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FUNCTIONAL ANALYSIS

The One-Third-Trick and Shift Operators

by

Richard LECHNER

Presented by Gilles PISIER

Summary. We obtain a representation as martingale transform operators for the rearrangement and shift operators introduced by T. Figiel. The martingale transforms and the underlying sigma algebras are obtained explicitly by combinatorial means. The known norm estimates for those operators are a direct consequence of our representation.

1. Introduction. The proof of the $T(1)$ theorem by T. Figiel proceeds by expanding the integral operator into an absolutely converging series of basic building blocks T_m and U_m, rearranging and shifting the Haar system. This involves the following norm estimates for those building blocks, which T. Figiel obtained by combinatorial means:

$$\|T_m : L^p_X \to L^p_X\| \le C(\log_2(2+|m|))^\alpha, \tag{1.1}$$

$$\|U_m : L^p_X \to L^p_X\| \le C(\log_2(2+|m|))^\beta, \tag{1.2}$$

where the constant $C > 0$ depends only on p, the UMD-constant of X and $0 < \alpha, \beta < 1$. For the original proof see [Fig88] and [FW01]. See also [NS97] and [Mül05]. For extensions to spaces of homogeneous type see [MP12].

The purpose of the present paper is to obtain a representation of T_m and U_m as the sum of roughly $\log_2(2+|m|)$ martingale transform operators. This is done by combinatorial analysis of the equations defining T_m and U_m.

Our combinatorial analysis exhibits the link of T. Figiel's rearrangement and shift operators to the so called one-third-trick originating in the work of [Wol82], [GJ82], [Dav80] and [CWW85].

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Related recent developments. Recently T. P. Hytönen [Hyt12] presented his own proof of T. Figiel's vector-valued $T(1)$ theorem (see [Fig90]). The basic aim of T. P. Hytönen in [Hyt12] is the same as ours, to find reductions of the general case to certain preferable situations where the so called Figiel compatibility condition is satisfied. To this end the problem in [Hyt12] is randomized and the properties of the so called random dyadic partitions of Nazarov, Treil and Volberg [NTV97, NTV03] are exploited.

By contrast, the proof in the present paper proceeds by finding explicitly those filtrations that turn a given bad combinatorial situation into a good one, such that Figiel's compatibility condition is satisfied. Our reduction is self-contained and develops specific combinatorics of colored dyadic intervals.

2. Preliminaries

The Haar system in $\mathbb{R}$. For any given finite interval $I \subset \mathbb{R}$ we define

$$h_I = \mathtt{1}_{I_0} - \mathtt{1}_{I_1},$$

where $\mathtt{1}_A$ denotes the characteristic function of a set A, $I_0 = [\inf I, (\inf I + \sup I)/2[$ and $I_1 = [(\inf I + \sup I)/2, \sup I[$. Let $\mathscr{D}_j = \{[2^{-j}k, 2^{-j}(k+1)[: k \in \mathbb{Z}\}$, $j \in \mathbb{Z}$, and define the collection of dyadic intervals by $\mathscr{D} = \bigcup_{j\in\mathbb{Z}} \mathscr{D}_j$. The L^∞-normalized Haar system is given by $\{h_I : I \in \mathscr{D}\}$.

Banach spaces with the UMD-property. By $L^p(\Omega, \mu; X)$ we denote the space of functions with values in X, Bochner-integrable with respect to μ. If $\Omega = \mathbb{R}$ and μ is the Lebesgue measure $|\cdot|$ on $\mathbb{R}$, then set $L^p_X(\mathbb{R}) = L^p(\mathbb{R}, |\cdot|; X)$, if unambiguous further abbreviated as L^p_X.

We say X is a *UMD space* if for every X-valued martingale difference sequence $\{d_j\}_j \subset L^p(\Omega, \mu; X)$, $1 < p < \infty$, and choice of signs $\varepsilon_j \in \{-1, 1\}$ one has

$$\Big\|\sum_j \varepsilon_j d_j\Big\|_{L^p(\Omega,\mu;X)} \le \mathscr{U}_p(X)\Big\|\sum_j d_j\Big\|_{L^p(\Omega,\mu;X)}, \tag{2.1}$$

where $\mathscr{U}_p(X)$ does not depend on ε_j or d_j. The constant $\mathscr{U}_p(X)$ is called the *UMD-constant*. We refer the reader to [Bur81].

Kahane's contraction principle. For every Banach space X, $1 \le p < \infty$, finite set $\{x_j\}_j \subset X$ and bounded sequence $\{c_j\}_j$ of scalars we have

$$\Big(\int_0^1 \Big\|\sum_j r_j(t) c_j x_j\Big\|_X^p\, dt\Big)^{1/p} \le \sup_j |c_j| \Big(\int_0^1 \Big\|\sum_j r_j(t) x_j\Big\|_X^p\, dt\Big)^{1/p}, \tag{2.2}$$

where $\{r_j\}_j$ denotes an independent sequence of Rademacher functions. For details see [Kah85].

Kahane's inequality. Given $1 \le p < \infty$, there exists a constant K_p such that for any Banach space X and any finite sequence $\{x_j\} \subset X$ we have

$$\Big(\int_0^1 \Big\| \sum_j r_j(t) x_j \Big\|_X^p \, dt\Big)^{1/p} \le K_p \int_0^1 \Big\| \sum_j r_j(t) x_j \Big\|_X \, dt, \tag{2.3}$$

where $\{r_j\}_j$ denotes an independent sequence of Rademacher functions. For details see [Kah85].

The martingale inequality of Stein — Bourgain's version. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m \subset \mathcal{F}$ denote an increasing sequence of σ-algebras. For every choice of $f_1, \ldots, f_m \in L^p(\Omega, \mu; X)$, $1 < p < \infty$, letting $r_1, \ldots, r_m$ denote independent Rademacher functions, we have

$$\int_0^1 \Big\| \sum_{i=1}^m r_i(t) \, \mathbb{E}(f_i \,|\, \mathcal{F}_i) \Big\|_{L^p(\Omega,\mu;X)} dt \le C \int_0^1 \Big\| \sum_{i=1}^m r_i(t) f_i \Big\|_{L^p(\Omega,\mu;X)} dt, \tag{2.4}$$

where C depends only on p and $\mathscr{U}_p(X)$.

The scalar-valued version of (2.4) by E. M. Stein can be found in [Ste70]. The vector-valued extension is due to J. Bourgain [Bou86]. A proof may be found in [FW01].

3. The one-third-trick. The one-third-trick originates in the work of [Wol82], [GJ82] and [CWW85].

3.1. Bilateral alternating one-third-shift. For every $j \in \mathbb{Z}$ let

$$s_j = (-1)^j \, 2^{-j}/3, \tag{3.1}$$

and define

$$s(I) = s_j \tag{3.2}$$

for all intervals I having measure $|I| = 2^{-j}$. Then define the *one-third-shift map*

$$\sigma(I) = I + s(I), \tag{3.3}$$

and the *one-third-shift operator*

$$S(h_I) = h_{\sigma(I)}. \tag{3.4}$$

For every given interval $I \in \mathscr{D}$ there exists a unique one-third-shifted interval $J \in \sigma(\mathscr{D})$ with $|J| = |I|/2$, contained in I. Observe that for every $j \in \mathbb{Z}$ and $I \in \mathscr{D}_j$ we have

$$\#\{J \in \sigma(\mathscr{D}_{j+1}) : J \cap I \neq \emptyset\} = 3,$$
$$\#\{J \in \sigma(\mathscr{D}_{j+1}) : J \subset I\} = 1.$$

