Indefinite Quasilinear Neumann Problem on Unbounded Domains

by

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Summary. We investigate the solvability of the quasilinear Neumann problem (1.1) with sub- and supercritical exponents in an unbounded domain $\Omega$. Under some integrability conditions on the coefficients we establish embedding theorems of weighted Sobolev spaces into weighted Lebesgue spaces. This is used to obtain solutions through a global minimization of a variational functional.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be an unbounded domain with a smooth noncompact boundary $\partial \Omega$. We are mainly concerned with the nonlinear Neumann problem

\begin{equation}
\begin{cases}
-\text{div}(\varrho(x)|\nabla u|^{p-2}\nabla u) \\
= a(x)|u|^{q-2}u - b(x)|u|^{s-2}u - c(x)|u|^{t-2}u & \text{in } \Omega, \\
\varrho(x)|\nabla u|^{p-2} \frac{\partial}{\partial \nu} u(x) + h(x)|u|^{p-2}u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $\nu$ is the outward normal vector to $\partial \Omega$. The exponents $p, q, s$ and $t$ satisfy the conditions: $1 < p < N$, $1 < s, q < p^* = Np/(N - p)$, $p^* < t$. The coefficient $\varrho$ belongs to $L^\infty(\Omega) \cap L^\infty(\partial \Omega)$ and $0 < \varrho_0 \leq \varrho(x)$ a.e. for some constant $\varrho_0$. The coefficients $a$ and $b$ are allowed to change signs while $c$ is assumed to be nonnegative and measurable on $\Omega$. This problem was considered in the paper [9]. The authors of that paper established the existence of a nonnegative nontrivial solution assuming that $c \in L^\infty(\Omega)$,
b ≥ 0 and the coefficients h, a, and b converge to 0 at a certain rate as |x| → ∞. In this paper we consider this problem under different assumptions than those in [9]. More specifically, we assume that

\[(A) \int_{\Omega} \frac{|a|}{c^{\frac{1}{q}}} dx < \infty, \text{Int(}\{x \in \Omega : a(x) > 0\}) \neq \emptyset.\]

\[(B) \int_{\Omega} \frac{|b|}{c^{\frac{1}{s}}} dx < \infty.\]

\[(H) \text{There exist constants } 0 < c_1 < c_2 \text{ such that}\]

\[\frac{c_1}{(1 + |x|)^{p-1}} \leq h(x) \leq \frac{c_2}{(1 + |x|)^{p-1}}\]

for a.e. \(x \in \partial \Omega\).

Problems of the form (1.1) originate in applied sciences: nonlinear elasticity [6], mathematical biology [2] and also in differential geometry [8], [10]. Since the pioneering paper [4] problems of this nature have attracted considerable attention. We refer to two extensive survey articles [3] and [14] for a review of the current bibliography (see also [5] and [13]).

Solutions to problem (1.1) will be found through a variational approach. To describe the variational setting we define a suitable Sobolev space in the following way. By \(C_0^\infty(\Omega)\) we denote the space \(C_0^\infty(\mathbb{R}^N)\) restricted to \(\Omega\). Let \(w_p(x) = 1/(1 + |x|)^p\) and let \(E_p = E_p(\Omega)\) be the Sobolev space obtained as the completion of the space \(C_0^\infty(\Omega)\) with respect to the norm

\[\|u\|_{1,p} = \left(\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |u|^p w_p(x) dx\right)^\frac{1}{p}.\]

By Lemma 1 in [11] the norm \(\| \cdot \|_{1,p}\) is equivalent to

\[\|u\|_{E_p} = \left(\int_{\Omega} |
abla u|^p g(x) dx + \int_{\partial \Omega} |u|^p h(x) dS_x\right)^\frac{1}{p}.\]

Given a nonnegative measurable function \(w(x)\) on \(\Omega\) we denote by \(L^r(\Omega, w)\) the weighted Lebesgue space equipped with norm

\[\|u\|_{L^r(\Omega, w)} = \left(\int_{\Omega} |u|^r w(x) dx\right)^\frac{1}{r}.\]

