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On the Hausdorff Dimension of Topological Subspaces

by

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Summary. It is shown that every Polish space X with $\dim_{\mathrm{T}} X \geq d$ admits a compact subspace Y such that $\dim_{\mathrm{H}} Y \geq d$ where $\dim_{\mathrm{T}}$ and $\dim_{\mathrm{H}}$ denote the topological and Hausdorff dimensions, respectively.

In this note we prove that every Polish space (i.e. complete and separable metric space) X has a compact subspace Y such that $\dim_{\mathrm{H}} Y \geq \dim_{\mathrm{T}} X$, where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{T}}$ denote the Hausdorff dimension and the topological dimension, respectively.

From Marczewski's theorem it follows that if X is a Polish space, then $\dim_{\mathrm{H}} X \geq \dim_{\mathrm{T}} X$ (see [8]). On the other hand, it is well known that for every positive integer n there exists an example of a Polish space, say E, such that $\dim_{\mathrm{T}} E = n$ and $\dim_{\mathrm{T}} F = 0$ for every compact subspace $F \subset E$ (see [2, 5]).

The present paper is closely related to [7]. It fills the gap contained in the proof of Lemma 2 there. Indeed, we tacitly assumed that the measure defined in that lemma was finite. Then we used Ulam's lemma (see [1]) valid only for finite measures.

Recall now the definition of the *upper Lebesgue integral*. Let $A \subset \mathbb{R}$ be a Borel set. Let $f : A \to [0, \infty]$ be given. Set

$$\mathcal{F}_f = \{g : A \to [0,\infty] : g \text{ is Borel measurable and } g(x) \geq f(x),\ x \in A\}$$

and define the upper Lebesgue integral by the formula

$$\overline{\int_A} f(x)\,dx = \inf_{g \in \mathcal{F}_f} \int_A g(x)\,dx.$$

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The following facts can be easily derived from the above definition:

- $\overline{\int}_A f(x)\,dx = \int_A f(x)\,dx$ if f is Borel measurable;
- $\overline{\int}_A f(x)\,dx \le \overline{\int}_A g(x)\,dx$ if $0 \le f(x) \le g(x)$ for $x \in A$;
- $\overline{\int}_A f(x)\,dx = \int_A g(x)\,dx$ for some $g \in \mathcal{F}_f$;
- $\overline{\int}_A f(x)\,dx > 0$ if $f(x) > 0$ for every $x \in A$;
- $\overline{\int}_A \liminf_{n\to\infty} f_n(x)\,dx \le \liminf_{n\to\infty} \overline{\int}_A f_n(x)\,dx$ for each sequence $(f_n)_{n\ge 1}$ of nonnegative functions.

By $B(x,r)$ (resp. $S(x,r)$) we denote the closed ball (resp. sphere) with centre x and radius r.

LEMMA 1. *If* $\dim_{\mathrm{T}} X \ge d+1$, *where* d *is an integer greater than or equal to* -1, *then there exists* $x_0 \in X$ *and* $\lambda_0 > 0$ *such that* $\dim_{\mathrm{T}} S(x_0,\lambda) \ge d$ *for every* $\lambda \in (0,\lambda_0]$.

The proof easily follows from the definition of topological dimension (see also [6]).

LEMMA 2. *Suppose that* $\dim_{\mathrm{T}} X \ge d$, *where* $d \in \mathbb{N} \cup \{0\}$. *Then there exists a Borel probability measure* μ *such that*

$$\mu(B(x,r)) \le Cr^d \quad \textit{for every } x \in X \textit{ and } r > 0, \tag{1}$$

where $C > 0$ *is a positive constant independent of* x *and* r.

Proof. We use induction on d. For $d = 0$ condition (1) obviously holds for every Borel probability measure μ. Assume that it holds for $d = m$. By Lemma 1 there exist $x_0 \in X$ and $\lambda_0 > 0$ such that $\dim_{\mathrm{T}} S(x_0,\lambda) \ge m$ for every $\lambda \in (0,\lambda_0]$. Fix $\lambda \in (0,\lambda_0]$ and set $X_\lambda = S(x_0,\lambda)$. By the induction hypothesis there exists a Borel probability measure $\widetilde{\mu}_\lambda$ on X_λ such that

$$\widetilde{\mu}_\lambda(B_\lambda(x,r)) \le C_\lambda r^m \quad \text{for every } x \in X_\lambda \text{ and } r > 0,$$

where $B_\lambda(x,r)$ stands for the closed ball in X_λ with centre $x \in X_\lambda$ and radius r, and C_λ is independent of x and r. Without loss of generality we may assume that $C_\lambda \ge 1$. Define the Borel measure $\mu_\lambda : \mathcal{B}(X) \to [0,1]$ by the formula

$$\mu_\lambda(A) = \widetilde{\mu}_\lambda(A \cap X_\lambda)/(2^m C_\lambda) \quad \text{for } A \in \mathcal{B}(X).$$

Clearly $\operatorname{supp}\mu_\lambda \subset X_\lambda$ and

$$\mu_\lambda(B(x,r)) \le r^m \quad \text{for every } x \in X \text{ and } r > 0. \tag{2}$$