So we can define $\omega(I)$ by

$$\omega(I) = J, \quad \text{where} \quad J \in \sigma(\mathscr{D}),\ |J| = |I|/2 \text{ and } J \subset I. \tag{3.5}$$

Note the basic properties summarized in:

Lemma 3.1. *The following statements are true:*

(i) *$\sigma(\mathscr{D})$ is a nested collection of dyadic intervals, and $\mathscr{D} \cap \sigma(\mathscr{D}) = \emptyset$.*
(ii) *$\omega : \mathscr{D} \to \sigma(\mathscr{D})$ is well defined and injective.*
(iii) *If $I \in \mathscr{D}$, then $\omega(I) \subset I$.*
(iv) *For every $I \in \mathscr{D}$ we have* $\operatorname{dist}(\omega(I), I^c) = |I|/6$.
(v) *If $I, J \in \mathscr{D}$ and $|I| = |J|$, then* $\operatorname{dist}(\omega(I), \omega(J)) < |\omega(I)|$ *if and only if $I = J$.*
(vi) *For all $I \in \mathscr{D}$ we have the identity*

$$\sigma(I) = \omega(I) \cup \big(\omega(I) + \operatorname{sign}(s(I)) \cdot |\omega(I)|\big).$$

Proof. The assertions are easily verified. ∎

We need some more notation. For all $j \in \mathbb{Z}$ and

$$u = \sum_{I \in \mathscr{D}} u_I h_I$$

let $(u)_j$ restrict the function u to level j, precisely

$$(u)_j = \sum_{I \in \mathscr{D}_j} u_I h_I. \tag{3.6}$$

If we define

$$\mathbb{I}(u)_j = \sum_{I \in \mathscr{D}_j} u_I 1_I, \tag{3.7}$$

then Kahane's contraction principle (2.2) and Kahane's inequality (2.3) yield

$$\int_0^1 \Big\| \sum_{j \in \mathbb{Z}} r_j(t)(u)_j \Big\|_{L^p_X} dt \approx \int_0^1 \Big\| \sum_{j \in \mathbb{Z}} r_j(t)\, \mathbb{I}(u)_j \Big\|_{L^p_X} dt. \tag{3.8}$$

The following theorem establishes that the one-third-shift operator $S : L^p_X \to L^p_X$ is an isomorphism.

Theorem 3.2. *Let $1 < p < \infty$ and X a Banach space with the* UMD-*property. Then there exists a constant $C > 0$ such that*

$$\frac{1}{C} \|u\|_{L^p_X} \le \|Su\|_{L^p_X} \le C \|u\|_{L^p_X}$$

for all $u \in L^p_X$. The constant C depends only on $\mathscr{U}_p(X)$.

Proof. Let $u = \sum_{I\in\mathscr{D}} u_I h_I |I|^{-1} \in L^p_X$ be fixed throughout this proof and set

$$v = \sum_{I\in\mathscr{D}} u_I h_{\omega(I)} |\omega(I)|^{-1},$$
$$(v)_j = \sum_{|\omega(I)|=2^{-j}} u_I h_{\omega(I)} |\omega(I)|^{-1}.$$

Note that $\{\omega(I) : I \in \mathscr{D}\}$ is nested (see Lemma 3.1(i) & (ii)). Observe that

$$\mathbb{I}(u)_j = \mathbb{E}(\mathbb{I}(v)_j \mid \mathscr{D}_j)$$

by Lemma 3.1(iii), so the UMD-property, Kahane's contraction principle (2.2) and Kahane's inequality (2.3) yield

$$\begin{aligned}\|u\|_{L^p_X} &\lesssim \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)(u)_j \Big\|_{L^p_X} dt \\ &= \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\, \mathbb{I}(u)_j \Big\|_{L^p_X} dt \\ &= \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\, \mathbb{E}(\mathbb{I}(v)_j \mid \mathscr{D}_j) \Big\|_{L^p_X} dt.\end{aligned}$$

Now we apply Stein's martingale inequality (2.4) followed by (3.8) to pass from $\mathbb{I}(v)_j$ to $(v)_j$, so

$$\begin{aligned}\int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\, \mathbb{E}(\mathbb{I}(v)_j \mid \mathscr{D}_j) \Big\|_{L^p_X} dt &\lesssim \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\, \mathbb{I}(v)_j \Big\|_{L^p_X} dt \\ &\approx \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)(v)_j \Big\|_{L^p_X} dt.\end{aligned}$$

Recalling definition (3.4) and applying Kahane's contraction principle, in view of $\omega(I) \subset \sigma(I)$ (see identity Lemma 3.1(vi)), we estimate

$$\int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)(v)_j \Big\|_{L^p_X} dt \le 2\int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)(Su)_j \Big\|_{L^p_X} dt,$$

and the UMD-property implies

$$\int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)(v)_j \Big\|_{L^p_X} dt \lesssim \|Su\|_{L^p_X}.$$

Thus, collecting the inequalities yields

$$\|u\|_{L^p_X} \lesssim \|Su\|_{L^p_X}.$$

One can repeat the preceding argument with the roles of u and Su interchanged to obtain the converse inequality

$$\|Su\|_{L^p_X} \lesssim \|u\|_{L^p_X}. \ \blacksquare$$

3.2. Unilateral one-third-shift. We now introduce modified versions σ_0 and σ_1 of the one-third-shift map σ. To this end we define $\sigma_0, \sigma_1 : \mathscr{D} \to \sigma(\mathscr{D})$ by

$$\sigma_0(I) = J, \quad \text{where} \quad J \in \sigma(\mathscr{D}),\ |J| = |I| \text{ and } \sup J \in I, \tag{3.9}$$

$$\sigma_1(I) = J, \quad \text{where} \quad J \in \sigma(\mathscr{D}),\ |J| = |I| \text{ and } \inf J \in I. \tag{3.10}$$

This induces the one-third-shift operators S_0 and S_1 given by the linear extension of

$$S_0(h_I) = h_{\sigma_0(I)}, \qquad I \in \mathscr{D}, \tag{3.11}$$

$$S_1(h_I) = h_{\sigma_1(I)}, \qquad I \in \mathscr{D}. \tag{3.12}$$

Observe that if $I \in \mathscr{D}_j$ and j is odd we have

$$\sigma_0(I) = \sigma(I) \quad \text{and} \quad \sigma_1(I) = \sigma(I) + |I|,$$

and if j is even

$$\sigma_0(I) = \sigma(I) - |I| \quad \text{and} \quad \sigma_1(I) = \sigma(I).$$

Furthermore, since

$$\sigma_i(\mathscr{D}_j) = \sigma(\mathscr{D}_j) \qquad \text{for all } i = 0, 1 \text{ and } j \in \mathbb{Z},$$

we deduce from Lemma 3.1(i) that $\sigma_i(\mathscr{D})$ is nested.

Finally, we can see that

$$|I \cap \sigma_0(I)| \geq \tfrac{1}{3}|I|, \qquad |I \cap \sigma_1(I)| \geq \tfrac{1}{3}|I|,$$

for all $I \in \mathscr{D}$. The proof of Theorem 3.2 with minor modifications yields Theorem 3.3 below.

THEOREM 3.3. *Let $1 < p < \infty$ and X a Banach space with the* UMD-*property. Then there exists a constant $C > 0$ such that*

$$\frac{1}{C}\|u\|_{L^p_X} \leq \|S_0 u\|_{L^p_X} \leq C\|u\|_{L^p_X},$$

$$\frac{1}{C}\|u\|_{L^p_X} \leq \|S_1 u\|_{L^p_X} \leq C\|u\|_{L^p_X},$$

for all $u \in L^p_X$. The constant C depends only on $\mathscr{U}_p(X)$.