We now define the underlying Sobolev space for problem (1.1) by \(E(\Omega) = E_p(\Omega) \cap L^t(\Omega, c)\) equipped with norm

\[\|u\|_E = \|u\|_{E_p} + \left(\int_{\Omega} |u|^t c(x) dx\right)^\frac{1}{t}.\]
Solutions to problem (1.1) will be obtained as critical points of the functional
\[ J(u) = \frac{1}{p} \left( \int_{\Omega} |\nabla u|^p g(x) \, dx + \int_{\partial\Omega} |u|^p h(x) \, dS_x \right) \]
\[ - \frac{1}{q} \int_{\Omega} |u|^q a(x) \, dx + \frac{1}{s} \int_{\Omega} |u|^s b(x) \, dx + \frac{1}{t} \int_{\Omega} |u|^t c(x) \, dx. \]

Throughout this paper, we denote strong convergence in a given Banach space \( X \) by “\( \to \)” and weak convergence by “\( \rightharpoonup \)”. The norms in the Lebesgue spaces \( L^q(\Omega) \) are denoted by \( \| \cdot \|_q \).

The paper is organized as follows. In Section 2 we present a compact embedding theorem for the space \( E(\Omega) \). This is used in Section 3 to establish the existence result for problem (1.1). In the proof of Theorem 1 in Section 3 we use some ideas from the paper [1].

2. Palais–Smale condition. First we establish the embedding of \( E(\Omega) \) into a weighted Lebesgue space.

**Lemma 2.1.** Let \( w \geq 0 \) be a function in \( L^\infty_{\text{loc}}(\Omega) \) such that
\[ \int_{\Omega} \frac{w^{\frac{t}{t-r}}}{c^{t-r}} \, dx < \infty, \]
where \( 1 < r < p^* < t \). Then \( E(\Omega) \) is compactly embedded into \( L^r(\Omega, w) \).

**Proof.** Let \( \delta > 0 \) and \( R > 0 \). By the Young inequality we have
\[ \int_{\Omega} |u|^r w \, dx \leq \delta \int_{\Omega} |u|^t c \, dx + C_1(\delta) \int_{\Omega} \frac{w^{\frac{t}{t-r}}}{c^{t-r}} \, dx \]
and
\[ \int_{\Omega_R} |u|^r w \, dx \leq \delta \int_{\Omega_R} |u|^t c \, dx + C_1(\delta) \int_{\Omega_R} \frac{w^{\frac{t}{t-r}}}{c^{t-r}} \, dx, \]
where \( \Omega_R = \Omega \cap (\mathbb{R}^N - B(0, R)) \) and \( C_1(\delta) > 0 \) is a constant depending on \( \delta, t \) and \( r \). Let \( \{u_m\} \) be a bounded sequence in \( E(\Omega) \). We may assume that \( u_m \to u \) in \( L^r_{\text{loc}}(\Omega) \). Applying (2.3) to \( u_m - u \) we get
\[ \int_{\Omega_R} |u_m - u|^r w \, dx \leq \delta \int_{\Omega} |u_m - u|^t c \, dx + C_1(\delta) \int_{\Omega_R} \frac{w^{\frac{t}{t-r}}}{c^{t-r}} \, dx \]
\[ \leq \delta \|u_m - u\|_E^t + C_1(\delta) \int_{\Omega_R} \frac{w^{\frac{t}{t-r}}}{c^{t-r}} \, dx. \]
By the Lebesgue dominated convergence theorem we have, for every $R > 0$,

$$\lim_{m \to \infty} \int_{\Omega \cap B(0,R)} |u_m - u|^r \, dx = 0.$$ 

Since $\int_{\Omega_R} (w^{\prime\prime(t-r)}/c^{\prime\prime(t-r)}) \, dx \to 0$ as $R \to \infty$, the compactness of the embedding of $E(\Omega)$ into $L^r(\Omega,w)$ follows. ■

