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GENERAL TOPOLOGY

A Remark on Variational Principles of Choban, Kenderov and Revalski

by

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Summary. We consider some variational principles in the spaces $C^*(X)$ of bounded continuous functions on metrizable spaces X, introduced by M. M. Choban, P. S. Kenderov and J. P. Revalski. In particular we give an answer (consistent with ZFC) to a question stated by these authors.

1. Introduction. We denote by $C^*(X)$ the Banach algebra of bounded continuous real-valued functions on a completely regular space X, equipped with the norm $\|f\| = \sup\{|f(x)| : x \in X\}$.

Choban, Kenderov and Revalski [CKR] considered a game $G(X)$ on X, a variation of the Banach–Mazur game, played by two players Σ and Ω, and showed (cf. [CKR, Theorem 3.1]) that if Ω has a winning strategy then for any lower-semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, bounded from below and not everywhere equal to $+\infty$, the set

$$S(f) = \{g \in C^*(X) : f + g \text{ attains its infimum on } X\} \tag{1}$$

is residual in $C^*(X)$.

These authors asked ([CKR, Question 5.4]) if this "generic variational principle" for X implies that Ω has a winning strategy in the game $G(X)$.

The following result contains some observations concerning this question, and in particular, its second part shows that a negative answer is consistent with the ZFC-axioms of set theory.

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Theorem 1.1. *There exists a subset X of the real line for which Ω has no winning strategy in the game $G(X)$, but for every lsc function f:*

(i) *any separable subset of $C^*(X)$ is contained in a separable Banach subalgebra Z of $C^*(X)$ such that $S(f) \cap Z$ is residual in Z,*
(ii) *it is consistent with ZFC that $S(f)$ is residual in $C^*(X)$.*

Remark 1.2. More specifically, the set X in Theorem 1.1 has the following property. Let $\mathcal{Z}$ be the collection of all separable uniformly closed subalgebras of $C^*(X)$ containing the constants and separating points from closed sets in X (each separable subset of $C^*(X)$ is contained in some member of $\mathcal{Z}$). Then, for any $Z \in \mathcal{Z}$, $S(f) \cap Z$ is residual in Z. We do not know if, in the realm of the usual set theory, one can define a non-residual set C in $C^*(X)$ such that each intersection $C \cap Z$ with $Z \in \mathcal{Z}$ is residual in Z.

2. Some background. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower-semi-continuous function on a completely regular space X, which is bounded from below and takes some real values. By [CKR, Proposition 2.4(a)], the set

$$\mathrm{Gr}(M_f) = \{(g, x) \in C^*(X) \times X : (f + g)(x) = \inf(f + g)\} \tag{2}$$

is closed in the product $C^*(X) \times X$.

Let us notice that for $S(f)$ defined in (1), we have

$$S(f) = \mathrm{proj}\,\mathrm{Gr}(M_f), \tag{3}$$

where proj is the projection from $C^*(X) \times X$ onto the first factor.

Given a compactification αX of X,

$$C^*_\alpha(X) = \{g \in C^*(X) : g \text{ extends continuously over } \alpha X\} \tag{4}$$

is a Banach subalgebra of $C^*(X)$ containing the constants and separating points from closed sets in X, and each subalgebra of $C^*(X)$ with these properties is of the form (4) (cf. [En, 3.12.22(e)].

If X is separable metrizable, then for each separable $A \subset C^*(X)$ there is a metrizable compactification αX of X such that $A \subset C^*_\alpha(X)$; since $C^*_\alpha(X)$ can be identified with $C^*(\alpha X)$, $C^*_\alpha(X)$ is separable (cf. [En, 3.12.22]).

Let X be a subset of the unit interval whose complement contains no Cantor sets. Then Theorem 4.1 and Proposition 5.1 in [CKR] show that $S(f)$ is everywhere of second category in $C^*(X)$. The reasoning of Choban, Kenderov and Revalski can be repeated to deduce that for any compactification αX of X, $S(f) \cap C^*_\alpha(X)$ is everywhere of second category in $C^*_\alpha(X)$.

Now let $E \subset [0, 1]$ be a λ-set of cardinality $\aleph_1$, i.e. each countable set in E is a relative G_δ-set in E (cf. [Ku, §40, III]). The set X in Theorem 1.1 is $X = [0, 1] \setminus E$. Assuming $\mathfrak{p} > \aleph_1$, X is an analytic set (cf. [Fr, 23.J]).

For this set X (and in fact, for any metrizable space), the game $G(X)$ considered by Choban, Kenderov and Revalski is equivalent to the game

$G^*(X)$ introduced by Michael [Mi, p. 520], and since X is not completely metrizable, Corollary 7.5 in [Mi] shows that Ω has no winning strategy in the game $G(X)$.