Proof. Define ω_0 and ω_1 by

$$\omega_0(I) = J, \quad \text{where} \quad J \in \sigma(\mathscr{D}),\ |J| = |I|/4 \text{ and } \sup J = \sup \sigma_0(I),$$

$$\omega_1(I) = J, \quad \text{where} \quad J \in \sigma(\mathscr{D}),\ |J| = |I|/4 \text{ and } \inf J = \inf \sigma_1(I),$$

for all $I \in \mathscr{D}$. Now all we need to do is repeat the proof of Theorem 3.3 with ω replaced by ω_δ in order to estimate S_δ, for each $\delta \in \{0, 1\}$. $\blacksquare$

4. The shift operator T_m. Here we define $16+4\log_2(|m|)$, $m \neq 0$, subcollections of the Haar system, so that on each such subcollection, T_m acts as a martingale transform operator on either the dyadic grid or the one-third-shifted dyadic grid. In Section 3 we established that changing the dyadic grid to the one-third-shifted dyadic grid is an isomorphism. Thus we may assume that T_m is representable as a martingale transform operator on each of the $16+4\log_2(|m|)$ subcollections, which yields the well known estimate

$$\|T_m : L^p_X \to L^p_X\| \leq C(\log_2(2+|m|))^\alpha,$$

for some $0<\alpha<1$, established by T. Figiel [Fig88].

Define the shift map τ_m, $m \in \mathbb{Z}$, by

$$\tau_m(I) = I + m|I| \tag{4.1}$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$. This induces the shift operator T_m, given by

$$T_m h_I = h_{\tau_m(I)} \tag{4.2}$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$. It is crucial that the one-third-shift operator S defined in (3.4) and the shift operator T_m commute, that is,

$$(S \circ T_m)(u) = (T_m \circ S)(u) \tag{4.3}$$

for all $u \in L^p_X$. Analogously, we have

$$(S_0 \circ T_m)(u) = (T_m \circ S_0)(u), \tag{4.4}$$

$$(S_1 \circ T_m)(u) = (T_m \circ S_1)(u), \tag{4.5}$$

for all $u \in L^p_X$ (see (3.9)–(3.12)).

We aim at splitting the dyadic intervals $\mathscr{D}$ into collections $\mathscr{B}_i^{(\delta)}$ such that we may bound $T_m \circ S^\delta$ on functions supported on $\sigma^\delta(\mathscr{B}_i^{(\delta)})$, $\delta \in \{0,1\}$. Note that if $\delta = 0$, then $S^\delta = \mathrm{Id}$ and $\sigma^\delta = \mathrm{Id}$.

Given a shift width $m \in \mathbb{Z}$, $m \neq 0$, we will partition the dyadic intervals $\mathscr{D}$ into $16+4\log_2(|m|)$ disjoint collections denoted by $\mathscr{B}_i^{(\delta)}$. The collections are constructed in such way that for each i and $\delta \in \{0,1\}$ fixed, whenever $I \in \mathscr{B}_i^{(\delta)}$, the intervals $\sigma^\delta(I)$ and $(\tau_m \circ \sigma^\delta)(I)$ share the same dyadic predecessor with respect to the collection $\sigma^\delta(\mathscr{B}_i^{(\delta)})$. The details are elaborated in Lemma 4.1 below.

LEMMA 4.1. *For every integer $m \in \mathbb{Z}$, $m \neq 0$ let τ_m denote the map given by*

$$\tau_m(I) = I + m|I|$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$. Then there exist a constant $K(m) \leq 7 + 2\log_2(|m|)$ and disjoint collections of dyadic intervals $\mathscr{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$, *with*

$$\mathscr{D} = \bigcup_{\delta\in\{0,1\}} \bigcup_{i=0}^{K(m)} \mathscr{B}_i^{(\delta)},$$

such that

$$\{I, \tau_m(I), I \cup \tau_m(I) : I \in \sigma^\delta(\mathscr{B}_i^{(\delta)})\} \tag{4.6}$$

is a nested collection of sets, for all $0 \le i \le K(m)$ *and* $\delta \in \{0,1\}$.

Proof. By symmetry we may assume that $m \ge 1$, and we set $K(m) = K(-m)$ if $m \le -1$. So fix a shift width $m \ge 2$ and a $\lambda \ge 4$ such that

$$2^{\lambda-3} \le m < 2^{\lambda-2}, \tag{4.7}$$

and define $L(m) = \lambda - 1$. If $m = 1$, then let $\lambda = 4$ and set $L(1) = 3$. Now we split $\mathscr{D}$ into disjoint collections $\mathscr{A}_i$, $0 \le i \le L(m)$, by omitting $L(m)$ consecutive levels of $\mathscr{D}$. More precisely, for every $0 \le i \le L(m)$ we define

$$\mathscr{A}_i = \bigcup_{j\in\mathbb{Z}} \{I \in \mathscr{D} : |I| = 2^{-(\lambda j+i)}\}. \tag{4.8}$$

Next we want to divide each $\mathscr{A}_i$ into two collections $\mathscr{A}_i^{(0)}$ and $\mathscr{A}_i^{(1)}$ so that every $I \in \mathscr{A}_i^{(0)}$ has the same predecessor in $\mathscr{A}_i^{(0)}$ as $\tau_m(I)$, and $\mathscr{A}_i^{(0)}$ is maximal. As a consequence, the collection $\mathscr{A}_i^{(1)}$ consists of all intervals I such that I and $\tau_m(I)$ do *not* share the same predecessor. But, if we apply the one-third-shift map σ to $\mathscr{A}_i^{(1)}$, then every $I \in \sigma(\mathscr{A}_i^{(1)})$ has the same predecessor in $\sigma(\mathscr{A}_i^{(1)})$ as $\tau_m(I)$.

We will now construct these two collections. To this end let $\mathscr{G}$ denote one of the collections $\mathscr{A}_i$, $\sigma(\mathscr{A}_i)$, $0 \le i \le L(m)$, and define

$$\begin{aligned} \mathscr{C}_0(\mathscr{G}, I) &= \{J \in \mathscr{G} : |J| = 2^{-\lambda}|I|,\ J \subset I \text{ and } \tau_m(J) \subset I\}, \\ \mathscr{C}_1(\mathscr{G}, I) &= \{J \in \mathscr{G} : |J| = 2^{-\lambda}|I|,\ J \subset I \text{ and } \tau_m(J) \cap I = \emptyset\}. \end{aligned} \tag{4.9}$$

Revisiting the definition of the one-third-shift maps σ, σ_1 and considering the restriction (4.7) one can see that

$$\sigma(\mathscr{C}_1(\mathscr{A}_i, I)) \subset \mathscr{C}_0(\sigma(\mathscr{A}_i), \sigma_1(I)) \tag{4.10}$$

for all $I \in \mathscr{A}_i$, $0 \le i \le L(m)$. Note that for $I \in \mathscr{D}_j$ we have $\sigma_1(I) = \sigma(I)$ if j is even and $\sigma_1(I) = \sigma(I + |I|)$ if j is odd. Thus (4.10) implies that all intervals $J \in \sigma(\mathscr{C}_1(\mathscr{A}_i, I))$ are such that J and $\tau_m(J)$ share $\sigma_1(I)$ as common predecessor with respect to the collection $\sigma(\mathscr{A}_i^{(1)})$. In Figure 1 one can see the action of the one-third-shift map σ on the collection $\mathscr{A}_i$.

