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PROBABILITY THEORY AND STOCHASTIC PROCESSES

Two Kinds of Invariance of Full Conditional Probabilities

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Summary. Let G be a group acting on Ω and $\mathscr{F}$ a G-invariant algebra of subsets of Ω. A full conditional probability on $\mathscr{F}$ is a function $P : \mathscr{F} \times (\mathscr{F}\backslash\{\emptyset\}) \to [0,1]$ satisfying the obvious axioms (with only finite additivity). It is weakly G-invariant provided that $P(gA \mid gB) = P(A \mid B)$ for all $g \in G$ and $A, B \in \mathscr{F}$, and strongly G-invariant provided that $P(gA \mid B) = P(A \mid B)$ whenever $g \in G$ and $A \cup gA \subseteq B$. Armstrong (1989) claimed that weak and strong invariance are equivalent, but we shall show that this is false and that weak G-invariance implies strong G-invariance for every Ω, $\mathscr{F}$ and P as above if and only if G has no non-trivial left-orderable quotient. In particular, $G = \mathbb{Z}$ provides a counterexample to Armstrong's claim.

A *full conditional* (finitely additive) *probability* on an algebra $\mathscr{F}$ of subsets of Ω is a function $P : \mathscr{F} \times (\mathscr{F}\backslash\{\emptyset\}) \to [0,1]$ such that:

(a) $P(- \mid B)$ is a finitely additive probability on $\mathscr{F}$ with $P(B \mid B) = 1$ for each fixed $B \in \mathscr{F}\backslash\{\emptyset\}$, and
(b) $P(A \mid B)P(B \mid C) = P(A \mid C)$ whenever $A \subseteq B \subseteq C \in \mathscr{F}$ with B non-empty.

See [1, 2, 5, 7] for some existence results.

Now suppose G is a group acting on Ω, and $\mathscr{F}$ is invariant under G (i.e., $gA \in \mathscr{F}$ whenever $g \in G$ and $A \in \mathscr{F}$). Then there are two ways to define the concept of G-invariance of P. We say P is *weakly* G-invariant provided that $P(gA \mid gB) = P(A \mid B)$ for all $g \in G$ and $A, B \in \mathscr{F}$ with $B \neq \emptyset$. We say P is *strongly* G-invariant provided that $P(gA \mid B) = P(A \mid B)$ for all $g \in G$ and $A, B \in \mathscr{F}$ with $A \cup gA \subseteq B$ and $B \neq \emptyset$.

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In [1, Prop. 1.3], it is claimed that weak and strong G-invariance are equivalent, but the proof is clearly incomplete. It is indeed easy to see that strong G-invariance implies weak G-invariance (see below), but our purpose here is to show that the converse is in general false.

It is worth noting that the proof in [1] of the very interesting fact that a strongly G-invariant probability exists on the G-invariant algebra $\mathscr{F}$ whenever G is supramenable (i.e., when for every subset A of G there is a G-invariant finitely additive probability μ on the powerset of G with $\mu(A) = 1$) fortunately does not depend on the faulty claim.

In fact, we can give a complete characterization of those groups G for which strong G-invariance implies weak G-invariance. As usual, a group G is *left-orderable* provided that there is a linear order $\leq$ on G such that if $x \leq y$, then $gx \leq gy$ for $g \in G$. (For more work on orders and preorders on groups, see [4].) A quotient of a group is *non-trivial* provided that it has at least two elements. We shall assume the Axiom of Choice.

