BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Vol. 54, No. 3-4, 2006

PARTIAL DIFFERENTIAL EQUATIONS

Indefinite Quasilinear Neumann Problem on Unbounded Domains

by

J. CHABROWSKI

Presented by Bogdan BOJARSKI

Summary. We investigate the solvability of the quasilinear Neumann problem (1.1) with sub- and supercritical exponents in an unbounded domain Ω. Under some integrability conditions on the coefficients we establish embedding theorems of weighted Sobolev spaces into weighted Lebesgue spaces. This is used to obtain solutions through a global minimization of a variational functional.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an unbounded domain with a smooth noncompact boundary $\partial\Omega$. We are mainly concerned with the nonlinear Neumann problem

$$(1.1)\qquad \begin{cases} -\operatorname{div}(\varrho(x)|\nabla u|^{p-2}\nabla u) \\ \qquad = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u - c(x)|u|^{t-2}u & \text{in } \Omega, \\ \varrho(x)|\nabla u|^{p-2}\dfrac{\partial}{\partial\nu}u(x) + h(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where ν is the outward normal vector to $\partial\Omega$. The exponents p, q, s and t satisfy the conditions: $1 < p < N$, $1 < s, q < p^* = Np/(N-p)$, $p^* < t$. The coefficient ϱ belongs to $L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ and $0 < \varrho_\circ \leq \varrho(x)$ a.e. for some constant $\varrho_\circ$. The coefficients a and b are allowed to change signs while c is assumed to be nonnegative and measurable on Ω. This problem was considered in the paper [9]. The authors of that paper established the existence of a nonnegative nontrivial solution assuming that $c \in L^\infty(\Omega)$,

2000 *Mathematics Subject Classification*: 35B33, 35J20, 35J65.

Key words and phrases: Neumann problem, sub- and supercritical exponents, indefinite weight functions.

$b \geq 0$ and the coefficients h, a and b converge to 0 at a certain rate as $|x| \to \infty$. In this paper we consider this problem under different assumptions than those in [9]. More specifically, we assume that

(A) $\int\limits_\Omega \frac{|a|^{\frac{t}{t-q}}}{c^{\frac{q}{t-q}}}\, dx < \infty$, $\mathrm{Int}(\{x \in \Omega : a(x) > 0\}) \neq \emptyset$.

(B) $\int\limits_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}}\, dx < \infty$.

(H) There exist constants $0 < c_1 < c_2$ such that

$$\frac{c_1}{(1+|x|)^{p-1}} \leq h(x) \leq \frac{c_2}{(1+|x|)^{p-1}}$$

for a.e. $x \in \partial\Omega$.

Problems of the form (1.1) originate in applied sciences: nonlinear elasticity [6], mathematical biology [2] and also in differential geometry [8], [10]. Since the pioneering paper [4] problems of this nature have attracted considerable attention. We refer to two extensive survey articles [3] and [14] for a review of the current bibliography (see also [5] and [13]).

Solutions to problem (1.1) will be found through a variational approach. To describe the variational setting we define a suitable Sobolev space in the following way. By $C_\delta^\infty(\Omega)$ we denote the space $C_0^\infty(\mathbb{R}^N)$ restricted to Ω. Let $w_p(x) = 1/(1+|x|)^p$ and let $E_p = E_p(\Omega)$ be the Sobolev space obtained as the completion of the space $C_\delta^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,p} = \Big(\int\limits_\Omega |\nabla u|^p\, dx + \int\limits_\Omega |u|^p w_p(x)\, dx\Big)^{\frac{1}{p}}.$$

By Lemma 1 in [11] the norm $\|\cdot\|_{1,p}$ is equivalent to

$$\|u\|_{E_p} = \Big(\int\limits_\Omega |\nabla u|^p \varrho(x)\, dx + \int\limits_{\partial\Omega} |u|^p h(x)\, dS_x\Big)^{\frac{1}{p}}.$$

Given a nonnegative measurable function $w(x)$ on Ω we denote by $L^r(\Omega, w)$ the weighted Lebesgue space equipped with norm

$$\|u\|_{L^r(\Omega,w)} = \Big(\int\limits_\Omega |u|^r w(x)\, dx\Big)^{\frac{1}{r}}.$$