Set

$$\beta = \overline{\int_{(0,\lambda_0]}} \mu_\lambda(X)\,d\lambda$$

and observe that $\beta > 0$, by the properties of the upper Lebesgue integral. From these properties it follows that there is a decreasing sequence $(X_n)_{n\geq 1}$ of closed subsets of X such that

$$\bar{\int}_{(0,\lambda_0]} \mu_\lambda(X_n)\,d\lambda > \beta/2$$

and X_n has a 2^{-n}-net for each $n \in \mathbb{N}$. Indeed, let $n = 1$ and let $\{x_k\}_{k\geq 1}$ be a dense subset of X. Applying the Fatou lemma for the upper Lebesgue integral to $f_n(\lambda) = \mu_\lambda(\bigcup_{i=1}^n B(x_i, 1/2))$ we obtain

$$\bar{\int}_{(0,\lambda_0]} \mu_\lambda\Big(\bigcup_{i=1}^{i_1} B(x_i, 1/2)\Big)\,d\lambda > \beta/2$$

for some $i_1 \in \mathbb{N}$. Set $X_1 = \bigcup_{i=1}^{i_1} B(x_i, 1/2)$. Further, assume that we have defined $X_1, \ldots, X_k$. As before we find i_{k+1} such that

$$\bar{\int}_{(0,\lambda_0]} \mu_\lambda\Big(X_k \cap \bigcup_{i=1}^{i_{k+1}} B(x_i, 1/2^{k+1})\Big)\,d\lambda > \beta/2.$$

Setting $X_{k+1} = X_k \cap \bigcup_{i=1}^{i_{k+1}} B(x_i, 1/2^{k+1})$ finishes the induction.

For $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$ we define

$$\alpha_{k,i} = \sup\{\mu_\lambda(X_k) : \lambda \in ((i-1)\lambda_0/k, i\lambda_0/k]\}.$$

Let

$$\nu_k = \frac{\lambda_0}{k}\sum_{i=1}^{k} \mu_{k,i},$$

where $\mu_{k,i} = \mu_{\lambda_{k,i}}$ with $\lambda_{k,i} \in ((i-1)\lambda_0/k, i\lambda_0/k]$ and

$$\mu_{\lambda_{k,i}}(X_k) \geq \alpha_{k,i}/2.$$

By the definition of the upper Lebesgue integral we have

$$2\nu_k(X_k) \geq \frac{\lambda_0}{k}\sum_{i=1}^{k} \alpha_{k,i} \geq \int_0^{\lambda_0} \mu_\lambda(X_k)\,d\lambda > \beta/2. \tag{3}$$

Now define a positive linear functional $\Lambda : C(X) \to \mathbb{R}$ by the formula

$$\Lambda(f) = \mathbb{L}\Big(\Big(\int_{X_k} f\,d\nu_k\Big)_{k\in\mathbb{N}}\Big) \quad \text{for } f \in C(X),$$

where $\mathbb{L}$ is a Banach limit and $C(X)$ stands for the space of continuous functions $f : X \to \mathbb{R}$. From (3) it follows that Λ is nontrivial. Let $K = \bigcap_{k=1}^\infty X_k$. Observe that K is a compact set and $\Lambda(f) = 0$ for $0 \leq f \leq \mathbf{1}_{X\setminus K}$.

Hence Λ is a Riesz functional. Let μ_* be the Borel measure such that

$$\Lambda(f) = \int_X f(x)\, \mu_*(dx) \quad \text{for } f \in C(X).$$

Obviously $\operatorname{supp} \mu_* \subset K$. We end the proof by showing that

$$\mu_*(B(x,r)) \le 2r^{m+1}$$

for all $x \in X$ and $r > 0$. This part of the proof is similar to the proof of Proposition 5 in [6]. It is incorporated here for convenience of the reader and to make the paper self-contained. Fix $x \in X$ and $r > 0$. For $k \in \mathbb{N}$ define

$$i(k) = \min J_k \quad \text{and} \quad I(k) = \max J_k,$$

where

$$J_k = \{1 \le i \le k : B(x,r) \cap S(x_0, \lambda_{k,i}) \ne \emptyset\}.$$

If $J_k = \emptyset$ we set $i(k) = I(k) = 0$. Further we have

$$\frac{\lambda_0}{k}\,(I(k) - i(k)) \le 2r + \frac{\lambda_0}{k}.$$

On the other hand, by the construction of the measures ν_k we obtain

$$\nu_k(B(x,r)) \le \frac{\lambda_0}{k}\, r^m (I(k) - i(k) + 1) \le 2r^{m+1} + \frac{\lambda_0}{k}\, 2r^m. \tag{4}$$

Fix now $\eta > 0$ and let $f \in C(X)$ be such that $f(y) = 1$ for $y \in B(x,r)$, $f(y) = 0$ for $y \notin B(x, r+\eta)$ and $0 \le f \le 1$. Then

$$\mu_*(B(x,r)) \le \Lambda(f) \le \limsup_{k\to\infty} \nu_k(B(x, r+\eta)).$$

By (4) and the fact that $\eta > 0$ may be arbitrarily small, we obtain

$$\mu_*(B(x,r)) \le 2r^{m+1}$$

and the proof is complete. ∎

We are in a position to formulate the main result of the paper.

THEOREM 1. *Suppose that* $\dim_{\mathrm{T}} X \ge d$, *where* $d \in \mathbb{N} \cup \{0\}$. *Then there exists a compact subspace* $Y \subset X$ *such that* $\dim_{\mathrm{H}} Y \ge d$.

Proof. From Lemma 3 it follows that there exists a Borel measure, say μ_*, and a positive constant $C > 0$ such that

$$\mu_*(B(x,r)) \le Cr^d \quad \text{for } x \in X \text{ and } r > 0.$$

Further there exists a compact set $Y \subset X$ such that $\mu_*(Y) > 0$, by Ulam's lemma. Then $\dim_{\mathrm{H}} Y \ge d$ by Frostman's lemma (see [3, 4]). ∎

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