**Remark 2.2.** It is known that $E_p(\Omega)$ is compactly embedded into $L^r(\Omega,w_\alpha)$, where $w_\alpha(x) = 1/(1 + |x|)^\alpha$ and $(*) \alpha > N(1-r/t)$ (see [12]). Lemma 2.1 gives the compact embeddings of the subspace $E(\Omega)$ of $E_p(\Omega)$ into weighted Lebesgue spaces. If $c(x) \geq c_\alpha > 0$ on $\Omega$ for some constant $c_\alpha$ (the function $c(x)$ can be unbounded on $\Omega$) and $\alpha$ satisfies $(*)$, then condition (2.1) holds with $w = w_\alpha$. Hence $E(\Omega)$ is compactly embedded into $L^r(\Omega,w_\alpha)$. We point out that we can deduce from Lemma 2.1 an embedding of $E(\Omega)$ into a weighted Lebesgue space with an unbounded weight function. For example, take $w(x) = (1 + |x|)^\alpha$ and $c(x) = (1 + |x|)^\beta$. If $\alpha, \beta > 0$ and $N + \alpha t/(t-r) - \beta r/(t-r) < 0$, then $E(\Omega)$ is compactly embedded into $L^r(\Omega,(1 + |x|)^\alpha)$.

**Lemma 2.3.** Suppose (A), (B) and (H) hold. Then the functional $J$ is bounded from below.

**Proof.** It follows from (A), (B) and the Young inequality that

$$J(u) \geq \frac{1}{p} \left( \int_\Omega |\nabla|^p \rho \, dx + \int_{\partial \Omega} |u|^p h \, dS_x \right) + \left( \frac{1}{t} - 2\delta \right) \int_\Omega |t|^lc \, dx$$

$$- C_1(\delta) \int_\Omega \frac{|a|^t_q}{c_{t-q}} \, dx - C_2(\delta) \int_\Omega \frac{|b|^t_s}{c_{t-s}} \, dx.$$ 

Taking $2\delta < 1/t$ yields the assertion. ■

We recall that a $C^1$-functional $\Phi : X \to \mathbb{R}$ on a Banach space $X$ satisfies the Palais–Smale condition at level $c$ (($\text{PS}_c$) condition for short) if each sequence $\{x_n\} \subset X$ such that $(*) \Phi(x_n) \to c$ and $(**) \Phi'(x_n) \to 0$ in $X^*$ is relatively compact in $X$. Finally, any sequence $\{x_n\}$ satisfying $(*)$ and $(**)$ is called a Palais–Smale sequence at level $c$ (($\text{PS}_c$) sequence for short).

**Lemma 2.4.** The functional $J$ is of class $C^1$.

**Proof.** We write

$$J(u) = \frac{1}{p} \left( \int_\Omega |\nabla|^p u \, dx + \int_{\partial \Omega} |u|^p h \, dS_x \right) - K_\alpha(u) + K_b(u) + K_c(u),$$

where

$$K_\alpha(u) = \frac{1}{q} \int_\Omega |u|^q a \, dx, \quad K_b(u) = \frac{1}{s} \int_\Omega |u|^s b \, dx, \quad K_c(u) = \frac{1}{t} \int_\Omega |t|^l c \, dx.$$
Now we show that these functionals are of class $C^1$ on $E(\Omega)$. We only consider the functional $K_b$. The Gateaux derivative is given by

\begin{equation}
\langle K'_b(u), \phi \rangle = \int_\Omega |u|^{s-2}u \phi b \, dx
\end{equation}

for $\phi \in E(\Omega)$. Indeed, by the mean value theorem we have, for $0 < t < 1$,

\begin{equation}
\left| \frac{|b|u + t\phi|^s - |b|u|^s}{|t|} \right| \leq s|b|(|u| + |\phi|)^{s-1}|\phi|.
\end{equation}

