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NUMBER THEORY

Parametric Solutions of the Diophantine Equation $A^2 + nB^4 = C^3$

by

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Summary. The Diophantine equation $A^2 + nB^4 = C^3$ has infinitely many integral solutions A, B, C for any fixed integer n. The case $n = 0$ is trivial. By using a new polynomial identity we generate these solutions, and then give conditions when the solutions are pairwise co-prime.

1. Introduction. In a series of papers [2]–[9], we studied different Diophantine equations, and have shown how to generate infinitely many co-prime integral solutions for each of them. In this paper, we want to study the Diophantine equation

$$A^2 + nB^4 = C^3, \tag{1.1}$$

where $n \in \mathbb{Z}$ is fixed, and A, B, C are integer variables. In (1.1), the case $n = 0$ is trivial. We use a new polynomial identity to generate infinitely many co-prime integral solutions A, B, C.

2. Diophantine equation $A^2 + nB^4 = C^3$. The Diophantine equation

$$A^2 + nB^4 = C^3, \tag{2.1}$$

where $n \in \mathbb{Z}$ is fixed, has not been studied fully. Beukers [1] has given solutions for (2.1) when $n = \pm 1$, but when $|n| \geq 2$, there are no publications. In an earlier paper [3], we studied the Diophantine equation

$$A^4 + nB^2 = C^3, \tag{2.2}$$

where $n \in \mathbb{Z}$ is fixed. Now, we want to establish the following theorem.

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THEOREM 2.1. *For any given $n \in \mathbb{Z}$, the Diophantine equation $A^2 + nB^4 = C^3$ has infinitely many integer solutions A, B, C satisfying the conditions $ABC \neq 0$ and* $\gcd(A, B, C) = 1$.

We observe that in (2.1), (2.2), and Theorem 2.1 the case $n = 0$ is trivial.

3. Proof of Theorem 2.1. To prove Theorem 2.1, we need the following lemma.

LEMMA 3.1. *For all $a, b \in \mathbb{Z}$,*

$$((a+b)(a^2 - 34ab + b^2))^2 + 108ab(a-b)^4 = (a^2 + 14ab + b^2)^3. \tag{3.1}$$

Proof. Take $c = (a+b)^2$ and $d = 12ab$ in the easily provable identity

$$c(c - 3d)^2 + d(3c - d)^2 = (c + d)^3.$$

After simplification we get (3.1). ∎

Proof of Theorem 2.1. In order to apply identity (3.1) to (2.1), we only need to find a, b such that $108ab = nv^4$ for some v. The full characterization of the integer solutions a, b, v of the equation $108ab = nv^4$ with $\gcd(a, b) = 1$ depends on the divisibility properties of n. The set of solutions needed for our purposes are of the form $a = u_1 d_1 p^4$, $b = u_2 d_2 q^4$, where $u_1 u_2 = 12$, $d_1 d_2 = n$ and $\gcd(u_1 d_1 p, u_2 d_2 q) = 1$.

In principle, we need not assume that $\gcd(u_1 d_1, q) = \gcd(u_2 d_2, p) = 1$ in order to have co-prime solutions of $108ab = nv^4$. However, because we put these expressions into the identity (3.1) the conditions are important in order to get co-prime solutions of (2.1). Now, we can take four pair of values for (a, b) to get the corresponding polynomials A_i, B_i, C_i where $i \in \{1, 2, 3, 4\}$, as solutions A, B, C in (2.1). With some additional conditions, in each case, we make the solutions co-prime.

(i) $(a, b) = (4d_1 p^4, 3d_2 q^4)$ gives

$$\begin{aligned} A_1 &= (4d_1 p^4 + 3d_2 q^4)(16d_1^2 p^8 - 408 d_1 d_2 p^4 q^4 + 9d_2^2 q^8), \\ B_1 &= 6pq(4d_1 p^4 - 3d_2 q^4), \\ C_1 &= 16d_1^2 p^8 + 168 d_1 d_2 p^4 q^4 + 9d_2^2 q^8. \end{aligned} \tag{3.2}$$

To obtain co-prime solutions of (2.1) take $\gcd(4d_1 p, 3d_2 q) = 1$ in (3.2).

(ii) $(a, b) = (3d_1 p^4, 4d_2 q^4)$ gives

$$\begin{aligned} A_2 &= (3d_1 p^4 + 4d_2 q^4)(9d_1^2 p^8 - 408 d_1 d_2 p^4 q^4 + 16d_2^2 q^8), \\ B_2 &= 6pq(3d_1 p^4 - 4d_2 q^4), \\ C_2 &= 9d_1^2 p^8 + 168 d_1 d_2 p^4 q^4 + 16d_2^2 q^8. \end{aligned} \tag{3.3}$$

To obtain co-prime solutions of (2.1) take $\gcd(3d_1 p, 4d_2 q) = 1$ in (3.3).

(iii) $(a, b) = (12d_1p^4, d_2q^4)$ gives

$$
\begin{aligned}
A_3 &= (12d_1p^4 + d_2q^4)(144d_1^2p^8 - 408d_1d_2p^4q^4 + d_2^2q^8),\\
B_3 &= 6pq(12d_1p^4 - d_2q^4),\\
C_3 &= 144d_1^2p^8 + 168d_1d_2p^4q^4 + d_2^2q^8.
\end{aligned}
\tag{3.4}
$$

Take $\gcd(12d_1p, d_2q) = 1$ in (3.4) to get co-prime solutions of (2.1).

(iv) $(a, b) = (d_1p^4, 12d_2q^4)$ gives

$$
\begin{aligned}
A_4 &= (d_1p^4 + 12d_2q^4)(d_1^2p^8 - 408d_1d_2p^4q^4 + 144d_2^2q^8),\\
B_4 &= 6pq(d_1p^4 - 12d_2q^4),\\
C_4 &= d_1^2p^8 + 168d_1d_2p^4q^4 + 144d_2^2q^8.
\end{aligned}
\tag{3.5}
$$

Take $\gcd(d_1p, 12d_2q) = 1$ in (3.5) to get co-prime solutions of (2.1). ■

Thus, we have found homogeneous polynomials $A, B, C \in \mathbb{Z}[p, q]$ of degrees 12, 6, 8 respectively such that $A(p, q)^2 + nB(p, q)^4 = C(p, q)^3$, and we have found conditions under which the solutions of (2.1) are co-prime.

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