THEOREM 1. *For any group G, the following statements are equivalent:*

(a) *Whenever G acts on Ω and $\mathscr{F}$ is a G-invariant algebra of subsets of Ω, every weakly G-invariant full conditional probability on $\mathscr{F}$ is strongly G-invariant.*
(b) *G has no non-trivial left-orderable quotient.*

In particular, every Abelian group with an element of infinite order provides a counterexample to the implication from weak to strong G-invariance. On the other hand, since no group generated by elements of finite order (say, a Euclidean isometry group, which is generated by reflections) has a left-orderable quotient, we have:

COROLLARY 1. *If G is generated by elements of finite order, then weak G-invariance of full conditional probability implies strong G-invariance.*

I am also grateful to a referee for the following application. If G is a finite-index subgroup of $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, then G is not left-orderable [9]. Moreover, all the non-trivial quotients of G are either finite or have finite index [6, Chapter IV], and hence in either case are not left-orderable. Thus:

COROLLARY 2. *If G is a finite-index subgroup of* $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, *then weak G-invariance of full conditional probability implies strong G-invariance.*

For the proof of Theorem 1, it is easier to work with probability exchange rates, introduced in [2], though we simplify their definition. A *probability exchange rate* on an algebra $\mathscr{F}$ of subsets of Ω is a function $r : \mathscr{F} \times (\mathscr{F} \backslash \{\emptyset\}) \to [0, \infty]$ such that:

(a) $r(-, A)$ is a finitely additive measure with $r(A, A) = 1$, and

(b) $r(A,B)r(B,C) = r(A,C)$ whenever $A,B,C \in \mathscr{F}$ with B and C non-empty and $r(A,B)r(B,C)$ well-defined.

Here, ab for $a,b \in [0,\infty]$ is *well-defined* provided that it is not the case that one of a,b is zero and the other is infinity.

It is easy to check (cf. [2]) that every full conditional probability P has an associated exchange rate defined by $r_P(A,B) = P(A \mid A \cup B)/P(B \mid A \cup B)$ (where $a/0 = \infty$ for $a > 0$), and every exchange rate r is associated with a conditional probability defined by $P_r(A \mid B) = r(A \cap B, B)$. Moreover, $r_{P_r} = r$ for any exchange rate r and $P_{r_P} = P$ for any full conditional probability P.

If G acts on Ω and $\mathscr{F}$ is G-invariant, we say that an exchange rate r is *weakly G-invariant* provided that $r(gA, gB) = r(A,B)$ for all $A,B \in \mathscr{F}$ with $A \neq \emptyset$, and is *strongly G-invariant* provided that $r(gA,B) = r(A,B)$ for all $A,B \in \mathscr{F}$ with $B \neq \emptyset$. Observe that $r(A,B) = 1/r(B,A)$ whenever A and B are non-empty, and hence strong G-invariance is equivalent to the condition that $r(A,gB) = r(A,B)$, and thus strong G-invariance implies weak G-invariance.

It is easy to check that a full conditional probability is weakly (respectively, strongly) G-invariant if and only if the associated exchange rate is weakly (respectively, strongly) G-invariant. It follows that, as claimed earlier, weak invariance implies strong invariance for full conditional probabilities.

Proof of Theorem 1. Assume (b) is false. Let $\leq$ be a linear order on a non-trivial quotient $H = G/N$ with $\leq$ compatible with left multiplication. Then G acts on H by left multiplication.

Write $a < b$ provided $a \leq b$ but $b \not\leq a$. Let $\mathscr{F}$ be the algebra on H generated by sets of the form $[a,b] = \{h \in H : a \leq h \leq b\}$. Every member of $\mathscr{F}$ is a finite union of $\leq$-intervals.

Write $A < b$ for $A \subseteq H$ provided that $a < b$ for all $a \in A$, and use similar notation for other comparisons. Set

$$P(A \mid B) = \begin{cases} 0 & \text{if there exists } b \in B \text{ with } A \cap B < b, \\ 1 & \text{otherwise.} \end{cases}$$