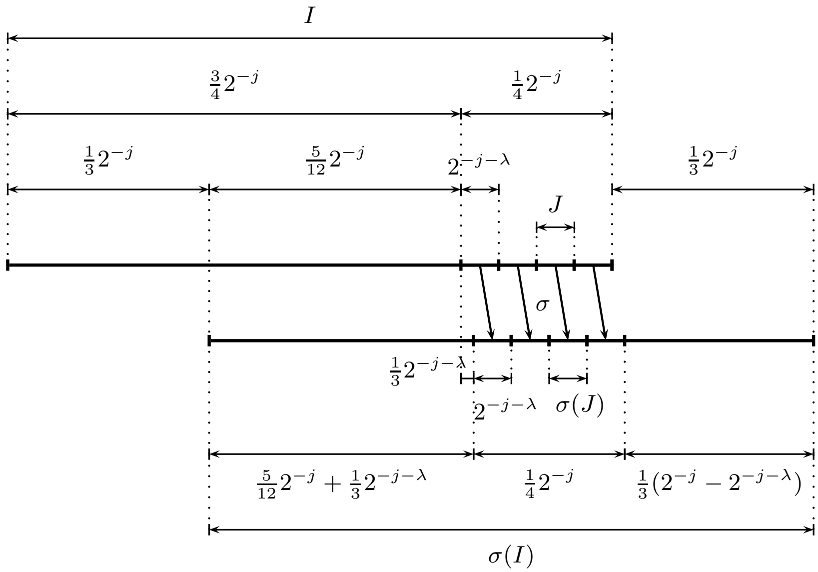

Fig. 1. The one-third-shift map σ acting on $I \in \mathscr{D}$, $|I| = 2^{-j}$ and $J \in \mathscr{D}$, $|J| = 2^{-j-\lambda}$, where $J \subset I$ and $\tau_m(J) \cap I = \emptyset$. In this picture λ is even.

Now define for every $0 \le i \le L(m)$ the following collections of dyadic intervals:

$$\begin{aligned} \mathscr{A}_i^{(0)} &= \bigcup \{\mathscr{C}_0(\mathscr{A}_i, I) : I \in \mathscr{A}_i\}, \\ \mathscr{A}_i^{(1)} &= \mathscr{A}_i \setminus \mathscr{A}_i^{(0)}. \end{aligned} \tag{4.11}$$

Finally, for all $0 \le i \le L(m)$ and $\delta \in \{0,1\}$ we split $\mathscr{A}_i^{(\delta)}$ into two disjoint collections

$$\mathscr{B}_i^{(\delta)} \quad \text{and} \quad \mathscr{B}_{i+L(m)+1}^{(\delta)} \tag{4.12}$$

such that

$$\mathscr{B}_i^{(\delta)} \cap \tau_m(\mathscr{B}_i^{(\delta)}) = \emptyset \tag{4.13}$$

for all $0 \le i \le K(m)$ and $\delta \in \{0,1\}$, where we set $K(m) = 2L(m) + 1$. Considering (4.7) and $L(m) = \lambda - 1$ we find that $K(m) \le 7 + 2\log_2(m)$. For this purpose consider the collection

$$\mathscr{E} = \{\tau_k(I) : I \in \mathscr{D},\ \inf I = 0,\ 0 \le k \le m-1\},$$

and observe that

$$\mathscr{D} = \bigcup_{\substack{j \in \mathbb{Z} \\ j \text{ even}}} \tau_{jm}(\mathscr{E}) \cup \bigcup_{\substack{j \in \mathbb{Z} \\ j \text{ odd}}} \tau_{jm}(\mathscr{E}) = \mathscr{D}_{\text{even}} \cup \mathscr{D}_{\text{odd}}.$$

Now define the collections

$$\mathscr{B}_i^{(\delta)} = \mathscr{A}_i^{(\delta)} \cap \mathscr{D}_{\text{even}}, \qquad \mathscr{B}_{i+L(m)+1}^{(\delta)} = \mathscr{A}_i^{(\delta)} \cap \mathscr{D}_{\text{odd}}, \tag{4.14}$$

for all $0 \le i \le L(m)$ and $\delta \in \{0,1\}$.

Taking into account (4.10), (4.9) and noting that $\tau_m(I) \in \mathscr{D}_{\text{odd}}$ if and only if $I \in \mathscr{D}_{\text{even}}$, we have verified (4.6), finishing this proof. ∎

Remark 4.2. Note that we actually proved the slightly stronger result

$$I \cup \tau_m(I) \subset \pi^{\lambda}(I) \tag{4.15}$$

for all $I \in \sigma^{\delta}(\mathscr{B}_i^{(\delta)})$, $0 \le i \le K(m)$, $\delta \in \{0,1\}$. For a given interval $I \in \sigma^{\delta}(\mathscr{D})$, $\pi(I)$ is the unique interval $J \in \sigma^{\delta}(\mathscr{D})$ such that $J \supset I$ and $|J| = 2\,|I|$.

As the combinatorial Lemma 4.1 exhibits the link between the shift map τ_m, the one-third-shift map σ and Figiel's compatibility condition (4.6), Theorem 4.3 below will translate the combinatorial results into analytical results, exhibiting the link between the shift operator T_m, the one-third-shift operator S and martingale transform operators.

In the following context it is understood that $1 < p < \infty$, X is a Banach space with the UMD-property and $m \in \mathbb{Z}$, $m \neq 0$. Now we define projections $P_i^{(\delta)} : L_X^p \to L_X^p$, associated with the collections $\mathscr{B}_i^{(\delta)}$ of Lemma 4.1 by

$$P_i^{(\delta)} u = \sum_{I \in \mathscr{B}_i^{(\delta)}} \langle u, h_I \rangle h_I |I|^{-1} \tag{4.16}$$

for all $0 \le i \le K(m)$, $\delta \in \{0,1\}$ and $u \in L_X^p$. The UMD-property of X implies uniform bounds on the projections $P_i^{(\delta)}$. Note the identity

$$u = \sum_{\delta \in \{0,1\}} \sum_{i=0}^{K(m)} P_i^{(\delta)} u \tag{4.17}$$

for all $u \in L_X^p$, since the collections $\mathscr{B}_i^{(\delta)}$, $0 \le i \le K(m)$, $\delta \in \{0,1\}$, form a partition of $\mathscr{D}$ (see Lemma 4.1).

Exploiting the fact that the one-third-shift operator S is an isomorphism on L_X^p (see Theorem 3.2), we will now estimate the shift operator T_m on the range of each $P_i^{(\delta)}$.

Theorem 4.3. *Let $1 < p < \infty$ and X be a Banach space with the* UMD-*property. Then for every $m \in \mathbb{Z}$, $0 \le i \le K(m)$ and $\delta \in \{0,1\}$ the inequality*

$$\|T_m \circ P_i^{(\delta)} u\|_{L_X^p} \le C \|P_i^{(\delta)} u\|_{L_X^p}, \tag{4.18}$$

holds true for all $u \in L_X^p$, where the constant C depends only on $\mathscr{U}_p(X)$. The projections $P_i^{(\delta)}$, $0 \le i \le K(m)$, $\delta \in \{0,1\}$, are defined according to (4.16), *and $K(m) \le 7 + 2\log_2(1 + |m|)$.*

Proof. Note that by symmetry once we establish (4.18) for $m \geq 1$, the theorem will be proved.

