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HOD-supercompactness, Indestructibility, and Level by Level Equivalence

by

Arthur W. APTER and Shoshana FRIEDMAN

Presented by Czesław BESSAGA

Summary. In an attempt to extend the property of being supercompact but not HOD-supercompact to a proper class of indestructibly supercompact cardinals, a theorem is discovered about a proper class of indestructibly supercompact cardinals which reveals a surprising incompatibility. However, it is still possible to force to get a model in which the property of being supercompact but not HOD-supercompact holds for the least supercompact cardinal κ_0, κ_0 is indestructibly supercompact, the strongly compact and supercompact cardinals coincide except at measurable limit points, and level by level equivalence between strong compactness and supercompactness holds above κ_0 but fails below κ_0. Additionally, we get the property of being supercompact but not HOD-supercompact at the least supercompact cardinal, in a model where level by level equivalence between strong compactness and supercompactness holds.

1. Introduction. In connection to his work in inner model theory, Woodin introduced the concept of N-supercompactness, where N is a proper class inner model of V. As mentioned in [Sar08], at a set theory seminar at Berkeley in 2005, Woodin asked if it were possible to construct a model of set theory in which κ is supercompact, but not HOD-supercompact.

We will extend Sargsyan's result of [Sar08], which answers Woodin's question with the following theorem.

THEOREM 1 ([Sar08]). *Suppose* $V \models$ ZFC + GCH + "*κ is a supercompact cardinal.*" *Then there is a forcing extension of V in which κ is supercompact, but not* HOD-*supercompact.*

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Note that the cardinal κ is HOD-*supercompact* iff κ is supercompact and for all strong limit cardinals λ, there exists an embedding $j : V \to M$ such that $\mathrm{cp}(j) = \kappa$, $j(\kappa) > \lambda$, $M^\lambda \subseteq M$, and $j(\mathrm{HOD}) \cap V_\lambda = \mathrm{HOD} \cap V_\lambda$. We follow standard notation in using $j(N)$, where N is a proper class, to mean $j(N) = \bigcup_{\alpha<\mathrm{ORD}} j(V_\alpha^N)$. Since $N = \mathrm{HOD}$ is a definable class, $j \restriction \mathrm{HOD} : \mathrm{HOD} \to j(\mathrm{HOD})$ is fully elementary.

There are a number of natural questions that arise as a result of this theorem. Three of the questions are as follows:

(1) *Can the property of being supercompact but not* HOD-*supercompact be extended to the class of supercompact cardinals, K, assuming K has more than one member, and each $\kappa \in K$ is indestructibly supercompact, i.e., is indestructible under κ-directed closed forcing as in* [Lav78]*?*

(2) *Can the property of being supercompact but not* HOD-*supercompact be extended to the least supercompact cardinal in a model where the supercompact and strongly compact cardinals coincide* (*except at measurable limit points*)*, and the least supercompact cardinal is also indestructibly supercompact?*

(3) *Can the property of being supercompact but not* HOD-*supercompact hold in a model containing supercompact cardinals which also satisfies level by level equivalence between strong compactness and supercompactness?*

These questions will be answered by theorems in the subsequent sections. The theorem in Section 2 (Theorem 4) is a serendipitous result that was discovered in the course of answering these questions.

We will take this opportunity to mention some preliminary material that will be used throughout this paper. If κ is a cardinal and $\mathbb{P}$ is a partial ordering, $\mathbb{P}$ is κ-*closed* if for every $\delta \le \kappa$, given any sequence $\langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$ such that $\beta < \gamma < \delta$ implies $p_\gamma \le p_\beta$ (a decreasing chain of length less than or equal to δ), there is some $p \in \mathbb{P}$ (a lower bound to this chain) such that $p \le p_\alpha$ for all $\alpha < \delta$. $\mathbb{P}$ is κ-*directed closed* if for every cardinal $\delta < \kappa$ and every directed set $D = \langle p_\alpha : \alpha < \delta \rangle$ of elements of $\mathbb{P}$, there is a lower bound $p \in \mathbb{P}$ for the members of D. $\mathbb{P}$ is κ-*strategically closed* if in the two-person game in which the players construct a decreasing sequence $\langle p_\alpha : \alpha \le \kappa \rangle$, where player I plays odd stages and player II plays even and limit stages, player II has a strategy which ensures the game can always be continued.

In [AS97], Shelah and the first author began the study of level by level equivalence between strong compactness and supercompactness by proving the following theorem.

THEOREM 2. *Let $V \models$ "ZFC + $\mathcal{K} \neq \emptyset$ is the class of supercompact cardinals." There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC + GCH + $\mathcal{K}$ is the class of supercompact cardinals + For every pair of regular cardinals $\kappa < \lambda$, κ is λ-strongly compact iff κ is λ-supercompact, except possibly if κ is a measurable limit of cardinals δ which are λ-supercompact."*

In any model witnessing the conclusions of Theorem 2, we will say that *level by level equivalence between strong compactness and supercompactness holds*. Note that the exception in Theorem 2 is provided by a theorem of Menas [Men75], who showed that if κ is a measurable limit of cardinals δ which are λ-strongly compact, then κ is λ-strongly compact but need not be λ-supercompact. Observe also that Theorem 2 is a strengthening of the result of Kimchi and Magidor [KM], who showed it is consistent for the classes of strongly compact and supercompact cardinals to coincide precisely, except at measurable limit points.