It follows from assumption (B) and the Hölder inequality that

\[
\int_\Omega |b|(|u| + |\phi|)^{s-1}|\phi| \, dx \leq \left( \int_\Omega \frac{|b| t^{-s}}{c^{\frac{s}{s-1}}} \, dx \right)^{\frac{t-s}{t}} \left( \int_\Omega |c|u|^{\frac{s-1}{s}}|\phi|^{\frac{t}{s}} \, dx \right)^{\frac{s}{t}}
\]

\[
+ \left( \int_\Omega \frac{|b| \frac{t}{s}}{c^{\frac{s}{s-1}}} \, dx \right)^{\frac{t-s}{t}} \left( \int_\Omega |\phi|^t c \, dx \right)^{\frac{s}{t}}
\]

\[
\leq \left( \int_\Omega \frac{|b| t^{-s}}{c^{\frac{s}{s-1}}} \, dx \right)^{\frac{t-s}{t}} \left( \int_\Omega |c|u|^{\frac{s-1}{s}} \, dx \right)^{\frac{s}{t}} \left( \int_\Omega |\phi|^t c \, dx \right)^{\frac{s}{t}}
\]

\[
+ \left( \int_\Omega \frac{|b| \frac{t}{s}}{c^{\frac{s}{s-1}}} \, dx \right)^{\frac{t-s}{t}} \left( \int_\Omega |\phi|^t c \, dx \right)^{\frac{s}{t}}.
\]

Since the right side of (2.5) is in $L^1(\Omega)$ formula (2.4) follows from the Lebesgue dominated convergence theorem. To complete the proof it is enough to show that $K'_b(u)$ is continuous on $E(\Omega)$. Let $u_n \to u$ in $E(\Omega)$. Since $u_n \to u$ in $L^t(\Omega, c)$ we may assume that, up to a subsequence, $c^{1/t}u_n \to c^{1/t}u$ a.e. on $\Omega$ and that there exists a function $\zeta \in L^t(\Omega)$ such that $|c^{1/t}u_n|, |c^{1/t}u| \leq \zeta$ a.e. on $\Omega$. By the Hölder inequality we have, for $\phi \in E(\Omega)$,

\[
|\langle K'_b(u_n), \phi \rangle - \langle K'_b(u), \phi \rangle| = \left| \int_\Omega (|u_n|^{s-2}u_n - |u|^{s-2}u) \phi b \, dx \right|
\]

\[
\leq \left( \int_\Omega |u_n|^{s-2}u_n - |u|^{s-2}u \frac{s-1}{s} |b| \, dx \right)^{\frac{s-1}{t}} \left( \int_\Omega |\phi|^s |b| \, dx \right)^{\frac{1}{t}}
\]

\[
\leq \left( \int_\Omega |u_n|^{s-2}u_n - |u|^{s-2}u \frac{s-1}{s} |b| \, dx \right)^{\frac{s-1}{t}} \left( \int_\Omega \frac{t}{s} \frac{s-1}{s} |b| \, dx \right)^{\frac{t-s}{s}} c \, dx
\]

\[
\times \left( \int_\Omega |\phi|^s |b| \, dx \right)^{\frac{1}{s}}.
\]

By the Lebesgue dominated convergence theorem the right side of this inequality converges to 0 uniformly in $\phi$ on bounded subsets of $E(\Omega)$. \blacksquare
Proposition 2.5. Suppose that assumptions (A), (B) and (H) hold. Assume additionally in the case $1 < q, s < 2$ that $a^+ \in L^\infty_{\text{loc}}(\Omega)$ and $b^- \in L^\infty_{\text{loc}}(\Omega)$. Then the functional $J$ satisfies the Palais–Smale condition.

Proof. Let $\{u_n\} \subset E(\Omega)$ be such that $J(u_n) \to c$ and $J'(u_n) \to 0$ in $E(\Omega)^*$. Using the Young inequality we have, for large $n$,

$$
\frac{1}{p} \left( \int_\Omega |\nabla u_n|^p \, dx + \int_{\partial \Omega} |u_n|^p h \, dS_x \right) + \frac{1}{t} \int_\Omega |u_n|^t \, c \, dx 
\leq c + 1 + \frac{1}{q} \int_\Omega |u_n|^q \, a \, dx + \frac{1}{s} \int_\Omega |u_n|^s \, b \, dx
$$

$$
\leq c + 1 + \delta \int_\Omega |u_n|^t \, c \, dx + C(\delta) \left( \int_\Omega \frac{|a|^\frac{t}{t-q}}{c^{\frac{t}{t-q}}} \, dx + \int_\Omega \frac{|b|^\frac{t}{t-s}}{c^{\frac{t}{t-s}}} \, dx \right).
$$