This is a full conditional probability. Clearly $P(B \mid B) = 1$. Suppose A_1 and A_2 are disjoint. Suppose first that $P(A_1 \mid B) = P(A_2 \mid B) = 0$. Then $A_1 \cap B < b_1$ and $A_2 < b_2$ for $b_1, b_2 \in B$, and so $A_1 \cup A_2 < \max(b_1, b_2)$ and $P(A_1 \cup A_2 \mid B) = 0$. Next, suppose exactly one of $P(A_i \mid B)$ is 1 and the other is zero. Clearly, then $P(A_1 \cup A_2 \mid B) = 1$. The remaining case to dispose of is where $P(A_1 \mid B) = P(A_2 \mid B) = 1$. Then $A_1 \cap B$ and $A_2 \cap B$ are finite unions of intervals. Let I_i be a subinterval of $A_i \cap B$ such that for all $a \in A_i \cap B$ there is a $b \in I_i$ with $a \leq b$. Since $P(A_i \mid B) = 1$, for every $b \in B$ there is an $a \in A_i \cap B$ with $b \leq a$. Thus, for every member b of I_1 there is a member a

of I_2 with $b \leq a$ and for every member b of I_2 there is a member a of I_1 with $b \leq a$. But this is impossible for disjoint intervals I_1 and I_2. Thus, condition (a) of full conditional probability is satisfied.

Now suppose $A \subseteq B \subseteq C$ with B non-empty. If $P(A \mid B) = 0$, then A has a strict upper bound in B, and hence in $C \supseteq B$, so $P(A \mid C) = 0$. If $P(B \mid C) = 0$, then B has a strict upper bound in C, and hence $A \subseteq B$ does as well, so $P(A \mid C) = 0$. In both cases $P(A \mid B)P(B \mid C) = 0 = P(A \mid C)$. The remaining case is where $P(A \mid B) = P(B \mid C) = 1$. To obtain a contradiction, suppose $P(A \mid C) = 0$. Then there is a $c \in C$ such that $A < c$. If $c \in B$, then $P(A \mid B) = 0$ and we have a contradiction. So $c \notin B$. Suppose there is a $b \in B$ with $c \leq b$. Then $c < b$ as $c \notin B$, and $A < b$, so $P(A \mid B) = 0$, a contradiction. So there is no such b and hence $B < c$ by totality of $\leq$. Thus, $P(B \mid C) = 0$, a contradiction. Thus, we must have $P(A \mid C) = 1$ and hence $P(A \mid B)P(B \mid C) = 1 = P(A \mid C)$. Hence, P is a full conditional probability.

It is obvious that P is weakly G-invariant. Now choose $a, b \in H$ with $a < b$. Let $h \in G$ be such that $hb = a$. Then $P(\{b\} \mid \{a, b\}) = 1$ but $P(h\{b\} \mid \{a, b\}) = P(\{a\} \mid \{a, b\}) = 0$ and so strong invariance fails.

Now for the converse, suppose (a) is false. By the correspondence between exchange rates and full conditional probabilities, there is a space Ω acted on by G and an exchange rate r on a G-invariant algebra $\mathscr{F}$ of subsets of Ω where r is weakly but not strongly G-invariant. Since r is not strongly G-invariant, there are $A, B \in \mathscr{F}$ and $g_0 \in G$ such that $r(g_0 A, B) \neq r(A, B)$. It follows that $r(g_0 A, A) \neq 1$, since if $r(g_0 A, A) = 1$ then $r(g_0 A, B) = r(g_0 A, A) r(A, B) = r(A, B)$.

Write $f \lesssim g$ for f, g in G if and only if $r(fA, gA) \leq 1$. This is a total, reflexive and transitive relation, i.e., a total preorder. Write $f \sim g$ if and only if $f \lesssim g$ and $g \lesssim f$. Observe that $g_0 \nsim e$.

The rest of the proof is an argument from [3] which we give in detail for the reader's convenience. By the Axiom of Choice, let $\prec$ be any well-order on G (no need for compatibility with multiplication). Write $f \equiv g$ if and only if $fh \sim gh$ for all $h \in G$. If $f \not\equiv g$, write $h(f, g) = \min_{\prec}\{h \in G : fh \nsim gh\}$. Define $f \leq g$ for $f, g \in G$ if and only if either (a) $f \equiv g$ or (b) $f \not\equiv g$ and $fh(f, g) \lesssim gh(f, g)$.