We also take this opportunity to discuss a generalization of Hamkins' Gap Forcing Theorem [Ham99], [Ham01] (as it is stated in [ACH07]), as its results are used throughout this paper. A forcing notion $\mathbb{P}$ (and the forcing extension to which it gives rise) *admits a closure point at δ* if it factors as $\mathbb{Q} * \dot{\mathbb{R}}$, where $\mathbb{Q}$ is nontrivial, $|\mathbb{Q}| \leq \delta$, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ-strategically closed." Our arguments, as do Sargsyan's [Sar08], rely on the following consequence of the main result of [Ham03].

THEOREM 3 ([Ham03]). *If $V \subseteq V[G]$ admits a closure point at δ and $j : V[G] \to M[j(G)]$ is an ultrapower embedding in $V[G]$ with $\delta < \mathrm{cp}(j)$, then $j{\restriction}V : V \to M$ is a definable class in V.*

This theorem follows from [Ham03, Theorem 3, Corollary 14]. If $j : V[G] \to M[j(G)]$ witnesses the λ-supercompactness of κ in $V[G]$, then by [Ham03, Corollary 4], the restriction $j{\restriction}V : V \to M$ witnesses the λ-supercompactness of κ in V. This theorem clearly can be applied to measurability embeddings as well, which gives us the result that if our forcing exhibits the closure point property at a sufficiently small cardinal, we can infer that the measurable and supercompact cardinals of the forcing extension already existed in the ground model.

2. Supercompactness but not HOD supercompactness, and the class of indestructibly supercompact cardinals. It would be natural to extend Sargsyan's result of [Sar08] to a proper class of indestructibly supercompact cardinals, as is postulated by Question 1. This was attempted in an earlier draft of this paper. The referee found a gap in the proof for good reason. In fact we have the following theorem, due to the first author.

THEOREM 4. *If $V \models$ ZFC + "$\exists$ a proper class of indestructibly supercompact cardinals", then $V =$ HOD.*

Proof. It suffices to show that every set of ordinals is coded. Let $A \subseteq \delta$ be a set of ordinals. We will show that there already is a coding for A in V. Let κ be the least indestructibly supercompact cardinal greater than δ. In a κ-directed closed way, we can force there to be a block of δ successor cardinals where GCH holds beyond κ (see the first paragraph of the proof of Theorem 2.1 of [Apt12] for details). Then on this block of δ cardinals where GCH holds, code A, by forcing GCH to fail at the αth successor cardinal of the block, according to whether α is in A, as in [Sar08]. This forcing can be done in a κ-directed closed way. Let us call the partial ordering to force GCH to hold on a block, followed by the coding, $\mathbb{P}$. By the indestructibility of κ, κ remains supercompact after forcing with $\mathbb{P}$. Let λ be sufficiently large, and $j : V^{\mathbb{P}} \to M$ be a λ-supercompactness embedding such that M contains this coding. Since δ is below the critical point of j which is κ, this coding of A reflects unboundedly in κ in $V^{\mathbb{P}}$. Since $\mathbb{P}$ is κ-directed closed, this coding actually reflects unboundedly in κ in V. Therefore, A is in HOD in V. Since A was arbitrary, $V =$ HOD. ■

REMARK 1. Theorem 4 does not require a proper class of indestructibly supercompact cardinals. A proper class of indestructibly strong cardinals (see [GS89]) or even a proper class of indestructibly strongly unfoldable cardinals (see [Joh08]) would suffice.

REMARK 2. If $V \models$ ZFC + "$\exists$ a proper class of indestructibly supercompact cardinals", then $V \models$ GA. This is the Ground Axiom of [Rei07]. The above proof essentially shows that $V \models$ CCA, which is the Continuum Coding Axiom of [Rei07]. Hence, by [Rei07], since $V \models$ CCA, $V \models$ GA.

REMARK 3. Even though Theorem 4 shows that Sargsyan's result of [Sar08] cannot be extended to a proper class of indestructibly supercompact cardinals, we can ask if this result can be extended to a proper class of supercompact cardinals which are not indestructibly supercompact. This is a question we are currently unable to answer (and in fact, we do not even know how to extend Sargsyan's result to two supercompact cardinals).

3. Indestructibility and HOD-supercompactness. We now come to Question 2, namely: *Can the property of being supercompact but not HOD-supercompact be extended to the least supercompact cardinal in a model where the supercompact and strongly compact cardinals coincide* (*except at measurable limit points*), *and the least supercompact cardinal is also indestructibly supercompact*? This question is answered in the affirmative by the following theorem.