Taking $\delta < 1/t$ we deduce that $\{u_n\}$ is bounded in $E(\Omega)$. Hence we may assume that $u_n \rightharpoonup u$ in $E(\Omega)$. First, we consider the case $2 < q, s$. Obviously in this case $t > 2$. We set

$$
F(x, u) = a^+(x) \frac{|u|^q}{q} - \frac{c(x)}{4t} |u|^t, \quad f(x, u) = F_u(x, u)
$$

and

$$
G(x, u) = b^-(x) \frac{|u|^s}{s} - \frac{c(x)}{4t} |u|^t, \quad g(x, u) = G_u(x, u).
$$

We now use the following inequality: for every $\alpha > 0$, $\beta > 0$ and $0 < l < r$ we have

$$
\alpha |u|^l - \beta |u|^r \leq C_{l_r} \alpha \left( \frac{\alpha}{\beta} \right)^\frac{l-r}{l}
$$

for every $u \in \mathbb{R}$, where the constant $C_{l_r} > 0$ depends only on $r$ and $l$. Applying this inequality we get

$$
f_u(x, u) = (q - 1)a^+(x)|u|^{q-2} - (t - 1) \frac{c(x)}{4} |u|^{t-2}
\leq C_{t,q} a^+(x) \left( \frac{4a^+(x)}{c(x)} \right)^{\frac{q-2}{t-q}}
$$

and

$$
g_u(x, u) = (s - 1)b^-(x)|u|^{s-2} - (t - 1) \frac{c(x)}{4} |u|^{t-2}
\leq C_{s,t} b^-(x) \left( \frac{4b^-(x)}{c(x)} \right)^{\frac{s-2}{t-s}}.
$$

Then it follows from (2.6) and (2.7) and the fact that $J'(u_n) \to 0$ in $E(\Omega)^*$
that

\[
\frac{1}{2} \int_{\Omega} \left( |u_n|^{t-2} u_n - |u_m|^{t-2} u_m \right) (u_n - u_m) c \, dx \\
+ \frac{1}{2} \int_{\Omega} \left( |u_n|^{s-2} u_n - |u_m|^{s-2} u_m \right) (u_n - u_m) b^+ \, dx \\
+ \frac{1}{2} \int_{\Omega} \left( |u_n|^{t-2} u_n - |u_m|^{t-2} u_m \right) (u_n - u_m) c \, dx
\]

\[
= \int_{\Omega} (f(x, u_n) - f(x, u_m))(u_n - u_m) \, dx
\]

\[
+ \int_{\Omega} (g(x, u_n) - g(x, u_m))(u_n - u_m) \, dx + o(1)
\]

\[
= \int_{\Omega} \left[ f_u(x, u_n + \sigma(u_n - u_m)) \right] d\sigma (u_n - u_m)^2 \, dx \\
+ \int_{\Omega} \left[ g_u(x, u_n + \sigma(u_n - u_m)) \right] d\sigma (u_n - u_m)^2 \, dx + o(1)
\]

\[
\leq C_{t,q} \int_{\Omega} a^+ \left( \frac{4a^+}{c} \right)^{q-2} (u_n - u_m)^2 \, dx + C_{s,t} \int_{\Omega} b^- \left( \frac{4b^-}{c} \right)^{s-2} \, dx + o(1).
\]

We may assume that \( u_n - u_m \to 0 \) in \( L^{q/2}(\Omega, a^+) \) as \( n, m \to \infty \). Since \((a^+/c)^{(q-2)/(t-q)} \in L^{q/(q-2)}(\Omega, a^+)\), we see that

\[
\lim_{n,m \to \infty} \int_{\Omega} a^+ \left( \frac{a^+}{c} \right)^{q-2} (u_n - u_m)^2 \, dx = 0.
\]