Recalling the properties of the partition $\mathscr{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$, of $\mathscr{D}$ (see Lemma 4.1), we know that the collection

$$\{I, \tau_m(I), I \cup \tau_m(I) : I \in \sigma^\delta(\mathscr{B}_i^{(\delta)})\} \tag{4.19}$$

is nested, for all $0 \leq i \leq K(m)$ and $\delta \in \{0,1\}$. Throughout this proof let $m \in \mathbb{Z}$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$ and $u \in P_i^{(\delta)}(L_X^p)$ be fixed. According to (4.16) we may assume that u has the representation

$$u = \sum_{I \in \mathscr{B}_i^{(\delta)}} u_I h_I |I|^{-1}.$$

For every $J \in \sigma^\delta(\mathscr{D})$ let

$$A^{(\delta)}(J) = J \cup \tau_m(J), \tag{4.20}$$

and define

$$\mathscr{A}_j^{(\delta)} = \{A^{(\delta)}(J) : J \in \sigma^\delta(\mathscr{D}_j) \cap \sigma^\delta(\mathscr{B}_i^{(\delta)})\} \tag{4.21}$$

for all $j \in \mathbb{Z}$. Then specify the filtration $\{\mathscr{F}_j^{(\delta)}\}_j$ by

$$\mathscr{F}_j^{(\delta)} = \sigma\text{-alg}\Big(\bigcup_{i \leq j} \mathscr{A}_i^{(\delta)}\Big), \tag{4.22}$$

and observe that due to (4.19) every $A^{(\delta)}(J)$, $J \in \sigma^\delta(\mathscr{D}_j)$, is an atom for $\mathscr{F}_j^{(\delta)}$. The sigma algebra $\sigma\text{-alg}(\bigcup_{i \leq j} \mathscr{A}_i^{(\delta)})$ is the smallest sigma algebra containing $\bigcup_{i \leq j} \mathscr{A}_i^{(\delta)}$. The one-third-shift operator is given by

$$S^\delta u = \sum_{I \in \mathscr{B}_i^{(\delta)}} u_I h_{\sigma^\delta(I)} |I|^{-1} = \sum_{J \in \sigma^\delta(\mathscr{B}_i^{(\delta)})} u_{\sigma^{-\delta}(J)} h_J |J|^{-1} \tag{4.23}$$

(see (3.4) for details). We recall the notation

$$(u)_j = \sum_{|I| = 2^{-j}} u_I h_I |I|^{-1} \quad \text{and} \quad \mathbb{I}(u)_j = \sum_{|I| = 2^{-j}} u_I 1_I |I|^{-1},$$

and note that

$$\|T_m S^\delta u\|_{L_X^p} \approx \int_0^1 \Big\| \sum_{j \in \mathbb{Z}} r_j(t)\, \mathbb{I}(T_m S^\delta u)_j \Big\|_{L_X^p} dt \tag{4.24}$$

(see (3.6)–(3.8)). If $I \in \mathscr{D}_j \cap \mathscr{B}_i^{(\delta)}$ then

$$\mathbb{E}(1_{\sigma^\delta(I)} \,|\, \mathscr{F}_j^{(\delta)}) = |A^{(\delta)}(\sigma^\delta(I))|^{-1} 1_{A^{(\delta)}(\sigma^\delta(I))} \geq \frac{1}{2}\, 1_{\tau_m(\sigma^\delta(I))},$$

therefore

$$\mathbb{I}(T_m S^\delta u)_j \le 2\,\mathbb{E}(\mathbb{I}(S^\delta u)_j \mid \mathscr{F}_j^{(\delta)}).$$

Thus, Kahane's contraction principle and Bourgain's version of Stein's martingale inequality yield

$$\begin{aligned}\int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\,\mathbb{I}(T_m\, S^\delta u)_j \Big\|_{L^p_X} dt &\le \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\, 2\,\mathbb{E}(\mathbb{I}(S^\delta u)_j \mid \mathscr{F}_j^{(\delta)}) \Big\|_{L^p_X} dt \\ &\lesssim \int_0^1 \Big\| \sum_{j\in\mathbb{Z}} r_j(t)\,\mathbb{I}(S^\delta u)_j \Big\|_{L^p_X} dt \\ &\approx \|S^\delta u\|_{L^p_X}.\end{aligned}$$

Combining the latter estimate with (4.24), Theorem 3.2 proves

$$\|T_m S^\delta u\|_{L^p_X} \lesssim \|u\|_{L^p_X}. \tag{4.25}$$

According to (4.3) the shift operator T_m and the one-third-shift operator S commute, so we have the identity

$$T_m u = (S^{-\delta} \circ T_m \circ S^\delta)(u),$$

and we obtain, by an application of Theorem 3.2,

$$\|T_m u\|_{L^p_X} \lesssim \|(T_m \circ S^\delta)(u)\|_{L^p_X}. \tag{4.26}$$

We conclude the proof by combining (4.26) and (4.25). ■

Remark 4.4. By slightly adjusting the construction of $\mathscr{B}_i^{(\delta)}$ we could replace Bourgain's version of Stein's martingale inequality by the martingale transforms in [Fig88, Proposition 2, Step 0] in order to obtain (4.25).

Within this remark we shall make use of the following notation. If $I \in \mathscr{D} \cup \sigma(\mathscr{D})$, then I_0 is the uniquely defined interval $J \in \mathscr{D} \cup \sigma(\mathscr{D})$ with $|J| = |I|/2$ such that $\inf J = \inf I$, and I_1 is the uniquely defined interval $J \in \mathscr{D} \cup \sigma(\mathscr{D})$ with $|J| = |I|/2$ such that $\sup J = \sup I$.

Let us now describe the aforementioned modifications. We have to replace λ by $\lambda + 1$ and redefine $\mathscr{C}_0$ and $\mathscr{C}_1$ as

$$\begin{aligned}\mathscr{C}_0(I, \mathscr{A}_i) &= \{J \in \mathscr{A}_i : |J| = 2^{-\lambda}|I|,\ J \subset I_0 \text{ and } \tau_m(J) \subset I_0\} \\ &\quad \cup \{J \in \mathscr{A}_i : |J| = 2^{-\lambda}|I|,\ J \subset I_1 \text{ and } \tau_m(J) \subset I_1\}, \\ \mathscr{C}_1(I, \mathscr{A}_i) &= \{J \in \mathscr{A}_i : |J| = 2^{-\lambda}|I|,\ J \subset I_0 \text{ and } \tau_m(J) \cap I_0 = \emptyset\} \\ &\quad \cup \{J \in \mathscr{A}_i : |J| = 2^{-\lambda}|I|,\ J \subset I_1 \text{ and } \tau_m(J) \cap I_1 = \emptyset\}\end{aligned}$$

(cf. (4.8) and (4.9)). This results in the collection

$$\{J_0, \tau_m(J)_0, J_1, \tau_m(J)_1, J \cup \tau_m(J) : J \in \sigma^\delta(\mathscr{B}_i^{(\delta)})\} \tag{4.27}$$

being nested for all $0 \le i \le K(m)$ and $\delta \in \{0,1\}$. With these modifications let us define

$$d_{J,1}^{(\delta)} = \frac{1}{2}(h_J + h_{\tau_m(J)}) \quad \text{and} \quad d_{J,2}^{(\delta)} = \frac{1}{2}(h_J - h_{\tau_m(J)}),$$

for all $J \in \sigma^\delta(\mathscr{B}_i^{(\delta)})$. Since (4.27) is nested, $\{d_{J,1}^{(\delta)}, d_{J,2}^{(\delta)} : J \in \sigma(\mathscr{B}_i^{(\delta)})\}$ forms a martingale difference sequence. Observe $h_J = d_{J,1}^{(\delta)} + d_{J,2}^{(\delta)}$ and $h_{\tau_m(J)} = d_{J,1}^{(\delta)} - d_{J,2}^{(\delta)}$, hence we may swap h_J and $h_{\tau_m(J)}$ without using Bourgain's version of Stein's martingale inequality.

