to the full shift on s symbols. Since $mP_{\mathrm{top}}(f|X_n,\varphi) = P_{\mathrm{top}}(f^m|X_n, S_m\varphi)$

(see [15, Theorem 9.8]), we can conclude that

$$h_\mu(f) + \int \varphi \, d\mu - \varepsilon \le \frac{1}{m} P_{\mathrm{top}}(f^m | X_n, S_m\varphi).$$

Recall that $S_m\varphi(z) = \sum_{i=0}^{m-1} \varphi(f^i(z))$. It now follows from Remark 1 and an elementary calculation that

$$\begin{aligned} (18) \qquad & h_\mu(f) + \int \varphi \, d\mu - \varepsilon \\ & \le \frac{1}{m} \lim_{k\to\infty} \frac{1}{k} \log\Big(\sum_{z\in \mathrm{Per}_{mk}(f)\cap X_n} \exp\Big(\sum_{i=0}^{k-1} S_m\varphi(f^{im}(z))\Big)\Big) \\ & \le \lim_{k\to\infty} \frac{1}{k} \log\Big(\sum_{z\in \mathrm{Per}_{k}(f)\cap X_n} \exp S_k\varphi(z)\Big). \end{aligned}$$

Combining (17) and (18) yields

$$h_\mu(f) + \int \varphi \, d\mu - \varepsilon \le \limsup_{k\to\infty} \frac{1}{k} \log \sum_{z\in \mathrm{Per}_k(\alpha, c_0)} \exp S_k\varphi(z).$$

Recall that by (2) the map $c \mapsto P_{\mathrm{P}}(\varphi, \alpha, c)$ is non-decreasing as $c \to 0^+$. Since $\varepsilon > 0$ is arbitrary the proof is complete. ∎

We can now give the proof of Theorem 2.

Proof of Theorem 2. Let $0 < \alpha$ and $0 < c \le 1$ be such that $\mathrm{Per}_n(\alpha, c) \ne \emptyset$ for some $n \in \mathbb{N}$. We first prove that

$$(19) \qquad P_{\mathrm{P}}(\varphi, \alpha, c) \le \sup_\nu \Big\{ h_\nu(f) + \int_\Lambda \varphi \, d\nu \Big\} \le P_{\mathrm{top}}(\varphi),$$

where the supremum is taken over all $\nu \in \mathcal{M}_{\mathrm{E}}$ with $\alpha \le \chi(\nu)$. Note that the supremum in (19) is not taken over the empty set. The right hand inequality in (19) is a consequence of the variational principle.

In order to prove the left hand inequality in (19) we define

$$\Lambda = \Lambda_{\alpha,c} := \overline{\bigcup_{n=1}^{\infty} \mathrm{Per}_n(\alpha, c)}.$$

It follows from a continuity argument that f is repelling on Λ. Furthermore, for every $n \ge 1$ with $\mathrm{Per}_n(\alpha, c) \ne \emptyset$ we have

$$(20) \qquad \mathrm{Per}_n(f) \cap \Lambda = \mathrm{Per}_n(\alpha, c).$$

Therefore, Lemma 1 implies that

$$(21) \qquad P_{\mathrm{P}}(\varphi, \alpha, c) \le P_{\mathrm{top}}(f|\Lambda, \varphi).$$

It follows from the variational principle that for every $\varepsilon > 0$ there is a $\mu \in \mathcal{M}_{\mathrm{E}}$

which is supported in Λ such that

$$P_{\mathrm{top}}(f|\Lambda,\varphi)-\varepsilon\le h_\mu(f)+\int\varphi\,d\mu\le P_{\mathrm{top}}(f|\Lambda,\varphi). \tag{22}$$

Since μ is ergodic we have $\chi(z)=\chi(\mu)$ for μ-almost every $z\in\Lambda$. It now follows from the continuity of $z\mapsto|f'(z)|$ and the definition of $\mathrm{Per}_n(\alpha,c)$ that $\alpha\le\chi(z)$ for all $z\in\Lambda$. We conclude that $\alpha\le\chi(\mu)$. Therefore, the left hand inequality in (19) follows from (21) and (22).

Next, we prove that

$$P_{\mathrm{top}}(\varphi)\le\lim_{c\to0}P_{\mathrm{P}}(\varphi,\alpha,c). \tag{23}$$

Let $0<\alpha<\alpha(\varphi)$ and $0<\varepsilon<\alpha(\varphi)-\alpha$. It follows from the definition of $\alpha(\varphi)$ (see (5)) that there exist $\mu\in\mathcal{M}_{\mathrm{E}}$ with $\chi(\mu)>\alpha(\varphi)-\varepsilon>\alpha$ such that

$$P_{\mathrm{top}}(\varphi)=h_\mu(f)+\int\varphi\,d\mu. \tag{24}$$

Therefore, Proposition 1 implies

$$h_\mu(f)+\int\varphi\,d\mu\le\lim_{c\to0}P_{\mathrm{P}}(\varphi,\alpha,c). \tag{25}$$

Since ε can be chosen arbitrarily small, (24) and (25) imply (23).