THEOREM 5. *Let $V \models$ ZFC + "K is the class of supercompact cardinals" + "κ_0 is the least supercompact cardinal." Then there is an Easton support iteration of length $\kappa_0 + 1$, $\mathbb{P} \in V$, such that:*

(1) *$V^{\mathbb{P}} \models$ "K is the class of supercompact cardinals."*
(2) *$V^{\mathbb{P}} \models$ "κ_0 is indestructibly supercompact, but not* HOD*-supercompact." In addition, $V^{\mathbb{P}} \models$ "Every supercompact cardinal greater than κ_0 is superdestructible."*
(3) *Suppose $V \models$ "$\delta > \kappa_0$ is δ^+-supercompact." Then $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds above κ_0 but fails below κ_0."*
(4) *$V^{\mathbb{P}} \models$ "The only strongly compact cardinals are the elements of K or their measurable limit points."*

Prior to beginning the proof of Theorem 5, we give some definitions. Note that as in the Main Theorem of [Ham98], a cardinal κ is said to be *superdestructible* if any partial ordering adding a subset of κ which is δ-closed for every $\delta < \kappa$ destroys κ's weak compactness. For κ a regular cardinal and γ an ordinal, $\mathrm{Add}(\kappa, \gamma)$ is the standard partial ordering for adding γ many Cohen subsets of κ.

The following partial ordering, a version of which is used by Sargsyan in [Sar08], is designed to code sets of ordinals into the continuum function, and hence into HOD.

DEFINITION 6. Suppose $\kappa < \lambda$ are cardinals and A is a subset of κ. Let λ_α be the $(\alpha + 1)$st strong limit cardinal strictly greater than λ. Let

$$\mathbb{S}_{\kappa,\lambda}(A) = \prod_{\alpha \in A} \mathrm{Add}(\lambda_\alpha, \lambda_\alpha^{++}).$$

If GCH holds above λ in V, then in $V^{\mathbb{S}_{\kappa,\lambda}(A)}$, $\alpha \in A$ iff $2^{\lambda_\alpha} = \lambda_\alpha^{++}$. This implies that in $V^{\mathbb{S}_{\kappa,\lambda}(A)}$, $A \in$ HOD.

We define a building block of our forcing, which we call the *lottery sum* after Hamkins [Ham00].

DEFINITION 7. The *lottery sum* of a collection A of partial orderings is defined as $\oplus A = \{\langle \mathbb{Q}, p \rangle : \mathbb{Q} \in A \text{ and } p \in \mathbb{Q}\} \cup \{\mathbf{1}\}$, ordered with $\mathbf{1}$ above everything and $\langle \mathbb{Q}, p \rangle \leq \langle \mathbb{Q}', q \rangle$ when $\mathbb{Q} = \mathbb{Q}'$ and $p \leq_{\mathbb{Q}} q$.

Since all compatible conditions in the generic object over $\oplus A$ must be in the same partial ordering, the forcing effectively holds a lottery among all the partial orderings in A. The generic object chooses the "winning" partial ordering $\mathbb{Q}$ and then forces with it.

We turn now to the proof of Theorem 5.

Proof of Theorem 5. Let V be as in the hypotheses for Theorem 5. Without loss of generality, by the results of [AS97], we assume as well that

$V \models$ GCH + "Level by level equivalence between strong compactness and supercompactness holds."

Let $S = \{\delta < \kappa_0 : \delta$ is a measurable limit of strong cardinals in $V\}$. By Lemma 2.1 of [AC01], since κ_0 is supercompact, S is unbounded in κ_0.

Let α' be the least V-strong cardinal above α. $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle : \alpha \in \kappa_0 + 1\rangle$ will be an Easton support iteration of length $\kappa_0 + 1$. We start with $\mathbb{P}_0 = \mathrm{Add}(\omega, 1)$. For $\alpha = \kappa_0$, $\dot{\mathbb{Q}}_\alpha$ will be a term for $\mathrm{Add}(\kappa_0, 1)$. Otherwise, for $\alpha < \kappa_0$, $\dot{\mathbb{Q}}_\alpha$ will be a term for trivial forcing unless $\alpha \in S$. If $\alpha \in S$, let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathrm{Add}}(\alpha, 1) * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_{\alpha,\alpha'}(\dot{X}) * \dot{\mathbb{P}}_{\omega,\gamma_\alpha}$, where $\dot{\mathbb{Q}}$ is a term for the lottery sum of all α-directed closed partial orderings of rank less than α', $\dot{X}$ is the name of the generic subset of α added by $\mathrm{Add}(\alpha, 1)$, $\dot{\mathbb{S}}_{\alpha,\alpha'}(\dot{X})$ is a term for the forcing in Definition 6, γ_α is the least inaccessible limit of strong cardinals greater than α, and $\dot{\mathbb{P}}_{\omega,\gamma_\alpha}$ is a term for the partial ordering to add a nonreflecting stationary subset of cofinality ω to γ_α. ∎