5. A martingale decomposition for U_m. Recall that in (4.1) we defined the shift map τ_m for every $m \in \mathbb{Z}$ by

$$\tau_m(I) = I + m|I|$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$. We introduce the operator U_m by setting

$$U_m h_I = 1_{\tau_m(I)} - 1_I \tag{5.1}$$

for all $I \in \mathscr{D} \cup \sigma(\mathscr{D})$. Although the operators T_m and U_m appear to be similar, U_m is much harder to analyze than T_m. This is rooted in the observation that $\{T_m h_I\}_{I\in\mathscr{A}}$ is a martingale difference sequence for any choice of $\mathscr{A} \subset \mathscr{D}$, whereas whether $\{U_m h_I\}_{I\in\mathscr{B}}$ forms a martingale difference sequence strongly depends on the choice of $\mathscr{B} \subset \mathscr{D}$. Essentially the same method we used to bound T_m for functions supported on the collections $\mathscr{B}_i^{(0)}$, $0 \le i \le K(m)$, qualifies for estimating U_m, since Lemma 4.1 ensures that $\{U_m h_I : I \in \mathscr{B}_i^{(0)}\}$ forms a martingale difference sequence. So the main obstacle is to estimate U_m on $\mathscr{B}_i^{(1)}$, since $\{U_m h_I : I \in \mathscr{B}_i^{(1)}\}$ is *not* a martingale difference sequence.

The remedy to this problem is the martingale difference sequence decomposition of U_m into

$$U_m = a_I + b_I + c_I, \qquad I \in \mathscr{B}_i^{(1,\varepsilon)},$$

for suitable subcollections $\mathscr{B}_i^{(1,\varepsilon)}$, $\varepsilon \in \{0,1\}$, of $\mathscr{B}_i^{(1)}$, such that the collections $\{a_I : I \in \mathscr{B}_i^{(1,\varepsilon)}\}$, $\{b_I : I \in \mathscr{B}_i^{(1,\varepsilon)}\}$ and $\{c_I : I \in \mathscr{B}_i^{(1,\varepsilon)}\}$ form martingale difference sequences. Since there are $56 + 14\log_2(|m|)$ subcollections $\mathscr{B}_i^{(\delta,\varepsilon)}$, we can obtain the estimate

$$\|U_m : L_X^p \to L_X^p\| \le C(\log_2(2+|m|))^\beta \quad \text{for some } 0 < \beta < 1,$$

established in [Fig88] by the same method we used for T_m in Section 4.

First, let us define the maps β_0, β_1 and β by

$$\beta_0(I) = \sigma_0(I) \setminus I, \tag{5.2}$$

$$\beta_1(I) = \sigma_1(I) \cap I, \tag{5.3}$$

$$\beta(I) = \beta_0(I) \cup \beta_1(I), \tag{5.4}$$

where σ_0 and σ_1 are given by (3.9) and (3.10). Secondly, set

$$\gamma_0(I) = \sigma_0(I) \cap I, \tag{5.5}$$

$$\gamma_1(I) = \sigma_1(I) \setminus I, \tag{5.6}$$

$$\gamma(I) = \gamma_0(I) \cup \gamma_1(I). \tag{5.7}$$

The functions σ_0, σ_1, β_0, β_1 and γ_0, γ_1 are visualized in Figure 2.

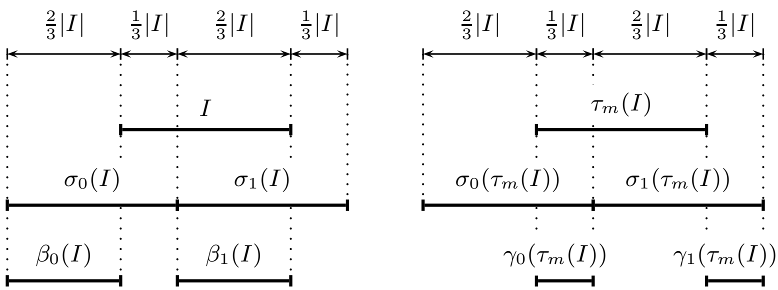

Fig. 2. The functions σ_0, σ_1, β_0, β_1 and γ_0, γ_1 applied to I or $\tau_m(I)$

Similar to the combinatorial Lemma 4.1 for the operator T_m we shall now display a combinatorial lemma for U_m, which will enable us to use the same method to estimate U_m as well. The estimates of U_m will be established in Theorem 5.2.

Lemma 5.1. *Let $m \in \mathbb{Z}$, $m \neq 0$ and let $\mathscr{B}_i^{(\delta)}$, $0 \leq i \leq K(m)$, $\delta \in \{0,1\}$, and $K(m)$ be defined as in Lemma 4.1. If we split each collection $\mathscr{B}_i^{(1)}$ into*

$$\mathscr{B}_i^{(1,\varepsilon)} = \mathscr{B}_i^{(1)} \cap \{\tau_k(I) : I \in \mathscr{D}, \inf I = 0, k \bmod 2 = \varepsilon\} \tag{5.8}$$

for all $0 \leq i \leq K(m)$, $\varepsilon \in \{0,1\}$, then

$$\{I, \tau_m(I), I \cup \tau_m(I) : I \in \mathscr{B}_i^{(0)}\}, \tag{5.9}$$

$$\{\sigma_0(I), \sigma_1(\tau_m(I)), \sigma_0(I) \cup \sigma_1(\tau_m(I)) : I \in \mathscr{B}_i^{(1,\varepsilon)}\}, \tag{5.10}$$

$$\{\beta_0(I), \beta_1(I), \beta_0(I) \cup \beta_1(I) : I \in \mathscr{B}_i^{(1,\varepsilon)}\}, \tag{5.11}$$

$$\{\gamma_0(I), \gamma_1(I), \gamma_0(I) \cup \gamma_1(I) : I \in \tau_m(\mathscr{B}_i^{(1,\varepsilon)})\} \tag{5.12}$$

are nested collections of sets, for all $0 \leq i \leq K(m)$ and $\varepsilon \in \{0,1\}$.

Proof. By symmetry we may assume $m \geq 1$. Let $m \in \mathbb{Z}$, $m \geq 1$, $0 \leq i \leq K(m)$ and $\varepsilon \in \{0,1\}$ be fixed throughout the rest of this proof. Given $I \in \sigma^\delta(\mathscr{D})$ its predecessor $\pi(I)$ is the uniquely determined interval $J \in \sigma^\delta(\mathscr{D})$ such that $J \supset I$ and $|J| = 2|I|$. Given intervals $J \in \mathscr{B}_i^{(1)}$ and $K = \pi^\lambda(J)$

observe that

$$(5.13)\qquad \tau_m(J)\cap K=\emptyset,\quad \inf J\geq \inf K+\frac{3}{4}|K|,\quad \tau_{-1}(J)\subset K.$$

First, note that (5.9) is covered by Lemma 4.1.

