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Characteristic Exponents of Rational Functions

by

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Summary. We consider two characteristic exponents of a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree $d \geq 2$. The exponent $\chi_a(f)$ is the average of $\log \|f'\|$ with respect to the measure of maximal entropy. The exponent $\chi_m(f)$ can be defined as the maximal characteristic exponent over all periodic orbits of f. We prove that $\chi_a(f) = \chi_m(f)$ if and only if $f(z)$ is conformally conjugate to $z \mapsto z^{\pm d}$.

1. Introduction and statement of results. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. In [BE], M. Barrett and A. Eremenko considered the value $K(f) = \max_{\hat{\mathbb{C}}} \|f'\|$. Here and below, $\|f'\|$ always denotes the derivative with respect to the spherical metric,

$$\|f'(z)\| = |f'(z)| \cdot \frac{1 + |z|^2}{1 + |f'(z)|^2}.$$

Among other issues, the authors of [BE] studied the behaviour of the value $K(\cdot)$ under iterations of a given function. More precisely, denote by f^n the nth iterate of f, and define

$$k_\infty(f) = \lim_{n\to\infty} \frac{1}{n} \log K(f^n).$$

A slightly different *maximum characteristic exponent* is defined as

$$\chi_m(f) = \sup_z \limsup_{n\to\infty} \frac{1}{n} \log \|(f^n)'(z)\|. \tag{1}$$

Clearly,

$$\chi_m(f) \leq k_\infty(f).$$

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According to a result of Przytycki [P] (reproved in [GPRR]), $k_\infty(f) = \chi_m(f)$ and one can replace $\sup_{z\in\hat{\mathbb{C}}}$ in (1) by the supremum over all periodic points:

$$\chi_m(f) = \sup_{z\in \mathrm{Per}(f)} \lim_{n\to\infty} \frac{1}{n} \log \|(f^n)'(z)\|. \tag{2}$$

Finally, let

$$\chi_a(f) = \int \log \|f'\|\, d\mu \tag{3}$$

be the average of $\log\|f'\|$ with respect to the unique measure of maximal entropy. Denoting $\alpha = \dim_{\mathrm{H}}(\mu)$ we can thus write

$$\chi_a(f) = \frac{\log d}{\alpha} \geq \frac{\log d}{2}. \tag{4}$$

So, we have the following inequalities:

$$\frac{1}{2}\log d \leq \chi_a(f) \leq \chi_m(f) \leq \log K(f). \tag{5}$$

In the first inequality of (5) equality holds only for Lattès maps (see [Z]). The authors of [BE] characterize the maps for which the third inequality becomes an equality: $\chi_m(f) = \log K(f)$ iff the set $M = \{z : \|f'(z)\| = K(f)\}$ contains a periodic orbit of f. They remark that, according to a suggestion of F. Przytycki, the method of my paper [Z] could probably be used to prove that the equality $\chi_a(f) = \chi_m(f)$ holds if and only if f is conformally conjugate to $z \mapsto z^{\pm d}$.

The aim of this note is to provide the proof of this fact. We have the following

THEOREM 1. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. Then $\chi_m(f) = \chi_a(f)$ if and only if f is conformally conjugate to one of the functions: $z \mapsto z^d$, $z \mapsto z^{-d}$.*

2. The proof of Theorem 1. The proof is based on the corresponding arguments in [Z]. We shall refer to several auxiliary facts proved in [Z].

As above, denote

$$\alpha = \dim_{\mathrm{H}}(\mu) = \frac{\log d}{\chi_a(f)}.$$

Set

$$\phi = \alpha \log \|f'\| - \log d. \tag{6}$$

Then $|\phi|^p$ is μ-integrable for all $p > 0$ (see e.g. [PUZ, Lemma 5]). Clearly,

$$\int \phi\, d\mu = 0.$$

DEFINITION 2. With the notation above, we say that $\psi \in L^2(\mu)$ is *cohomologous to zero* in $L^2(\mu)$ if there exists $u \in L^2(\mu)$ such that $\psi = u \circ f - u$.

Following [Z] we shall distinguish two cases, depending on whether ϕ is cohomologous to 0 or not.

Definition 3. A rational function f is called *exceptional* if ϕ is cohomologous to zero in $L^2(\mu)$.

The exceptional maps have been classified in [Z]. It turns out that for an exceptional map we have $\alpha = 1$ or $\alpha = 2$ and the map is critically finite, with parabolic orbifold (see [Z] for details). The only exceptional maps with $\alpha = 1$ are (up to a conjugacy by a Möbius transformation): $f(z) = z^{\pm d}$ and $f(z) = \pm$ Chebyshev polynomial. For $\pm$ Chebyshev polynomial of degree d it is easy to see that $\chi_m(f) = 2\log d > \chi_a(f) = \log d$. Obviously, we have $\chi_a(f) = \chi_m(f)$ for $f(z) = z^{\pm d}$.

The case $\alpha = 2$ corresponds to so-called Lattès examples. It has been treated in [BE]. In this case, $\frac{1}{2}\log d = \chi_a(f) < \chi_m(f) = \log d$.

Thus, the rest of the proof of Theorem 1 relies on the following.

Proposition 4. *If a rational function f of degree $d \geq 2$ is not exceptional then $\chi_a(f) < \chi_m(f)$.*

Proof. We recall the notation and some facts from [Z]. First, $J = J(f)$ denotes the Julia set of the map f. It can be defined as the topological support of the measure μ of maximal entropy. As in [Z], we work in the natural extension $(\tilde{J}, \tilde{\mu}, \tilde{f})$. See e.g. [PU] for the definition of the natural extension and its properties. The set $\tilde{J}$ consists of two-sided infinite sequences (trajectories)

$$(\ldots, x_{-k}, x_{-(k-1)}, \ldots, x_0, x_1, \ldots, x_k, \ldots)$$

such that $f(x_j) = x_{j+1}$ for all $j \in \mathbb{Z}$. The invertible map $\tilde{f} : \tilde{J} \to \tilde{J}$ is the left shift. Let $\pi : \tilde{J} \to J$ be the projection onto the 0th coordinate. The measure $\tilde{\mu}$ is invariant for the automorphism $\tilde{f}$ and $\pi_*\tilde{\mu} = \mu$.