In a similar manner we show that

\[
\lim_{n,m \to \infty} \int_{\Omega} b^- \left( \frac{b^-}{c} \right)^{s-2} (u_n - u_m)^2 \, dx = 0.
\]

To estimate from below the terms on the left side of (2.8) we use the following inequalities: for all \( x, y \in \mathbb{R}^N \),

\[
(|x|^{r-2} x - |y|^{r-2} y, x - y) \geq C_r |x - y|^r \quad \text{if} \; r \geq 2,
\]

\[
(2.9)
\]
and for all \( x, y \in \mathbb{R}^N \),

\[
C_r \frac{|x - y|^2}{(|x| + |y|)^2} \leq (|x|^{r-2}x - |y|^{r-2}y, x - y) \quad \text{if } r < 2,
\]

where \( C_r > 0 \) is a constant. If \( p > 2 \) by (2.9) we have

\[
C_p \int_{\Omega} |\nabla u_n - \nabla u_m|^2 \varrho \, dx \leq \left( \int_{\Omega} |\nabla u_n|^p \nabla u_m - |\nabla u_m|^p \nabla u_n, \nabla u_n - \nabla u_m \right) \, dx.
\]

In this way we estimate the remaining terms of the left side of (2.8). If \( 1 < p < 2 \), we use (2.10) to obtain

\[
\int_{\Omega} |\nabla u_n - \nabla u_m|^p \varrho \, dx
\]

\[
\leq \int_{\Omega} \frac{|\nabla u_n - \nabla u_m|^p}{(|\nabla u_n| + |\nabla u_m|)^{2-p}p} \left( |\nabla u_n| + |\nabla u_m| \right)^{2-p} \varrho \, dx
\]

\[
\leq \left( \int_{\Omega} \frac{|\nabla u_n - \nabla u_m|^2}{(|\nabla u_n| + |\nabla u_m|)^{2-p}p} \varrho \, dx \right)^{\frac{p}{2}} \left( \int_{\Omega} (|\nabla u_n| + |\nabla u_m|)^p \varrho \, dx \right)^{\frac{2-p}{2}}.
\]

Since the sequence \( \left\{ \int_{\Omega} |\nabla u_n|^p \varrho \, dx \right\} \) is bounded we derive from this that

\[
\left( \int_{\Omega} |\nabla u_n - \nabla u_m|^p \varrho \, dx \right)^{\frac{2}{p}} \leq C_1 \int_{\Omega} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m, \nabla u_n - \nabla u_m) \varrho \, dx
\]

for some constant \( C_1 > 0 \). It is now clear that, up to a subsequence, \( u_n \to u \) in \( E(\Omega) \).

We now consider the case \( 1 < q < 2 \). Let us denote by \( I_{nm} \) the left hand side of inequality (2.8) without the integral involving \( c \). We rewrite (2.8) in the following way:

\[
I_{nm} + \int_{\Omega} (|u_n|^{t-2}u_n - |u_m|^{t-2}u_m, u_n - u_m) c \, dx
\]

\[
= \int_{\Omega} (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m, u_n - u_m) a^+ \, dx
\]

\[
+ \int_{\Omega} (|u_n|^{s-2}u_n - |u_m|^{s-2}u_m, u_n - u_m) b^- \, dx + o(1)
\]

if \( 1 < s < 2 \). By Lemma 2.1 the last two integrals converge to 0 as \( n, m \to \infty \).