We now define the underlying Sobolev space for problem (1.1) by $E(\Omega) = E_p(\Omega) \cap L^t(\Omega, c)$ equipped with norm

$$\|u\|_E = \|u\|_{E_p} + \Big(\int\limits_\Omega |u|^t c(x)\, dx\Big)^{\frac{1}{t}}.$$

Solutions to problem (1.1) will be obtained as critical points of the functional

$$J(u) = \frac{1}{p}\Big(\int_\Omega |\nabla u|^p \varrho(x)\,dx + \int_{\partial\Omega} |u|^p h(x)\,dS_x\Big) - \frac{1}{q}\int_\Omega |u|^q a(x)\,dx + \frac{1}{s}\int_\Omega |u|^s\, b(x)\,dx + \frac{1}{t}\int_\Omega |u|^t c(x)\,dx.$$

Throughout this paper, we denote strong convergence in a given Banach space X by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^q(\Omega)$ are denoted by $\|\cdot\|_q$.

The paper is organized as follows. In Section 2 we present a compact embedding theorem for the space $E(\Omega)$. This is used in Section 3 to establish the existence result for problem (1.1). In the proof of Theorem 1 in Section 3 we use some ideas from the paper [1].

2. Palais–Smale condition. First we establish the embedding of $E(\Omega)$ into a weighted Lebesgue space.

LEMMA 2.1. *Let $w \ge 0$ be a function in $L^\infty_{\rm loc}(\Omega)$ such that*

$$\int_\Omega \frac{w^{\frac{t}{t-r}}}{c^{\frac{r}{t-r}}}\,dx < \infty, \tag{2.1}$$

where $1 < r < p^* < t$. Then $E(\Omega)$ is compactly embedded into $L^r(\Omega, w)$.

Proof. Let $\delta > 0$ and $R > 0$. By the Young inequality we have

$$\int_\Omega |u|^r w\,dx \le \delta \int_\Omega |u|^t c\,dx + C_1(\delta)\int_\Omega \frac{w^{\frac{t}{t-r}}}{c^{\frac{r}{t-r}}}\,dx \tag{2.2}$$

and

$$\int_{\Omega_R} |u|^r w\,dx \le \delta \int_{\Omega_R} |u|^t c\,dx + C_1(\delta)\int_{\Omega_R} \frac{w^{\frac{t}{t-r}}}{c^{\frac{r}{t-r}}}\,dx, \tag{2.3}$$

where $\Omega_R = \Omega \cap (\mathbb{R}^N - B(0,R))$ and $C_1(\delta) > 0$ is a constant depending on δ, t and r. Let $\{u_m\}$ be a bounded sequence in $E(\Omega)$. We may assume that $u_m \to u$ in $L^r_{\rm loc}(\Omega)$. Applying (2.3) to $u_m - u$ we get

$$\int_{\Omega_R} |u_m - u|^r w\,dx \le \delta \int_\Omega |u_m - u|^t c\,dx + C_1(\delta)\int_{\Omega_R} \frac{w^{\frac{t}{t-r}}}{c^{\frac{r}{t-r}}}\,dx \le \delta\|u_m - u\|_E^t + C_1(\delta)\int_{\Omega_R} \frac{w^{\frac{t}{t-r}}}{c^{\frac{r}{t-r}}}\,dx.$$

By the Lebesgue dominated convergence theorem we have, for every $R > 0$,

$$\lim_{m\to\infty} \int_{\Omega\cap B(0,R)} |u_m - u|^r \, dx = 0.$$

Since $\int_{\Omega_R} (w^{t/(t-r)}/c^{r/(t-r)}) \, dx \to 0$ as $R \to \infty$, the compactness of the embedding of $E(\Omega)$ into $L^r(\Omega, w)$ follows. ■

REMARK 2.2. It is known that $E_p(\Omega)$ is compactly embedded into $L^r(\Omega, w_\alpha)$, where $w_\alpha(x) = 1/(1 + |x|)^\alpha$ and $(*)$ $\alpha > N(1 - r/t)$ (see [12]). Lemma 2.1 gives the compact embeddings of the subspace $E(\Omega)$ of $E_p(\Omega)$ into weighted Lebesgue spaces. If $c(x) \geq c_\circ > 0$ on Ω for some constant $c_\circ$ (the function $c(x)$ can be unbounded on Ω) and α satisfies $(*)$, then condition (2.1) holds with $w = w_\alpha$. Hence $E(\Omega)$ is compactly embedded into $L^r(\Omega, w_\alpha)$. We point out that we can deduce from Lemma 2.1 an embedding of $E(\Omega)$ into a weighted Lebesgue space with an unbounded weight function. For example, take $w(x) = (1 + |x|)^\alpha$ and $c(x) = (1 + |x|)^\beta$. If $\alpha, \beta > 0$ and $N + \alpha t/(t - r) - \beta r/(t - r) < 0$, then $E(\Omega)$ is compactly embedded into $L^r(\Omega, (1 + |x|)^\alpha)$.