Lemma 5.1. *$V^{\mathbb{P}} \models$ "κ_0 is a supercompact cardinal whose supercompactness is indestructible under κ_0-directed closed forcing."*

Proof. It suffices to show that κ_0 can be made indestructibly supercompact by forcing with $\mathbb{P}_{\kappa_0} * \dot{\mathrm{Add}}(\kappa_0, 1)$. Let $G * g \subseteq \mathbb{P}_{\kappa_0} * \dot{\mathrm{Add}}(\kappa_0, 1)$ be V-generic. Now we want to show that $V[G][g] \models$ "κ_0 is supercompact" and $V[G][g] \models$ "κ_0's supercompactness is indestructible by κ_0-directed closed forcing." Fix any $\mathbb{Q} \in V[G][g]$ which is κ_0-directed closed. Fix any $\dot{\mathbb{Q}}$, a name for $\mathbb{Q}$, for which $1 \Vdash$ "$\dot{\mathbb{Q}}$ is κ_0-directed closed." Let $g^* \subseteq \mathbb{Q}$ be $V[G][g]$-generic. We want to show $V[G][g][g^*] \models$ "κ_0 is λ-supercompact" for arbitrary cardinals $\lambda > \kappa_0$. Fix any $\lambda > \kappa_0$, λ a cardinal such that $\dot{\mathbb{Q}} \in H_{\lambda^+}$ and let $\theta \gg 2^{\lambda^{<\kappa_0}}$. Fix in V a θ-supercompactness embedding $j : V \to M$. Since $\theta \geq 2^{\kappa_0}$, $M \models$ "κ_0 is measurable". Also, if $V \models$ "$\delta < \kappa_0$ is a strong cardinal", $M \models$ "$j(\delta) = \delta$ is a strong cardinal." Thus, $\kappa_0 \in j(S)$, so κ_0 is a nontrivial stage of forcing in M. Therefore, below a condition in M which opts for $\dot{\mathbb{Q}}$ in the stage κ_0 lottery, we infer that $j(\mathbb{P}_{\kappa_0} * \dot{\mathrm{Add}}(\kappa_0, 1) * \dot{\mathbb{Q}})$ is forcing equivalent to $\mathbb{P}_{\kappa_0} * \dot{\mathrm{Add}}(\kappa_0, 1) * \dot{\mathbb{Q}} * \dot{\mathbb{S}}_{\kappa_0,\kappa_0'}(\dot{X}) * \dot{\mathbb{P}}_{\omega,\gamma_{\kappa_0}} * \dot{\mathbb{P}}_{\mathrm{tail}} * \dot{\mathrm{Add}}(j(\kappa_0), 1) * j(\dot{\mathbb{Q}})$, with $\dot{X}$ being the name for the generic subset added by $\mathrm{Add}(\kappa_0, 1)$, $\dot{\mathbb{P}}_{\mathrm{tail}}$ being a name for the forcing from $(\gamma_{\kappa_0}, j(\kappa_0))$, and the first nontrivial stage of forcing in $\dot{\mathbb{S}}_{\kappa_0,\kappa_0'}(\dot{X}) * \dot{\mathbb{P}}_{\omega,\gamma_{\kappa_0}} * \dot{\mathbb{P}}_{\mathrm{tail}}$ taking place well beyond θ. This is since in M, there are no strong cardinals in $(\kappa_0, \theta]$, because if there were, $M \models$ "κ_0 is supercompact up to a strong cardinal." By the proof of Lemma 2.4 of [AC01], this implies that $M \models$ "κ_0 is fully supercompact." Therefore, by reflection in V, κ_0 is a limit of supercompact cardinals, contradicting that κ_0 is the least supercompact cardinal in V.

Force to add $G^* \subseteq \dot{\mathbb{S}}_{\kappa_0,\kappa_0'}(\dot{X}) * \dot{\mathbb{P}}_{\omega,\gamma_{\kappa_0}} * \dot{\mathbb{P}}_{\mathrm{tail}}$, a $V[G][g][g^*]$-generic object. Then j lifts to $j : V[G] \to M[j(G)]$ in $V[G][g][g^*][G^*]$ and $j(G) =$