Secondly, we will show that (5.10) is nested. Henceforth, we shall abbreviate $\mathscr{B}_i^{(1,\varepsilon)}$ by $\mathscr{B}$. Let $I,J\in\mathscr{B}$ be such that

$$\big(\sigma_0(J)\cup\sigma_1(\tau_m(J))\big)\cap\big(\sigma_0(I)\cup\sigma_1(\tau_m(I))\big)\neq\emptyset.$$

Note that if $|I|=|J|$, then $I=J$ by definition of $\mathscr{B}$. So assume that $|J|<|I|$. Defining $K:=\pi^\lambda(J)$ and recalling (5.13) we see $\inf J\geq\inf K+\frac{3}{4}|K|$, therefore $\sigma_0(J)\subset\sigma_1(K)$. Furthermore, considering the constraints for the parameters λ and m we obtain

$$\sup\sigma_1(\tau_m(J))\leq\sup K+\frac{2}{3}|J|+m|J|<\sup K+\frac{1}{3}|K|,$$

thus $\sigma_1(\tau_m(J))\subset\sigma_1(K)$ as well. So we have proved that $\sigma_0(J)\cup\sigma_1(J)\subset\sigma_1(K)$. The nestedness of the one-third-shifted intervals implies either $\sigma_1(K)\subset\sigma_0(I)$ or $\sigma_1(K)\subset\sigma_1(I)$, hence

$$\sigma_0(J)\cup\sigma_1(J)\subset\sigma_0(I)\quad\text{or}\quad\sigma_0(J)\cup\sigma_1(J)\subset\sigma_1(I).$$

Thirdly, we prove the nestedness of (5.11). To this end, let $I,J\in\mathscr{B}$ be such that

$$\beta(J)\cap\beta(I)\neq\emptyset.$$

As in the proof of (5.10) we see that $|I|=|J|$ implies $I=J$. Let us now assume $|J|<|I|$,

$$(\mathcal{A})\qquad \beta(J)\cap\beta(I)\neq\emptyset\quad\text{and}\quad\beta(J)\cap\beta(I)^c\neq\emptyset.$$

Considering that $\beta(I)=\beta_0(I)\cup\beta_1(I)$ and $\beta(J)\subset J\cup\tau_{-1}(J)$, assumption $(\mathcal{A})$ is covered by the following four cases:

(1) $\inf\beta_0(I)\in J\cup\tau_{-1}(J)$,
(2) $\sup\beta_0(I)\in J\cup\tau_{-1}(J)$,
(3) $\inf\beta_1(I)\in J\cup\tau_{-1}(J)$,
(4) $\sup\beta_1(I)\in J\cup\tau_{-1}(J)$.

Assume case (1) is true. Define $K=\pi^\lambda(J)$ and note that $|K|\leq|I|$ by definition of $\mathscr{B}$. Since the endpoint of the one-third-shifted interval $\sigma_0(I)$ is contained in a non-shifted smaller interval K, we obtain

$$(5.14)\qquad \inf\beta_0(I)=\inf K+\frac{z}{3}|K|$$

for some $z\in\{1,2\}$, depending on the direction of the one-third-shift. We know from (5.13) that $\inf J\geq\inf K+\frac{3}{4}|K|$, hence

$$\inf J\cup\tau_{-1}(J)\geq\inf K+\frac{11}{16}|K|>\inf K+\frac{2}{3}|K|,$$

which contradicts (5.14). If we assume case (2) is true, then $\sup\beta_0(I) = \inf I$ and we find that $\inf J = \inf I$. Consequently, $\inf J = \inf \pi^\lambda(J)$, which contradicts $J \in \mathscr{B}$. Case (3) is analogous to (1), and case (4) is analogous to (2). Altogether we have proved that assumption ($\mathcal{A}$) is false, therefore, considering that $\operatorname{dist}(\beta_0(I), \beta_1(I)) \geq \frac{1}{3}|I|$ and $\operatorname{diam}(\beta(J)) \leq \frac{1}{8}|I|$ yields

$$\beta(J) \subset \beta_0(I) \quad \text{or} \quad \beta(J) \subset \beta_1(I)$$

for all $I, J \in \mathscr{B}$ with $|J| < |I|$ such that $\beta(J) \cap \beta(I) \neq \emptyset$.

The proof that (5.12) is nested uses essentially the same argument presented in the proof of (5.11), therefore we omit the details. Both of these cases use that if $J \in \tau_m(\mathscr{B})$, then $\gamma(J) \subset J \cup \tau_1(J)$, $\pi^\lambda(J) = \pi^\lambda(\tau_1(J))$ and $\inf \tau_1(J) \neq \inf \pi^\lambda(\tau_1(J))$. ■

With $m \in \mathbb{Z}$ fixed, we introduce the functions

$$a_I = \mathbb{1}_{\sigma_1(\tau_m(I))} - \mathbb{1}_{\sigma_0(I)}, \qquad I \in \mathscr{D}, \tag{5.15}$$

$$b_I = \mathbb{1}_{\beta_0(I)} - \mathbb{1}_{\beta_1(I)}, \qquad I \in \mathscr{D}, \tag{5.16}$$

$$c_I = \mathbb{1}_{\gamma_0(\tau_m(I))} - \mathbb{1}_{\gamma_1(\tau_m(I))}, \qquad I \in \mathscr{D} \tag{5.17}$$

(see Figure 3).

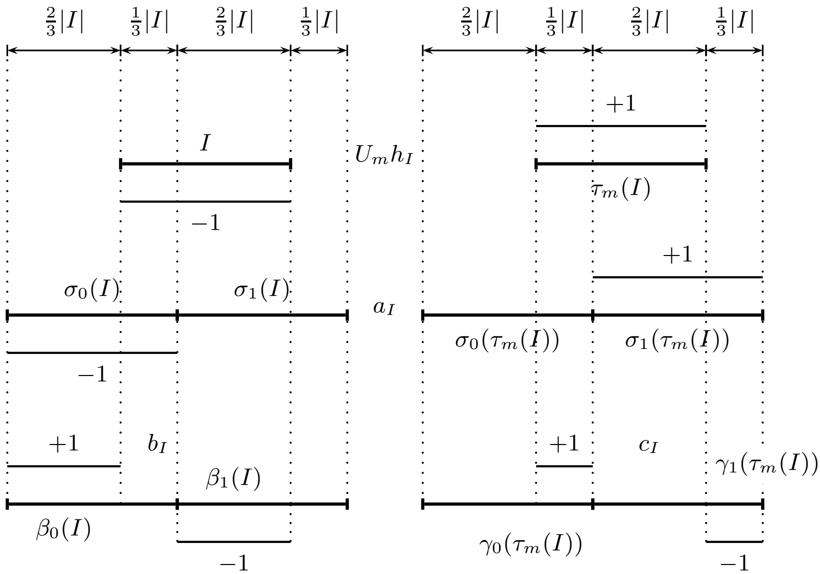

Fig. 3. Martingale decomposition of U_m

Accordingly, we define operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ as the linear extensions of

$$\mathcal{A}h_I = a_I, \quad I \in \mathscr{D}, \tag{5.18}$$

$$\mathcal{B}h_I = b_I, \quad I \in \mathscr{D}, \tag{5.19}$$

$$\mathcal{C}h_I = c_I, \quad I \in \mathscr{D}. \tag{5.20}$$

Note the identities

$$\begin{aligned} U_m h_I &= a_I + b_I + c_I, && I \in \mathscr{D}, \\ U_m u &= \mathcal{A}u + \mathcal{B}u + \mathcal{C}u, && u \in L^p_X, \\ \mathcal{A}u &= (U_{m+1} \circ S_0)(u), && u \in L^p_X, \end{aligned} \tag{5.21}$$

as well as the estimates

$$\begin{aligned} |b_I| &\le |h_I| + |S_0(h_I)|, && I \in \mathscr{D}, \\ |c_I| &\le |T_m(S_0 h_I)| + |T_m(S_1 h_I)|, && I \in \mathscr{D} \end{aligned} \tag{5.22}$$

(see Figure 3).