LEMMA 2.3. *Suppose* (A), (B) *and* (H) *hold. Then the functional J is bounded from below.*

Proof. It follows from (A), (B) and the Young inequality that

$$J(u) \geq \frac{1}{p}\Big(\int_\Omega |\nabla|^p \varrho \, dx + \int_{\partial\Omega} |u|^p h \, dS_x\Big) + \left(\frac{1}{t} - 2\delta\right) \int_\Omega |u|^t c \, dx$$
$$- C_1(\delta) \int_\Omega \frac{|a|^{\frac{t}{t-q}}}{c^{\frac{q}{t-q}}} \, dx - C_2(\delta) \int_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}} \, dx.$$

Taking $2\delta < 1/t$ yields the assertion. ■

We recall that a C^1-functional $\Phi : X \to \mathbb{R}$ on a Banach space X satisfies the *Palais–Smale condition* at level c ($(\mathrm{PS})_c$ condition for short) if each sequence $\{x_n\} \subset X$ such that $(*)$ $\Phi(x_n) \to c$ and $(**)$ $\Phi'(x_n) \to 0$ in X^* is relatively compact in X. Finally, any sequence $\{x_n\}$ satisfying $(*)$ and $(**)$ is called a *Palais–Smale sequence* at level c ($(\mathrm{PS})_c$ sequence for short).

LEMMA 2.4. *The functional J is of class C^1.*

Proof. We write

$$J(u) = \frac{1}{p}\Big(\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} |u|^p h \, dS_x\Big) - K_a(u) + K_b(u) + K_c(u),$$

where

$$K_a(u) = \frac{1}{q} \int_\Omega |u|^q a \, dx, \quad K_b(u) = \frac{1}{s} \int_\Omega |u|^s b \, dx, \quad K_c(u) = \frac{1}{t} \int_\Omega |u|^t c \, dx.$$

Now we show that these functionals are of class C^1 on $E(\Omega)$. We only consider the functional K_b. The Gateaux derivative is given by

$$\langle K_b'(u), \phi\rangle = \int_\Omega |u|^{s-2} u\phi b\, dx \tag{2.4}$$

for $\phi \in E(\Omega)$. Indeed, by the mean value theorem we have, for $0 < t < 1$,

$$\left| \frac{b|u + t\phi|^s - b|u|^s}{|t|} \right| \le s|b|(|u| + |\phi|)^{s-1}|\phi|. \tag{2.5}$$

It follows from assumption (B) and the Hölder inquality that

$$\begin{aligned}
\int_\Omega |b|(|u| + |\phi|)^{s-1}|\phi|\, dx &\le \left(\int_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}}\, dx\right)^{\frac{t-s}{t}} \left(\int_\Omega c|u|^{\frac{s-1}{s}t}|\phi|^{\frac{t}{s}}\, dx\right)^{\frac{s}{t}} \\
&\quad + \left(\int_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}}\, dx\right)^{\frac{t-s}{t}} \left(\int_\Omega |\phi|^t c\, dx\right)^{\frac{s}{t}} \\
&\le \left(\int_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}}\, dx\right)^{\frac{t-s}{t}} \left(\int_\Omega c|u|^t\, dx\right)^{\frac{s-1}{t}} \left(\int_\Omega |\phi|^t c\, dx\right)^{\frac{1}{t}} \\
&\quad + \left(\int_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}}\, dx\right)^{\frac{t-s}{t}} \left(\int_\Omega |\phi|^t c\, dx\right)^{\frac{s}{t}}.
\end{aligned}$$