$G * g * g^* * G^*$. In order to complete the proof of the lemma we need to lift again. We can if $j''(g * g^*) \in M[j(G)]$. It is by the usual argument, since $g * g^* \in M[j(G)]$ and $M[j(G)]^\theta \subseteq M[j(G)]$ in $V[G][g][g^*][G^*]$. The fact that $M[j(G)]^\theta \subseteq M[j(G)]$ follows since $\mathbb{P}_{\kappa_0} * \dot{\mathrm{Add}}(\kappa_0, 1) * \dot{\mathbb{Q}}$ is θ-c.c. (because $|\mathbb{P}_{\kappa_0} * \dot{\mathrm{Add}}(\kappa_0, 1) * \dot{\mathbb{Q}}| < \theta$) and $\dot{\mathbb{S}}_{\kappa_0,\kappa_0'}(\dot{X}) * \dot{\mathbb{P}}_{\omega,\gamma_{\kappa_0}} * \dot{\mathbb{P}}_{\mathrm{tail}}$ is θ-strategically closed in both $M[G][g][g^*]$ and $V[G][g][g^*]$ (because the first stage of nontrivial forcing in $\dot{\mathbb{S}}_{\kappa_0,\kappa_0'}(\dot{X}) * \dot{\mathbb{P}}_{\omega,\gamma_{\kappa_0}} * \dot{\mathbb{P}}_{\mathrm{tail}}$ takes place well beyond θ). Since $j''(g * g^*) \in M[j(G)]$, there is a master condition p^* below $j''(g * g^*)$; it exists because $j(\kappa_0) > \theta$, $j(\mathrm{Add}(\kappa_0, 1) * \dot{\mathbb{Q}})$ is $j(\kappa_0)$-directed closed, and $M[j(G)]^\theta \subseteq M[j(G)]$. We can therefore force to add an H which is $V[G][g][g^*][G^*]$-generic such that $p^* \in H$. Now we can lift j to $j : V[G][g][g^*] \to M[j(G)][H]$ in $V[G][g][g^*][G^*][H]$. Since we chose θ to be sufficiently large, $G^* * H$ is $V[G][g][g^*]$-generic over a partial ordering which does not add any ultrafilters over $P_{\kappa_0}(\lambda)$. We thus have $V[G][g][g^*] \models$ "κ_0 is λ-supercompact." ∎

Lemma 5.2. *In $V^{\mathbb{P}}$, for any $\kappa \in K$, $\kappa > \kappa_0$, κ is superdestructible.*

Proof. Since for any $\kappa \in K$ with $\kappa > \kappa_0$, $|\mathbb{P}| < \kappa$, by the Main Theorem of [Ham98], κ is superdestructible in $V^{\mathbb{P}}$. ∎

Lemma 5.3. *K is the class of supercompact cardinals in $V^{\mathbb{P}}$.*

Proof. Since $\mathbb{P}$ can be defined to have cardinality κ_0, by the Lévy–Solovay results [LS67], the class of supercompact cardinals above κ_0 remains the same in V and $V^{\mathbb{P}}$. By Lemma 5.1, κ_0 remains supercompact. It thus suffices to show that no new supercompact cardinals are created by $\mathbb{P}$. Since $\mathbb{P}$ admits a closure point at ω, by an application of Theorem 3, no new supercompact cardinals were created by $\mathbb{P}$. Hence K is the class of supercompact cardinals in $V^{\mathbb{P}}$. ∎

Lemma 5.4. *$V^{\mathbb{P}} \models$ "κ_0 is not* HOD*-supercompact."*

Proof. This proof follows closely the proof of Lemma 2.2 in [Sar08]. Let $G \subseteq \mathbb{P}$ be V-generic. Factor $\mathbb{P} = \mathbb{P}_{\kappa_0} * \dot{\mathbb{Q}}_{\kappa_0}$. Let $G = G_{\kappa_0} * g$ be the corresponding generic objects. Assume κ_0 is HOD-supercompact in $V[G] = W$ and let $\mathrm{HOD} = \mathrm{HOD}^W$. Fix a strong limit cardinal $\lambda > \kappa_0$ such that $\mathrm{HOD}^{W_\lambda} = \mathrm{HOD} \cap W_\lambda$. Let $j : W \to M$ be a λ-supercompactness embedding such that $j(\mathrm{HOD}) \cap W_\lambda = \mathrm{HOD} \cap W_\lambda = \mathrm{HOD}^{W_\lambda}$. By Theorem 3, $i = j \restriction V$ is definable in V, j is the lift of i, and i is a λ-supercompactness embedding. Let $N = i(V) = \bigcup_{\alpha < \mathrm{ORD}} i(V_\alpha)$. If H is the N-generic object for $i(\mathbb{P})$, then $M = N[H] = N[j(G)]$. We also see that $H \cap \mathbb{P}_{\kappa_0} = G_{\kappa_0}$.

In addition, for any $\delta < \kappa_0$ such that $V \models$ "δ is a strong cardinal", $N \models$ "$i(\delta) = \delta$ is a strong cardinal." Since $\lambda > 2^{\kappa_0}$ and $N^\lambda \subseteq N$, $N \models$ "κ_0 is measurable." Thus, in N, κ_0 is a nontrivial stage of forcing. This means that we can let g' be the generic for $\mathrm{Add}(\kappa_0, 1)$ given by H. Then in $N[H] = M$, g' is ordinal definable. But because κ_0 is HOD-supercompact, $j(\mathrm{HOD}) \cap W_\lambda = \mathrm{HOD} \cap W_\lambda = \mathrm{HOD}^{W_\lambda}$. This implies $g' \in \mathrm{HOD}^{W_\lambda}$. Thus g' is ordinal definable in $W_\lambda = V_\lambda^{V[G]} = V_\lambda^{V[G_{\kappa_0}][g]}$. Thus g' had to have been added over $V_\lambda[G_{\kappa_0}]$, and more particularly, g' is added over $V_\lambda[G_{\kappa_0}]$ by homogeneous forcing. This fact, along with g' being ordinal definable in W_λ, implies that g' is in $V_\lambda[G_{\kappa_0}]$. This is impossible as g' is a $V[G_{\kappa_0}]$-generic object for $\mathrm{Add}(\kappa_0, 1)$. Therefore κ_0 is not HOD-supercompact. ∎