We now apply the argument from the previous case to the terms on the left side. In this way we again show that \( u_n \to u \) in \( E(\Omega) \). Finally, if \( 2 < s \) we
modify (2.8) in the following way:

\[
I_{mn} + \frac{1}{2} \int_{\Omega} (|\nabla u_n|^{t-2}\nabla u_n - |\nabla u_m|^{t-2}\nabla u_m, \nabla u_n - \nabla u_m)c\, dx \\
= \int_{\Omega} (|u_n|^{q-2}u_n - |u_m|^{q-2}u_m, u_n - u_m)a^+\, dx \\
+ \int_{\Omega} \int_0^1 \tilde{f}_u(x, u_n + t(u_n - u_m)) dt (u_n - u_m)^2\, dx + o(1),
\]

where \( \tilde{f}_u = \tilde{F} \) and \( \tilde{F}(x, u) = a^+(x)|u|^q - c(x)|u|^t/2t \). To complete the proof we repeat the argument from the previous part. 

3. Main result. By Lemma 2.3 the functional \( J \) is bounded from below on \( E(\Omega) \). We put

\[ m = \inf_{u \in E(\Omega)} J(u). \]

By the Ekeland variational principle [7] there exists a Palais–Smale sequence \( \{u_n\} \) at level \( m \) (see also [15, Corollary 2.5]). It then follows from Proposition 2.5 that, up to a subsequence, \( u_n \to u \) in \( E(\Omega) \). Obviously \( u \) is a nontrivial solution of (1.1) provided \( m < 0 \). In Theorem 1 below we formulate conditions guaranteeing that \( m < 0 \). It is clear that \( |u| \) is also a minimizer of \( J \). Therefore we can assume that \( u \) is nonnegative on \( \mathbb{R}^N \). We put

\[ A(v) = \int_{\Omega} |v|^q a(x)\, dx \quad \text{and} \quad C(v) = \int_{\Omega} |v|^t c(x)\, dx. \]

**Theorem 3.1.** Suppose that (A), (B) and (H) hold and moreover that \( a^+, b^- \in L^\infty_{\text{loc}}(\Omega) \).

(i) If (\( * \)) \( q < \min(p, s, t) \), then problem (1.1) has a solution.

If (\( * \)) is not satisfied we assume that \( V = \text{Int}(\text{supp} a^+ - \text{supp} b) \neq \emptyset \). We then have two cases:

(ii) If \( q < p \), then problem (1.1) has a nontrivial solution.

(iii) If \( p < q \) and there exists a \( C^1 \) function \( v \) with \( \text{supp} v \subset V \) such that

\[
(t - p) \frac{p-q}{p^{\frac{t}{t-p}}} (q - p) \frac{p-q}{p^{\frac{t}{t-p}}} \frac{(||v||_{L^p(E)})^{\frac{t-q}{t-p}}}{C(v)^{\frac{p-q}{p^{\frac{t}{t-p}}}} < A(v)},
\]

then problem (1.1) has a nontrivial solution.

**Proof.** (i) Let \( v \) be a \( C^1 \) function with \( v \neq 0 \) and \( \text{supp} v \subset \{x \in \Omega : a(x) > 0\} \). Since \( q < \min(p, s, t) \) we see that \( J(\sigma v) < 0 \) for \( \sigma > 0 \) sufficiently small and so \( m < 0 \)
(ii) We choose \( v \) as in (i) but with \( \text{supp} \, v \subset V \). Then \( J(\sigma v) < 0 \) for \( \sigma > 0 \) sufficiently small and so \( m < 0 \).

(iii) Let \( v \) be a function satisfying (3.1) and let

\[
f(\sigma) = \frac{\sigma^{p-q}}{p} \|v\|^{p}_{1,p} + \frac{\sigma^{t-q}}{t} C(v).
\]

Then we have

\[
\inf_{\sigma > 0} f(\sigma) = (t-p) \frac{\sigma^{p-q}}{p \gamma r^q \beta^k} \left( \frac{t-q}{t-\beta} \right) \left( \frac{\|v\|^q_{e^r}}{C(v) \gamma r^q \beta^k} \right) < A(v).
\]

Since \( v \) satisfies (3.1) there exists \( \sigma > 0 \) such that

\[
\frac{\sigma^{p-q}}{p} \|v\|^{p}_{1,p} + \frac{\sigma^{t-q}}{t} C(v) < A(v)
\]

and consequently \( m < 0 \).

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Received September 7, 2006;
received in final form November 10, 2006