Observe that $\leq$ is a total preorder. Reflexivity is immediate, and totality follows from the fact that if $f \not\equiv g$, then $f \leq g$ or $g \leq f$ depending whether $fh(f, g) \lesssim gh(f, g)$ or $gh(f, g) \lesssim fh(f, g)$, respectively. Only transitivity remains. It is easy to see that if $f \equiv g$ and $g \leq k$ then $f \leq k$ and that if $f \leq g$ and $g \equiv k$ then $f \leq k$. Suppose $f \not\equiv g$, $g \not\equiv k$, $f \leq g$ and $g \leq k$. Let $h_1 = h(f, g)$ and $h_2 = h(g, k)$. Then $fh_1 \lesssim gh_1$ and $gh_2 \lesssim kh_2$.

Let $h = \min_{\prec}(h_1, h_2)$. We then have $fh \lesssim gh$ since $h \leq h_1$ and $gh \lesssim kh$ since $h \leq h_2$. So $fh \lesssim kh$. Moreover, $fh' \sim gh' \sim kh'$ for all $h' \prec h$. Suppose

$fh \sim kh$. Since $fh \lesssim gh \lesssim kh$ we then have $fh \sim gh$ and $gh \sim kh$, which is impossible by choice of h, h_1, h_2. So $fh \nsim kh$ and hence $f \leq k$.

It is clear that $\leq$ is invariant under left multiplication. Observe also that the conjunction $f \leq g$ and $g \leq f$ holds if and only if $f \equiv g$. Let $N = \{f \in G : f \equiv e\}$. This is a normal subgroup of G. For fix $f \in N$ and $h \in G$. Then for any $g \in G$ we have $fhg \sim hg$ since $f \equiv e$, and so by left invariance of $\lesssim$ we have $h^{-1}fhg \sim h^{-1}hg = g$. Thus $h^{-1}fh \equiv e$, so $h^{-1}fh \in N$, and so N is normal. (This is the crucial point. If we had $\{f \in G : f \sim e\}$ normal, we could have worked more simply with $\lesssim$ instead of $\leq$.)

Observe that $g_0 \notin N$, since $g_0 \nsim e$. Thus N is a proper subgroup of G. Observe that if $a \in N$ and $f \in G$, then $ah \sim h$ for all $h \in G$, and so $fah \sim fh$. Thus, if $a \in N$, we have $fa \equiv f$.

Next define $fN \leq gN$ if and only if $f \leq g$. Then $\leq$ is a well-defined total order on G/N. For suppose $fa = f'$ and $gb = g'$ for $a, b \in N$ and $f, g \in G$. Then $f \equiv f'$ and $f \equiv g'$, and so $f \leq g$ if and only if $f' \leq g'$, and we have well-definition. The totality, reflexivity and transitivity of $\leq$ on G/N follows from that of $\leq$ on G. And if $fN \leq gN$ and $gN \leq fN$, then $f \equiv g$, and so $g^{-1}f \equiv e$ and hence $g^{-1}f \in N$ and $fN = gN$. ∎

Theorem 1 shows that if G has a non-trivial left-orderable quotient then there is a space Ω acted on by G and a G-invariant algebra $\mathscr{F}$ on Ω that provides a counterexample to the weak-to-strong G-invariance implication for full conditional probabilities. The counterexample crucially used an algebra $\mathscr{F}$ of finite unions of intervals.

It is also interesting whether the weak-to-strong implication holds in the special case where $\mathscr{F}$ is the powerset algebra $\mathcal{P}\Omega$. The answer to this is negative.