LEMMA 5.5. *Suppose $V \models$ "$\delta > \kappa_0$ is δ^+-supercompact." Then $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds above κ_0 but fails below κ_0."*

Proof. Since $\mathbb{P}$ can be defined to have cardinality κ_0, and level by level equivalence holds in V, by the results of [LS67], level by level equivalence holds above κ_0 in $V^{\mathbb{P}}$. Further, by the results of [LS67], $V^{\mathbb{P}} \models$ "δ is δ^+-supercompact."

Level by level equivalence does not hold below κ_0 in $V^{\mathbb{P}}$ because κ_0 is indestructibly supercompact and there exists a cardinal $\delta > \kappa_0$ which is δ^+-supercompact. Thus, by Theorem 5 of [AH02], $\{\delta < \kappa_0 : \delta$ is δ^+-strongly compact but δ is not δ^+-supercompact$\}$ is unbounded in κ_0. This proves Lemma 5.5. ∎

LEMMA 5.6. *$V^{\mathbb{P}} \models$ "κ_0 is the least strongly compact cardinal."*

Proof. By Lemma 5.1, it suffices to show that $V^{\mathbb{P}} \models$ "No cardinal $\delta < \kappa_0$ is strongly compact." By the definition of $\mathbb{P}$, unboundedly many $\gamma < \kappa_0$ contain nonreflecting stationary sets of ordinals of cofinality ω. By Theorem 4.8 of [SRK78] and the succeeding remarks, no cardinal $\delta < \kappa_0$ is strongly compact in $V^{\mathbb{P}}$. ∎

LEMMA 5.7. *$V^{\mathbb{P}} \models$ "The only strongly compact cardinals are the elements of K or their measurable limit points."*

Proof. By Lemma 5.6, in $V^{\mathbb{P}}$, κ_0 is the least strongly compact cardinal. By level by level equivalence between strong compactness and supercompactness, in V, the only strongly compact cardinals are the elements of K or their measurable limit points. Since $|\mathbb{P}| = \kappa_0$, by the results of [LS67], in $V^{\mathbb{P}}$, the only strongly compact cardinals are the elements of K or their measurable limit points. ∎

Lemmas 5.1–5.7 prove Theorem 5. ∎

If there is no $\delta > \kappa_0$ in V such that δ is δ^+-supercompact, then Theorem 5 is still true. This is shown by modifying the definition of $\mathbb{P}$ to force a failure of level by level equivalence between strong compactness and supercompactness below κ_0 (e.g., by keeping the definition of $\mathbb{P}$ as it was originally, except that after forcing with $\mathrm{Add}(\omega, 1)$, we iteratively force to add a nonreflecting stationary set of ordinals of cofinality ω to each measurable cardinal below the least cardinal $\delta < \kappa_0$ which is δ^+-supercompact). The other clauses in the statement of Theorem 5 remain true with the same proofs as before (although some may only be vacuously true).

4. Level by level equivalence and HOD-supercompactness. We now come to Question 3, namely: *Can Sargsyan's result hold in a model containing supercompact cardinals which also satisfies level by level equivalence between strong compactness and supercompactness?*

We answer this question in the affirmative with the following theorem.

THEOREM 8. *Let $V \models$ ZFC + "K is the class of supercompact cardinals" + "κ_0 is the least supercompact cardinal." Then there is an Easton support iteration of length $\kappa_0 + 1$, $\mathbb{P} \in V$, such that $V^{\mathbb{P}} \models$ ZFC + "K is the class of supercompact cardinals" + "Level by level equivalence between strong compactness and supercompactness holds" + "κ_0 is supercompact, but not HOD-supercompact."*

Proof. Without loss of generality, as in the proof of Theorem 5, we assume as well that $V \models$ GCH + "Level by level equivalence between strong compactness and supercompactness holds."

As in the proof of Theorem 5, let $S = \{\delta < \kappa_0 : \delta$ is a measurable limit of strong cardinals in $V\}$. $\mathbb{P} = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle : \alpha \in \kappa_0 + 1\rangle$ will be an Easton support iteration of length $\kappa_0 + 1$. We start with $\mathbb{P}_0 = \mathrm{Add}(\omega, 1)$. For κ_0, $\dot{\mathbb{Q}}_{\kappa_0}$ will be a term for $\mathrm{Add}(\kappa_0, 1)$. Otherwise $\dot{\mathbb{Q}}_\alpha$ will be a term for trivial forcing unless $\alpha \in S$. If $\alpha \in S$, let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathrm{A}}\mathrm{dd}(\alpha, 1) * \dot{\mathbb{S}}_{\alpha,\alpha'}(\dot{X})$, where $\dot{X}$ is the name of the generic subset of α added by $\mathrm{Add}(\alpha, 1)$.