We close this section by recalling that a *Suslin set* in a completely metrizable (but not necessarily separable) space H is a set obtained by the Suslin operation on closed subsets of H (separable Suslin sets coincide with analytic ones). The Suslin sets are open modulo first category sets, the projection on H of a Suslin set in the product of H with a compact metrizable space is a Suslin set, and in particular, if $f : H \to T$ is a Borel map into a compact metrizable space and $L \subset T$ is analytic then $f^{-1}(L)$ is Suslin (cf. [Ku, §11, VII], [JR, 2.3, 2.6.6, 2.9.4], [Ke]).

3. Proof of Theorem 1.1(i). Let $X = [0,1]\setminus E$ be the complement of a λ-set E of cardinality $\aleph_1$ in $[0,1]$ (cf. Section 2). As was noticed in Section 2, Ω has no winning strategy in the game $G(X)$.

Let (cf. (2))

$$\overline{\mathrm{Gr}(M_f)} = \text{the closure of } \mathrm{Gr}(M_f) \text{ in } C^*(X)\times[0,1], \tag{5}$$

and, for $g \in C^*(X)$,

$$K(g) = \{x \in [0,1] : (g,x) \in \overline{\mathrm{Gr}(M_f)}\}. \tag{6}$$

Since $\mathrm{Gr}(M_f)$ is closed in $C^*(X)\times X$ (cf. Section 2), we have, by (1),

$$C^*(X)\setminus S(f) = \{g \in C^*(X) : K(g) \subset E\}. \tag{7}$$

The map $g \mapsto K(g)$ is upper-semicontinuous, and hence it is a Borel map from $C^*(X)$ to the space $\mathcal{K}[0,1]$ of compact subsets of $[0,1]$, equipped with the Vietoris topology (cf. [Ke, 12.C]; recall that $\mathcal{K}[0,1]$ is a compact metrizable space).

Let αX be any compact metrizable extension of X. By comments in Section 2, it is enough to make sure that the $C^*_\alpha(X)\setminus S(f)$ is of first category in $C^*_\alpha(X)$ (cf. (4)).

To that end, let us consider $g_1, g_2, \ldots$ dense in $C^*_\alpha(X)\setminus S(f)$. Since compact subsets of E are countable, we infer from (7) that the set

$$A = \bigcup_i K(g_i) \subset E \text{ is countable.} \tag{8}$$

Therefore, since $\{L \in \mathcal{K}[0,1] : L\setminus A \neq \emptyset\}$ is an analytic set, we conclude that

$$A^* = \{g \in C^*_\alpha(X) : K(g) \subset A\} \tag{9}$$

is the complement of a Suslin set in $C^*_\alpha(X)$, hence open modulo first category sets in $C^*_\alpha(X)$.

Moreover, as was explained in Section 2, $S(f) \cap C^*_\alpha(X)$ is everywhere of second category in $C^*_\alpha(X)$, and in effect, we conclude that

(10) $$A^* \text{ is of first category in } C^*_\alpha(X).$$

Because E is a λ-set, (8) yields an existence of open sets $V_1 \supset V_2 \supset \cdots$ in $[0,1]$ such that $A = E \cap \bigcap_n V_n$. Since the map $g \mapsto K(g)$ is upper-semicontinuous, each set

$$V^*_n = \{g \in C^*_\alpha(X) : K(g) \subset E \cap V_n\}$$

is an open set in $C^*_\alpha(X) \setminus S(f)$ (cf. (7)) and

$$A^* = \bigcap_n V^*_n$$

(cf. (9)). By (8), (9), $g_i \in A^*$, so A^* is a dense G_δ-set in $C^*_\alpha(X) \setminus S(f)$ and hence $(C^*_\alpha(X) \setminus S(f)) \setminus A^*$ is of first category in $C^*_\alpha(X) \setminus S(f)$. This, combined with (10), implies that $S(f) \cap C^*_\alpha(X)$ is residual in $C^*_\alpha(X)$.

4. Proof of Theorem 1.1(ii). Let E and X be as in Section 3. As was recalled in Section 2, under $\mathfrak{p} > \aleph_1$, X is an analytic set and hence $\mathrm{Gr}(M_f) \subset C^*(X) \times [0,1]$ is Suslin in $C^*(X) \times X$. Therefore, by (1), $S(f)$ is Suslin in $C^*(X)$, hence open modulo first category sets. Since, as was established in [CKR], $S(f)$ is everywhere of second category in $C^*(X)$, we conclude that $S(f)$ is residual in $C^*(X)$.

Basing on the idea of this paper, it is possible to give another negative answer to a question from [CKR]. If the mapping M_f is non-empty-valued for a residual part of the space $C^*(X)$, then it is also single-valued and continuous for residually many g. This follows from quasicontinuity of M_f and metrizability of X. Therefore it settles the similar question related to strong minima as well.

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