We will introduce projections $P_i^{(0)}$, $P_i^{(1,\varepsilon)}$, $0 \le i \le K(m)$, $\varepsilon \in \{0,1\}$, and estimate the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ on their respective images (see Theorem 5.2). Recall that

$$P_i^{(0)} u = \sum_{I \in \mathscr{B}_i^{(0)}} \langle u, h_I \rangle h_I |I|^{-1}, \quad 0 \le i \le K(m),$$

was already defined in (4.16). Accordingly, we set

$$P_i^{(1,\varepsilon)} u = \sum_{I \in \mathscr{B}_i^{(1,\varepsilon)}} \langle u, h_I \rangle h_I |I|^{-1}, \quad 0 \le i \le K(m),\ \varepsilon \in \{0,1\}, \tag{5.23}$$

for all $u \in L^p_X$, where $\mathscr{B}_i^{(1,\varepsilon)}$ is defined in (5.8). Note that $P_i^{(1)} = P_i^{(1,0)} + P_i^{(1,1)}$, where $P_i^{(1)}$ was defined in (4.16). Thus, (4.17) implies the identity

$$u = \sum_{i=0}^{K(m)} P_i^{(0)} u + P_i^{(1,0)} u + P_i^{(1,1)} u, \quad u \in L^p_X. \tag{5.24}$$

Consider the splitting of $\mathscr{D}$ into the sets $\mathscr{B}_i^{(\delta)}$, $0 \le i \le K(m)$, $\delta \in \{0,1\}$ (see Lemma 4.1 for details), which we used in Theorem 4.3 to treat the operator T_m. Retracing our steps in the proof of Theorem 4.3 we shall find that with the result in Lemma 5.1 we could actually repeat this proof with the operator $T_m \circ P_i^{(\delta)}$, $\delta \in \{0,1\}$, replaced by one of the operators $U_m \circ P_i^{(0)}$ or $\mathcal{A} \circ P_i^{(1,\varepsilon)}$, $\varepsilon \in \{0,1\}$. The residual terms $\mathcal{B} \circ P_i^{(1,\varepsilon)}$ and $\mathcal{C} \circ P_i^{(1,\varepsilon)}$ occurring in the martingale decomposition of U_m are dominated by $(\mathrm{Id} + S_0) \circ P_i^{(1,\varepsilon)}$ and $(T_m \circ (S_0 + S_1)) \circ P_i^{(1,\varepsilon)}$, respectively.

THEOREM 5.2. *Let $m \in \mathbb{Z}$, $m \neq 0$ and let $P_i^{(0)}$, $P_i^{(1,\varepsilon)}$, $\varepsilon \in \{0,1\}$, be defined according to* (4.16) *and* (5.23). *Then*

$$U_m u = \sum_{i=0}^{K(m)} \Big\{ (U_m \circ P_i^{(0)})(u) + \sum_{\varepsilon \in \{0,1\}} ((\mathcal{A} + \mathcal{B} + \mathcal{C}) \circ P_i^{(1,\varepsilon)})(u) \Big\} \tag{5.25}$$

for all $u \in L_X^p$. Furthermore, we have the estimates

$$\|(U_m \circ P_i^{(0)})(u)\|_{L_X^p} \le C \|P_i^{(0)} u\|_{L_X^p}, \tag{5.26}$$

$$\|(\mathcal{A} \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p} \le C \|P_i^{(1,\varepsilon)} u\|_{L_X^p}, \tag{5.27}$$

$$\|(\mathcal{B} \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p} \le C \|P_i^{(1,\varepsilon)} u\|_{L_X^p}, \tag{5.28}$$

$$\|(\mathcal{C} \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p} \le C \|P_i^{(1,\varepsilon)} u\|_{L_X^p}, \tag{5.29}$$

for all $u \in L_X^p$, $0 \le i \le K(m)$ and $\varepsilon \in \{0,1\}$. The constant C depends only on $\mathscr{U}_p(X)$ and we have the estimate $K(m) \le 7 + 2\log_2(|m|)$.

Proof. Let $m \in \mathbb{Z}$, $m \neq 0$, $0 \le i \le K(m)$ and $\varepsilon \in \{0,1\}$ be fixed throughout the proof. By symmetry we may assume that $m \ge 1$.

Note that (5.21) and (5.24) imply identity (5.25).

From Lemma 5.1 we know that each of the collections

$$\{U_m h_I : I \in \mathscr{B}_i^{(0)}\}, \qquad \{\mathcal{A} h_I : I \in \mathscr{B}_i^{(1,\varepsilon)}\},$$
$$\{\mathcal{B} h_I : I \in \mathscr{B}_i^{(1,\varepsilon)}\}, \qquad \{\mathcal{C} h_I : I \in \mathscr{B}_i^{(1,\varepsilon)}\}$$

forms a martingale difference sequence. Hence, in order to estimate $\{U_m h_I : I \in \mathscr{B}_i^{(0)}\}$ and $\{\mathcal{A} h_I : I \in \mathscr{B}_i^{(1,\varepsilon)}\}$ we can essentially repeat the proof of Theorem 4.3, thereby obtaining (5.26) and (5.27). Since $|b_I| \le |h_I| + |S_0(h_I)|$ (see (5.22)), we deduce from the UMD-property, Kahane's inequality (2.3) and Kahane's contraction principle (2.2) that

$$\|(\mathcal{B} \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p} \lesssim \|P_i^{(1,\varepsilon)} u\|_{L_X^p} + \|(S_0 \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p}.$$

The latter estimate together with Theorem 3.3 implies (5.28). Observe $|c_I| \le |T_m(S_0 h_I)| + |T_m(S_1 h_I)|$ (see (5.22)), thus the UMD-property, Kahane's inequality and Kahane's contraction principle imply

$$\|(\mathcal{C} \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p} \lesssim \|(S_0 \circ T_m \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p} + \|(S_1 \circ T_m \circ P_i^{(1,\varepsilon)})(u)\|_{L_X^p}.$$

Note that (5.10) implies that

$$\{\sigma_\delta(I), \sigma_\delta(\tau_m(I)), \sigma_\delta(I) \cup \sigma_\delta(\tau_m(I)) : I \in \mathscr{B}_i^{(1,\varepsilon)}\}$$

is nested, hence we can replicate the proof of Theorem 4.3 once more. ∎

From Theorems 4.3 and 5.2, one can obtain the estimates stated in Theorem 5.3 below by exploiting the type and cotype inequalities for T_m, and only the cotype inequality for U_m. Inserting the estimates of Theorems 4.3

and 5.2 into [Fig88, Lemma 1] one can obtain [Fig88, Theorem 1], stated below for the sake of completeness.

THEOREM 5.3 ([Fig88]). *Let $1 < p < \infty$, and X be a Banach space with the UMD-property. For $m \in \mathbb{Z}$ let τ_m denote the shift map defined by*

$$I \mapsto I + m|I|.$$

Let T_m, U_m denote the linear extensions of the maps

$$T_m h_I = h_{\tau_m(I)}$$

and

$$U_m h_I = \mathbb{1}_{\tau_m(I)} - \mathbb{1}_I,$$

respectively. Then

$$\|T_m : L^p_X \to L^p_X\| \le C(\log_2(2+|m|))^\alpha, \tag{5.30}$$

$$\|U_m : L^p_X \to L^p_X\| \le C(\log_2(2+|m|))^\beta, \tag{5.31}$$

where the constant $C > 0$ depends only on $\mathscr{U}_p(X)$ and $0 < \alpha, \beta < 1$. Moreover, if L^p_X has type $\mathcal{T}$ and cotype $\mathcal{C}$, then one can take $\alpha = 1/\mathcal{T} - 1/\mathcal{C}$ and $\beta = 1 - 1/\mathcal{C}$.

These bounds are used to establish equivalence of some bases to the Haar system, e.g. the Franklin system (cf. [Fig88]). In [Fig90], T. Figiel bounds singular integral operators S acting on vector-valued L^p spaces by means of the estimates (5.30) and (5.31). This is achieved by decomposing S into weighted series of the rearrangement operators T_m and U_m.

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Richard Lechner
Institute of Analysis
Johannes Kepler University Linz
Altenbergerstrasse 69
A-4040 Linz, Austria
E-mail: Richard.Lechner@jku.at

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