Example 1. Let $G = \mathbb{Z}$. It is known [7] that there is a strongly $\mathbb{Z}$-invariant full conditional probability P_0 on $\mathcal{P}\mathbb{Z}$ (indeed, on $\mathcal{P}\mathbb{R}$). For $A, B \in \mathcal{P}Z$ with $B \neq \emptyset$ set

$$P(A \mid B) = \begin{cases} 0 & \text{if } \sup A \cap B < \sup B < \infty, \\ 1 & \text{if } \sup A \cap B = \sup B < \infty, \\ P_0(A \cap \mathbb{Z}^+ \mid B \cap \mathbb{Z}^+) & \text{if } \sup B = \infty. \end{cases}$$

This is a full conditional probability. For clearly $P(- \mid B)$ is a finitely additive probability with $P(B \mid B) = 1$. Note that by strong $\mathbb{Z}$-invariance and finite additivity $P_0(A \mid B) = 0$ whenever A is finite and B is infinite.

Suppose $A \subseteq B \subseteq C$ with B non-empty. Suppose first that $\sup B < \infty$. If $\sup C = \infty$, then $C \cap \mathbb{Z}^+$ is infinite while both $B \cap \mathbb{Z}^+$ and $A \cap \mathbb{Z}^+$ are finite, and so $P(A \mid B)P(B \mid C) = 0 = P(A \mid C)$. If $\sup C < \infty$, then $P(A \mid B)P(B \mid C) = 1 = P(A \mid C)$ if $\sup A = \sup C$, and $P(A \mid B)P(B \mid C) = 0 = P(A \mid C)$ if $\sup A < \sup C$.

Suppose $\sup B = \infty$. Then

$$P(A \mid B)P(B \mid C) = P_0(A \cap \mathbb{Z}^+ \mid B \cap \mathbb{Z}^+)P_0(B \cap \mathbb{Z}^+ \mid C \cap \mathbb{Z}^+) \\ = P_0(A \cap \mathbb{Z}^+ \mid C \cap \mathbb{Z}^+) = P(A \mid C).$$

Since $P(\{0\} \mid \{0,1\}) = 0$ and $P(\{1\} \mid \{0,1\}) = 1$, we do not have strong $\mathbb{Z}$-invariance. But we do have weak $\mathbb{Z}$-invariance. All we need to show is that $P(1 + A \mid 1 + B) = P(A \mid B)$. If $\sup B < \infty$ this is obvious. Suppose $\sup B = \infty$. Then

$$P(1 + A \mid 1 + B) = P_0((1 + (A \cap \mathbb{Z}^+)) \cup A_0 \mid (1 + (B \cap \mathbb{Z}^+)) \cup B_0)$$

where A_0 and B_0 have at most one element each. But $B \cap \mathbb{Z}^+$ is infinite, so $P_0(A_0 \mid (1+(B\cap\mathbb{Z}^+))\cup B_0) = 0$ and $P_0(1+(B\cap\mathbb{Z}^+) \mid (1+(B\cap\mathbb{Z}^+))\cup B_0) = 1$, so

$$P_0((1 + (A \cap \mathbb{Z}^+)) \cup A_0 \mid (1 + (B \cap \mathbb{Z}^+)) \cup B_0) \\ = P_0(1 + (A \cap \mathbb{Z}^+) \mid 1 + (B \cap \mathbb{Z}^+)) \\ = P_0(A \cap \mathbb{Z}^+ \mid B \cap \mathbb{Z}^+) = P(A \mid B),$$

by strong, and hence weak, $\mathbb{Z}$-invariance of P_0.

We end with two interesting questions.

QUESTION 1. For what groups G is there a weakly G-invariant probability P on a powerset algebra where P is not strongly G-invariant?

QUESTION 2. Is there a group G and a space $(\Omega, \mathscr{F})$ that admits a weakly G-invariant probability but no strongly G-invariant probability? What if $\mathscr{F} = \mathcal{P}\Omega$?

As a special case, because of the Sierpiński–Mazurkiewicz Paradox [8, p. 9], there is no full conditional probability on $\mathcal{P}\mathbb{R}^2$ that is strongly invariant under rigid motions (since there is no rigid-motion invariant finitely additive probability on the Sierpiński–Mazurkiewicz set), but it is not known whether there is a weakly invariant one (though of course there is none that is weakly invariant under all isometries, by Corollary 1).

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