Since the right side of (2.5) is in $L^1(\Omega)$ formula (2.4) follows from the Lebesgue dominated convergence theorem. To complete the proof it is enough to show that $K_b'(u)$ is continuous on $E(\Omega)$. Let $u_n \to u$ in $E(\Omega)$. Since $u_n \to u$ in $L^t(\Omega, c)$ we may assume that, up to a subsequence, $c^{1/t}u_n \to c^{1/t}u$ a.e. on Ω and that there exists a function $\zeta \in L^t(\Omega)$ such that $|c^{1/t}u_n|, |c^{1/t}u| \le \zeta$ a.e. on Ω. By the Hölder inequality we have, for $\phi \in E(\Omega)$,

$$\begin{aligned}
|\langle K_b'(u_n), \phi\rangle - \langle K_b'(u), \phi\rangle| &= \left| \int_\Omega (|u_n|^{s-2}u_n - |u|^{s-2}u)\phi b\, dx \right| \\
&\le \left(\int_\Omega \left||u_n|^{s-2}u_n - |u|^{s-2}u\right|^{\frac{s}{s-1}} |b|\, dx\right)^{\frac{s-1}{s}} \left(\int_\Omega |\phi|^s |b|\, dx\right)^{\frac{1}{s}} \\
&\le \left(\int_\Omega \left||u_n|^{s-2}u_n - |u|^{s-2}u\right|^{\frac{s}{s-1}\frac{t}{s}} c\, dx\right)^{\frac{s-1}{t}} \left(\int_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}}\right)^{\frac{t-s}{t}\frac{s-1}{s}} \\
&\quad \times \left(\int_\Omega |\phi|^s |b|\, dx\right)^{\frac{1}{s}}.
\end{aligned}$$

By the Lebesgue dominated convergence theorem the right side of this inequality converges to 0 uniformly in ϕ on bounded subsets of $E(\Omega)$. ■

Proposition 2.5. *Suppose that assumptions* (A), (B) *and* (H) *hold. Assume additionally in the case* $1 < q, s < 2$ *that* $a^+ \in L^\infty_{\rm loc}(\Omega)$ *and* $b^- \in L^\infty_{\rm loc}(\Omega)$. *Then the functional* J *satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\} \subset E(\Omega)$ be such that $J(u_n) \to c$ and $J'(u_n) \to 0$ in $E(\Omega)^*$. Using the Young inequality we have, for large n,

$$\begin{aligned}\frac{1}{p}\Big(\int_\Omega |\nabla u_n|^p \varrho\, dx + \int_{\partial\Omega} |u_n|^p h\, dS_x\Big) + \frac{1}{t}\int_\Omega |u_n|^t c\, dx \\ \le c + 1 + \frac{1}{q}\int_\Omega |u_n|^q |a|\, dx + \frac{1}{s}\int_\Omega |u_n|^s |b|\, dx \\ \le c + 1 + \delta\int_\Omega |u_n|^t c\, dx + C(\delta)\bigg(\int_\Omega \frac{|a|^{\frac{t}{t-q}}}{c^{\frac{q}{t-q}}}\, dx + \int_\Omega \frac{|b|^{\frac{t}{t-s}}}{c^{\frac{s}{t-s}}}\, dx\bigg).\end{aligned}$$

Taking $\delta < 1/t$ we deduce that $\{u_n\}$ is bounded in $E(\Omega)$. Hence we may assume that $u_n \rightharpoonup u$ in $E(\Omega)$. First, we consider the case $2 < q, s$. Obviously in this case $t > 2$. We set

$$F(x,u) = a^+(x)\,\frac{|u|^q}{q} - \frac{c(x)}{4t}\,|u|^t, \qquad f(x,u) = F_u(x,u)$$

and

$$G(x,u) = b^-(x)\,\frac{|u|^s}{s} - \frac{c(x)}{4t}\,|u|^t, \qquad g(x,u) = G_u(x,u).$$