LEMMA 8.1. *$V^{\mathbb{P}} \models$ "κ_0 is a supercompact cardinal."*

Proof. Let $\lambda > \kappa_0$ be an arbitrary cardinal, and let $j : V \to M$ be a θ-supercompactness embedding with $\theta \gg 2^{\lambda^{<\kappa_0}}$. Since as in the proof of Lemma 5.1, κ_0 is a measurable limit of strong cardinals in M, the stage κ_0 forcing in $M^{\mathbb{P}_{\kappa_0}}$ is nontrivial, and we have $j(\mathbb{P}) = \mathbb{P}_{\kappa_0} * \dot{\mathrm{A}}\mathrm{dd}(\kappa_0, 1) * \dot{\mathbb{S}}_{\kappa_0,\kappa_0'}(\dot{X}) * \dot{\mathbb{P}}_{\mathrm{tail}}$, with $\dot{X}$ the name for the generic subset added by $\mathrm{Add}(\kappa_0, 1)$, and $\dot{\mathbb{P}}_{\mathrm{tail}}$ a term for the rest of the forcing up to and including $j(\kappa_0)$. Since as in the proof of Lemma 5.1, the first nontrivial stage of forcing in $\dot{\mathbb{S}}_{\kappa_0,\kappa_0'}(\dot{X}) * \dot{\mathbb{P}}_{\mathrm{tail}}$ is beyond $\max(\lambda, \theta) = \theta$, we may abuse notation and write $j(\mathbb{P})$ as $\mathbb{P}_{\kappa_0} * \dot{\mathrm{A}}\mathrm{dd}(\kappa_0, 1) * \dot{\mathbb{P}}_{\mathrm{tail}}$, where the first nontrivial stage of forcing in $\dot{\mathbb{P}}_{\mathrm{tail}}$ is beyond θ. The arguments of Lemma 5.1 now show that $V^{\mathbb{P}} \models$ "κ_0 is λ-supercompact." Since λ was arbitrary, this completes the proof of Lemma 8.1. ∎

Lemma 8.2. *K is the class of supercompact cardinals in $V^{\mathbb{P}}$.*

Proof. Since $\mathbb{P}$ can be defined to have cardinality κ_0, the proof given in Lemma 5.3 remains valid and shows that K is the class of supercompact cardinals in $V^{\mathbb{P}}$. ∎

Lemma 8.3. *Suppose $\delta < \kappa_0$ and $\lambda > \delta$ is regular. If $V \models$ "δ is λ-supercompact", then $V^{\mathbb{P}} \models$ "δ is λ-supercompact."*

Proof. We divide into two cases.

Case 1: Suppose δ is a limit of strong cardinals. Then λ is less than the least V-strong cardinal δ' above δ. This is because otherwise, δ would be supercompact up to a strong cardinal. As we observed in the proof of Lemma 5.1, this means that δ would be fully supercompact, a contradiction to the fact that κ_0 is the least supercompact cardinal.

Factor $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta, 1) * \dot{\mathbb{S}}_{\delta,\delta'}(\dot{X}) * \dot{\mathbb{P}}^{\delta+1}$. Since $\lambda < \delta'$, the first stage of nontrivial forcing in $\dot{\mathbb{S}}_{\delta,\delta'}(\dot{X}) * \dot{\mathbb{P}}^{\delta+1}$ is beyond λ. Thus, $V^{\mathbb{P}} \models$ "δ is λ-supercompact" iff $V^{\mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta,1)} \models$ "δ is λ-supercompact."

Let U be a normal, fine ultrafilter over $P_\delta(\lambda)$ such that for $j : V \to M$ the associated elementary embedding, $M \models$ "δ is not λ-supercompact." Note that $M \models$ "No cardinal $\gamma \in (\delta, \lambda]$ is strong." This is since otherwise, $M \models$ "δ is supercompact up to a strong cardinal and hence is fully supercompact", which contradicts that $M \models$ "δ is not λ-supercompact." As in the proof of Lemma 5.1, δ is a nontrivial stage of forcing in M. This means that $j(\mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta, 1)) = \mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta, 1) * \dot{\mathbb{Q}} * \dot{\mathrm{Add}}(j(\delta), 1)$, where the first stage of nontrivial forcing in $\dot{\mathbb{Q}}$ is above λ.