Finally, we prove (b). Let $\alpha>0$ be such that (11) holds. For $n\ge1$ and $c>0$ with $\mathrm{Per}_n(\alpha,c)\ne\emptyset$ we define the measure $\sigma_n=\sigma_n(\alpha,c,\varphi)\in\mathcal{M}$ by

$$\sigma_n=\frac{1}{\sum_{z\in\mathrm{Per}_n(\alpha,c)}\exp(S_n\varphi(z))}\sum_{z\in\mathrm{Per}_n(\alpha,c)}\exp(S_n\varphi(z))\delta_z, \tag{26}$$

where δ_z denotes the Dirac measure supported at z. Note that every measure $\sigma_n=\sigma_n(\alpha,c,\varphi)$ defined in (26) is in the convex hull of the set $\{\delta_z\colon z\in\mathrm{Per}_n(\alpha,c)\}$. Consider a subsequence $(\sigma_{n_k})_k$ converging to some measure $\mu_{\alpha,c}=\mu_{\alpha,c}(\varphi)\in\mathcal{M}$ in the weak* topology. It follows that $\chi(\mu_{\alpha,c})\ge\alpha$. Note that f is expanding on $\Lambda_{\alpha,c}=\overline{\bigcup_{n=1}^\infty\mathrm{Per}_n(\alpha,c)}$. Thus, there exists an expansivity constant $\delta=\delta(\alpha,c)$ for $f|\Lambda_{\alpha,c}$. In particular, for every $n\in\mathbb{N}$ and every $0<\varepsilon\le\delta$ the set $\mathrm{Per}_n(\alpha,c)$ is (n,ε)-separated. As in the proof of [15, Theorem 9.10] it follows that

$$\limsup_{n\to\infty}\frac{1}{n}\log\sum_{x\in\mathrm{Per}_n(\alpha,c)}\exp(S_n\varphi(x))\le h_{\mu_{\alpha,c}}(f)+\int_X\varphi\,d\mu_{\alpha,c}. \tag{27}$$

By construction, we have

$$P_{\mathrm{top}}(\varphi)=\lim_{c\to0}\Big(h_{\mu_{\alpha,c}}(f)+\int_X\varphi\,d\mu_{\alpha,c}\Big). \tag{28}$$

As c decreases to zero, there exists a subsequence $(\mu_{\alpha,c_k})_k$ converging to some measure $\mu=\mu(\varphi)\in\mathcal{M}$ in the weak* topology. Using the upper semicontinuity of the entropy map and (28), we conclude that

$$\lim_{c_k\to 0}\Big(h_{\mu_{\alpha,c_k}}(f)+\int_X \varphi\,d\mu_{\alpha,c_k}\Big)=h_\mu(f)+\int_X \varphi\,d\mu=P_{\rm top}(\varphi).$$

It remains to show that $\chi(\mu)\ge\alpha$. This is trivial in case the sets Λ_{α,c_k} do not accumulate at critical points. To handle the case that the sets Λ_{α,c_k} possibly accumulate at a critical point $\gamma\in X$ we consider a decreasing sequence $(r_i)_i$ of positive numbers converging to 0 and a decreasing sequence of functions $(\phi_i)_i$ in $C(X,\mathbb{R})$ such that:

(i) $\phi_i\ge\log|f'|$ and $\phi_i(z)=\log|f'(z)|$ for all $z\in X\setminus B(\gamma,r_i)$.
(ii) $\phi_i(\gamma)\le -i$.

In particular, ϕ_i converges pointwise to $\log|f'|$. Fix $i\in\mathbb{N}$. Since μ_{α,c_k} converges to μ in the weak* topology, we conclude that

$$\int_X \phi_i\,d\mu=\lim_{k\to\infty}\int_X \phi_i d\mu_{\alpha,c_k}\ge\liminf_{k\to\infty}\chi(\mu_{\alpha,c_k})\ge\alpha. \tag{29}$$

It now follows from (3) and the monotone convergence theorem that

$$\chi(\mu)=\lim_{i\to\infty}\int_X \phi_i\,d\mu\ge\alpha.$$

One can choose μ to be ergodic by using an ergodic decomposition argument. The case when the sets Λ_{α,c_k} accumulate at finitely many critical points can be treated entirely analogously. ∎

REMARK. We note that in the proof of Theorem 2 we have used similar techniques to those in our paper [6] in the case of C^2-diffeomorphisms, as well as ideas from [3] where the topological entropy (i.e. $\varphi=0$) of surface diffeomorphisms is studied.

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