We now use the following inequality: for every $\alpha > 0$, $\beta > 0$ and $0 < l < r$ we have

$$\alpha|u|^l - \beta|u|^r \le C_{lr}\alpha\Big(\frac{\alpha}{\beta}\Big)^{\frac{l}{r-l}}$$

for every $u \in \mathbb{R}$, where the constant $C_{lr} > 0$ depends only on r and l. Applying this inequality we get

$$\begin{aligned}f_u(x,u) &= (q-1)a^+(x)|u|^{q-2} - (t-1)\,\frac{c(x)}{4}\,|u|^{t-2} \\ &\le C_{t,q}a^+(x)\Big(\frac{4a^+(x)}{c(x)}\Big)^{\frac{q-2}{t-q}}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}g_u(x,u) &= (s-1)b(x)^-|u|^{s-2} - (t-1)\,\frac{c(x)}{4}\,|u|^{t-2} \\ &\le C_{s,t}b^-(x)\Big(\frac{4b^-(x)}{c(x)}\Big)^{\frac{s-2}{t-s}}.\end{aligned} \tag{2.7}$$

Then it follows from (2.6) and (2.7) and the fact that $J'(u_n) \to 0$ in $E(\Omega)^*$

that

$$
\begin{aligned}
(2.8)\qquad & \int_\Omega (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m)(\nabla u_n - \nabla u_m)\varrho\,dx \\
&+ \int_{\partial\Omega} (|u_n|^{p-2}u_n - |u_m|^{p-2}u_m)(u_n - u_m)h\,dx \\
&+ \int_\Omega (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m)(u_n - u_m)a^-\,dx \\
&+ \int_\Omega (|u_n|^{s-2}u_n - |u_m|^{s-2}u_m)(u_n - u_m)b^+\,dx \\
&+ \frac{1}{2}\int_\Omega (|u_n|^{t-2}u_n - |u_m|^{t-2}u_m)(u_n - u_m)c\,dx \\
= & \int_\Omega (f(x,u_n) - f(x,u_m))(u_n - u_m)\,dx \\
&+ \int_\Omega (g(x,u_n) - g(x,u_m))(u_n - u_m)\,dx + o(1) \\
= & \int_\Omega \int_0^1 f_u(x, u_n + \sigma(u_n - u_m))\,d\sigma\,(u_n - u_m)^2\,dx \\
&+ \int_\Omega \int_0^1 g_u(x, u_n + \sigma(u_n - u_m))\,d\sigma\,(u_n - u_m)^2\,dx + o(1) \\
\le & C_{t,q}\int_\Omega a^+\left(\frac{4a^+}{c}\right)^{\frac{q-2}{t-q}}(u_n - u_m)^2\,dx + C_{s,t}\int_\Omega b^-\left(\frac{4b^-}{c}\right)^{\frac{s-2}{t-s}}dx + o(1).
\end{aligned}
$$

We may assume that $u_n - u_m \rightharpoonup 0$ in $L^{q/2}(\Omega, a^+)$ as $n, m \to \infty$. Since $(a^+/c)^{(q-2)/(t-q)} \in L^{q/(q-2)}(\Omega, a^+)$, we see that

$$\lim_{n,m\to\infty}\int_\Omega a^+\left(\frac{a^+}{c}\right)^{\frac{q-2}{t-q}}(u_n - u_m)^2\,dx = 0.$$

In a similar manner we show that

$$\lim_{n,m\to\infty}\int_\Omega b^-\left(\frac{b^-}{c}\right)^{\frac{s-2}{t-s}}(u_n - u_m)^2\,dx = 0.$$

To estimate from below the terms on the left side of (2.8) we use the following inequalities: for all $x, y \in \mathbb{R}^N$,

$$(2.9)\qquad (|x|^{r-2}x - |y|^{r-2}y, x - y) \ge C_r|x - y|^r \quad \text{if } r \ge 2,$$

and for all $x, y \in \mathbb{R}^N$,

$$C_r \frac{|x-y|^2}{(|x|+|y|)^{2-r}} \le (|x|^{r-2}x - |y|^{r-2}y, x-y) \quad \text{if } r < 2, \tag{2.10}$$

where $C_r > 0$ is a constant. If $p > 2$ by (2.9) we have

$$C_p \int_\Omega |\nabla u_n - u_m|^2 \varrho\, dx \le \int_\Omega (|\nabla u_n|^p \nabla u_m - |\nabla u_m|^p \nabla u_m, \nabla u_n - \nabla u_m)\, dx.$$