Let $G * g \subseteq \mathbb{P}_\delta * \dot{\mathrm{Add}}(\delta, 1)$ be V-generic. We now argue as in the proof of Lemma 1.2 of [Apt05], from which we quote freely. The usual diagonalization argument (as given, e.g., in the construction of the generic object G_1 in Lemma 2.4 of [AC01]) may be used to build in $V[G][g]$ an $M[G][g]$-generic object g^* over $\mathbb{Q}$ and lift j to $j : V[G] \to M[G][g][g^*]$. (This is since GCH in both $V[G][g]$ and $M[G][g]$ at and above δ combined with the fact that $\mathbb{Q}$ is λ^+-directed closed in both $V[G][g]$ and $M[G][g]$ allow us to let $\langle D_\alpha : \alpha < \lambda^+\rangle \in V[G][g]$ be an enumeration of the dense open subsets of $\mathbb{Q}$ present in $M[G][g]$ and then construct g^* in $V[G][g]$ to meet each D_α. Full details may be found in [AC01].) A master condition for $j''g$ may now be constructed in $V[G][g]$ as in the proof of Lemma 5.1. The diagonalization argument just mentioned then once again allows us to construct in $V[G][g]$ an $M[G][g][g^*]$-generic object H over $\mathrm{Add}(j(\delta), 1)$ containing this master condition and fully lift j in $V[G][g]$ to $j : V[G][g] \to M[G][g][g^*][H]$.

CASE 2: Suppose δ is not a limit of strong cardinals. Note that δ is not itself a strong cardinal. This is since otherwise, by GCH and the fact that $\lambda > \delta$, δ is both (at least) 2^δ-supercompact and strong. Hence, by Lemma 2.1 of [AC01], δ is a limit of strong cardinals, contrary to our assumptions.

Let $\mathbb{P} = \mathbb{P}_\delta * \dot{\mathbb{P}}^\delta$. The supremum of the strong cardinals less than or equal to δ is below δ, since δ is not a strong cardinal. Hence, $|\mathbb{P}_\delta| < \delta$ by the definition of the forcing. By the results of [LS67], $V^{\mathbb{P}_\delta} \models$ "δ is λ-supercompact." No nontrivial forcing is done at stage δ, since δ is neither a strong cardinal nor a limit of strong cardinals. The next stage of nontrivial forcing is beyond λ, because otherwise δ would be supercompact up to a strong cardinal, and thus would be fully supercompact. Since the next stage of nontrivial forcing is beyond λ, $\dot{\mathbb{P}}^\delta$ is therefore sufficiently directed closed in $V^{\mathbb{P}_\delta}$ so that δ will remain λ-supercompact in $V^{\mathbb{P}_\delta * \dot{\mathbb{P}}^\delta} = V^{\mathbb{P}}$. ∎

LEMMA 8.4. $V^{\mathbb{P}} \models$ *"Level by level equivalence between strong compactness and supercompactness holds."*

Proof. Since level by level equivalence between strong compactness and supercompactness holds in V and $|\mathbb{P}| = \kappa_0$, by the results of [LS67], $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds above κ_0." By Lemma 8.1, $V^{\mathbb{P}} \models$ "κ_0 is supercompact." It thus suffices to show that $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds below κ_0" to prove this lemma.

Towards this end, let $\delta < \kappa_0$ and $\lambda > \delta$ be such that $V^{\mathbb{P}} \models$ "δ is λ strongly compact and λ is regular." By the definition of $\mathbb{P}$ and the results of [Ham99], [Ham01], and [Ham03], $V \models$ "δ is λ-strongly compact." Because level by level equivalence between strong compactness and supercompactness holds in V, either $V \models$ "δ is λ-supercompact", or $V \models$ "δ is a measurable limit of cardinals which are λ-supercompact." By Lemma 8.3, either $V^{\mathbb{P}} \models$ "δ is λ-supercompact", or $V^{\mathbb{P}} \models$ "δ is a measurable limit of cardinals which are λ-supercompact." Therefore, $V^{\mathbb{P}} \models$ "Level by level equivalence between strong compactness and supercompactness holds below κ_0." This proves Lemma 8.4. ∎

We note that in Lemma 8.4, $\lambda < \delta'$ as well. This is since otherwise, some cardinal $\gamma < \kappa_0$ is supercompact up to a strong cardinal and hence is fully supercompact.

The proof that $V^{\mathbb{P}} \models$ "κ_0 is not HOD-supercompact" is the same as the proof of Lemma 5.4. Thus, Lemmas 8.1–8.4 complete the proof of Theorem 8. ∎

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Arthur W. Apter
Department of Mathematics
Baruch College of CUNY
New York, NY 10010, U.S.A.
and
The CUNY Graduate Center, Mathematics
365 Fifth Avenue
New York, NY 10016, U.S.A.
E-mail: awapter@alum.mit.edu
URL: http://faculty.baruch.cuny.edu/aapter

Shoshana Friedman
Department of Mathematics and Computer Science
Kingsborough Community College-CUNY
2001 Oriental Blvd, Brooklyn, NY 11235, U.S.A.
E-mail: shoshana.friedman4@gmail.com

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