In this way we estimate the remaining terms of the left side of (2.8). If $1 < p < 2$, we use (2.10) to obtain

$$\begin{aligned} &\int_\Omega |\nabla u_n - \nabla u_m|^p \varrho\, dx \\ &\quad\le \int_\Omega \frac{|\nabla u_n - \nabla u_m|^p}{(|\nabla u_n| - |\nabla u_m|)^{\frac{2-p}{2}p}} (|\nabla u_n| + |\nabla u_m|)^{\frac{2-p}{2}p} \varrho\, dx \\ &\quad\le \left(\int_\Omega \frac{|\nabla u_n - \nabla u_m|^2}{(|\nabla u_n| + |\nabla u_m|)^{2-p}} \varrho\, dx \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_n| + |\nabla u_m|)^p \varrho\, dx \right)^{\frac{2-p}{2}}. \end{aligned}$$

Since the sequence $\{\int_\Omega |\nabla u_n|^p \varrho\, dx\}$ is bounded we derive from this that

$$\begin{aligned} &\left(\int_\Omega |\nabla u_n - u_m|^p \varrho\, dx \right)^{\frac{2}{p}} \\ &\qquad\le C_1 \int_\Omega (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m, \nabla u_n - \nabla_m) \varrho\, dx \end{aligned}$$

for some constant $C_1 > 0$. It is now clear that, up to a subsequence, $u_n \to u$ in $E(\Omega)$.

We now consider the case $1 < q < 2$. Let us denote by I_{nm} the left hand side of inequality (2.8) without the integral involving c. We rewrite (2.8) in the following way:

$$\begin{aligned} I_{nm} + \int_\Omega (|u_n|^{t-2}u_n - |u_m|^{t-2}u_m, u_n - u_m) c\, dx &= \int_\Omega (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m, u_n - u_m) a^+\, dx \\ &\quad + \int_\Omega (|u_n|^{s-2}u_n - |u_m|^{s-2}u_m, u_n - u_m) b^-\, dx + o(1) \end{aligned} \tag{2.11}$$

if $1 < s < 2$. By Lemma 2.1 the last two integrals converge to 0 as $n, m \to \infty$. We now apply the argument from the previous case to the terms on the left side. In this way we again show that $u_n \to u$ in $E(\Omega)$. Finally, if $2 < s$ we modify (2.8) in the following way:

$$I_{mn} + \frac{1}{2}\int_{\Omega}(|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_m|^{t-2}\nabla u_m, \nabla u_n - \nabla u_m)c\,dx$$
$$= \int_{\Omega}(|u_n|^{q-2}u_n - |u_m|^{q-2}u_m, u_n - u_m)a^+\,dx$$
$$+ \int_{\Omega}\int_0^1 \widetilde{f}_u(x, u_n + t(u_n - u_m))\,dt(u_n - u_m)^2\,dx + o(1),$$

where $\widetilde{f}_u = \widetilde{F}$ and $\widetilde{F}(x,u) = a^+(x)|u|^q/q - c(x)|u|^t/2t$. To complete the proof we repeat the argument from the previous part. ∎

3. Main result. By Lemma 2.3 the functional J is bounded from below on $E(\Omega)$. We put

$$m = \inf_{u\in E(\Omega)} J(u).$$

By the Ekeland variational principle [7] there exists a Palais–Smale sequence $\{u_n\}$ at level m (see also [15, Corollary 2.5]). It then follows from Proposition 2.5 that, up to a subsequence, $u_n \to u$ in $E(\Omega)$. Obviously u is a nontrivial solution of (1.1) provided $m < 0$. In Theorem 1 below we formulate conditions guaranteeing that $m < 0$. It is clear that $|u|$ is also a minimizer of J. Therefore we can assume that u is nonnegative on $\mathbb{R}^N$. We put

$$A(v) = \int_{\Omega} |v|^q a(x)\,dx \quad \text{and} \quad C(v) = \int_{\Omega} |v|^t c(x)\,dx.$$

THEOREM 3.1. *Suppose that* (A), (B) *and* (H) *hold and moreover that* $a^+, b^- \in L^\infty_{\mathrm{loc}}(\Omega)$.

(i) *If* (∗) $q < \min(p,s,t)$, *then problem* (1.1) *has a solution.*

If (∗) *is not satisfied we assume that* $V = \mathrm{Int}(\operatorname{supp} a^+ - \operatorname{supp} b) \neq \emptyset$. *We then have two cases*:

(ii) *If* $q < p$, *then problem* (1.1) *has a nontrivial solution.*

(iii) *If* $p < q$ *and there exists a* C^1 *function* v *with* $\operatorname{supp} v \subset V$ *such that*

$$(3.1) \qquad (t-p)\,\frac{t^{\frac{p-q}{t-p}}}{p^{\frac{t-q}{t-p}}}\,\frac{(q-p)^{\frac{p-q}{t-p}}}{(t-q)^{\frac{t-q}{t-p}}}\,\frac{(\|v\|^p_{E_p})^{\frac{t-q}{t-p}}}{C(v)^{\frac{p-q}{t-p}}} < A(v),$$

then problem (1.1) *has a nontrivial solution.*

Proof. (i) Let v be a C^1 function with $v \not\equiv 0$ and $\operatorname{supp} v \subset \{x \in \Omega : a(x) > 0\}$. Since $q < \min(p,s,t)$ we see that $J(\sigma v) < 0$ for $\sigma > 0$ sufficiently small and so $m < 0$.

(ii) We choose v as in (i) but with $\operatorname{supp} v \subset V$. Then $J(\sigma v) < 0$ for $\sigma > 0$ sufficiently small and so $m < 0$.

(iii) Let v be a function satisfying (3.1) and let

$$f(\sigma) = \frac{\sigma^{p-q}}{p} \|v\|_{1,p}^p + \frac{\sigma^{t-q}}{t} C(v).$$

Then we have

$$\inf_{\sigma>0} f(\sigma) = (t-p) \frac{t^{\frac{p-q}{t-p}}}{p^{\frac{t-q}{t-p}}} \frac{(q-p)^{\frac{p-q}{t-p}}}{(t-q)^{\frac{t-q}{t-p}}} \frac{(\|v\|_{E_p}^p)^{\frac{t-q}{t-p}}}{C(v)^{\frac{p-q}{t-p}}} < A(v).$$

Since v satisfies (3.1) there exists $\sigma > 0$ such that

$$\frac{\sigma^{p-q}}{p} \|v\|_{E_p}^p + \frac{\sigma^{t-q}}{t} C(v) < A(v)$$

and consequently $m < 0$. ■

References

[1] S. Alama and G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign*, J. Funct. Anal. 141 (1996), 159–215.

[2] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. Math. 30 (1978), 33–76.

[3] Th. Bartsch, Z. Q. Wang and M. Willem, *The Dirichlet problem for superlinear elliptic equations*, in: Handbook of Differential Equations, Vol. 2, M. Chipot and P. Quittner (eds.), Elsevier, 2005, 1–55.

[4] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. 36 (1983), 437–477.

[5] J. Chabrowski and M. Willem, *Least energy solutions of a critical Neumann problem with weight*, Calc. Var. 15 (2002), 121–131.

[6] P. G. Ciarlet, *Mathematical Elasticity*, Vol. I, *Three-Dimensional Elasticity*, North-Holland, Amsterdam, 1988.

[7] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. 47 (1974), 324–353.

[8] J. F. Escobar, *Sharp constant in a Sobolev trace inequality*, Indiana Univ. Math. J. 37 (1988), 687–698.

[9] D. A. Kandilakis and A. N. Lyberopoulos, *Indefinite quasilinear elliptic problems with subcritical and supercritical nonlinearities on unbounded domains*, J. Differential Equations 2006, 1–25, doi:10.1016/jde.2006.03.008.

[10] J. L. Kazdan, *Prescribing the Curvature of a Riemannian Manifold*, CBMS Reg. Conf. Ser. Math. 57, Amer. Math. Soc., Providence, RI, 1985.

[11] K. Pflüger, *Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition*, Electron. J. Differential Equations, 1998, no. 10, 13 pp.

[12] —, *Compact traces in weighted Sobolev spaces*, Analysis 18 (1998), 65–83.

[13] X. J. Wang, *Neumann problems of semilinear elliptic equations involving critical Sobolev exponents*, J. Differential Equations 93 (1991), 283–310.

[14] J. D. Rossi, *Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem*, in: Handbook of Differential Equations, Vol. 2, M. Chipot and P. Quittner (eds.), Elsevier, 2005, 311–406.

[15] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

J. Chabrowski
Department of Mathematics
University of Queensland
St. Lucia 4072, Qld, Australia
E-mail: jhc@maths.uq.edu.au

Received September 7, 2006;
received in final form November 10, 2006 (7549)