Let $B = B(p, r)$ be a ball in $\hat{\mathbb{C}}$. We denote by $2B$ the ball $B(p, 2r)$.

Let f_ν^{-n} be a branch of f^{-n}. We say that this branch is (K, δ, B)-*good* if f_ν^{-n} is well-defined in $2B$ and $\operatorname{diam}(f_\nu^{-n}(B)) < K\exp(-n\delta)$.

We recall, in a more convenient form, the Basic Lemma from [Z].

Lemma 5 (Basic Lemma). *Let f be a rational function of degree $d \geq 2$. There exist $\delta > 0$ such that for every $\tilde{\varepsilon} > 0$ there exist $M \in \mathbb{Z}_+$ and $K > 0$ such that the following holds. If B is a ball in $\hat{\mathbb{C}}$ and there are no critical values of f^M in $2B$ then there is a subset $\tilde{K}_B \subset \tilde{B} = \pi^{-1}(B)$ of $\tilde{\mu}$-measure greater than $(1 - \tilde{\varepsilon})\mu(B)$ such that, for every $k \in \mathbb{Z}_+$ and for every $(\ldots, x_{-k}, x_{-k+1}, \ldots, x_0, x_1, \ldots) \in \tilde{K}_B$, we have*

$$x_{-k} = f_\nu^{-k}(x_0)$$

for some (K, δ, B)-good branch of f^{-k}.

To every $\tilde{\varepsilon} > 0$ one can associate a family $\mathcal{B}$ of balls in the following way: Choose some $\tilde{\varepsilon} > 0$ and let M be the value assigned to $\tilde{\varepsilon}$ in the Basic Lemma. Let $p_1, \ldots, p_s$ be the critical values of f^M, and let $B_1, \ldots, B_s$ be the balls centred at p_i's with some radius r. Let $\mathcal{B}$ be a cover of $\hat{\mathbb{C}} \setminus \bigcup_{i=1}^{s} B_i$ with balls of radius $r/4$. Clearly, we can assume that for every $B \in \mathcal{B}$,

$$2B \cap \{p_1, \ldots, p_s\} = \emptyset, \tag{7}$$

so that for each $B \in \mathcal{B}$ Lemma 5 applies. Moreover, since the measure μ is atomless, we can require that r is small enough, so that

$$\tilde{\mu}\Big(\bigcup_{B\in\mathcal{B}} \tilde{K}_B\Big) > 1 - 2\tilde{\varepsilon}. \tag{8}$$

Note that the family $\mathcal{B}$, the radius r and the set $\bigcup_{B\in\mathcal{B}} \tilde{K}_B$ depend on $\tilde{\varepsilon}$.

Recall that the function ϕ is given by (6). Since ϕ is not cohomologous to 0 in $L^2(\mu)$, we know that the sequence $\phi, \phi \circ f, \phi \circ f^2, \ldots$ satisfies the Central Limit Theorem. This means that if we set $S_n\phi := \phi + \phi \circ f + \cdots + \phi \circ f^{n-1}$, the sequence of random variables defined in the measure space $(\hat{\mathbb{C}}, \mathcal{B}(\hat{\mathbb{C}}), \mu)$ by

$$X_n = \frac{S_n\phi}{\sigma\sqrt{n}}$$

tends to $N(0,1)$ in distribution. Here, $\sigma^2 \neq 0$ is the so-called asymptotic variance. See e.g. [PUZ, Section 4] for the proof of the Almost Sure Invariance Principle in this context, or [DPU] for the proof of CLT relying on Gordin's method, or [Du] for a higher dimensional generalisation.

It follows that for every $A > 0$,

$$\mu(\{\tilde{x} \in \tilde{J} : S_n\phi(\tilde{x}) > A\sigma\sqrt{n}\} \to 1 - \Psi(A) > 0$$

where Ψ is the distribution function of the normal distribution $N(0,1)$.

Now, we fix some positive A. Next, we fix $\tilde{\varepsilon}$ satisfying $1 - \Psi(A) > 4\tilde{\varepsilon}$. Let $\mathcal{B}$ be the family of balls assigned to $\tilde{\varepsilon}$ as described above. Using (8) and the invariance of the measure μ we see that there exists a ball $B \in \mathcal{B}$ such that the inequality

$$\tilde{\mu}(\{\tilde{x} \in \tilde{J} : S_n\phi(\tilde{x}) > A\sigma\sqrt{n} \text{ and } \tilde{f}^n(\tilde{x}) \in \tilde{K}_B\}) > \beta > 0$$

holds with some fixed $\beta > 0$ and for infinitely many n's.

We now fix such a ball B. By the topological exactness of the map $f : J \to J$ we have $f^l(\frac{1}{4}B) \supset J$ for some $l \in \mathbb{N}$. Let $q_1, \ldots, q_m$ be the critical values of f^l, and put $D_i = B(q_i, \rho)$ for $i = 1, \ldots, m$. Choose ρ small enough to have

$$\tilde{\mu}\Big(\Big\{\tilde{x} \in \tilde{J} : \pi(\tilde{x}) \notin \bigcup_{i=1}^{m} D_i,\ S_{n-l}\phi(\tilde{x}) > A\sigma\sqrt{n-l},\ \tilde{f}^{n-l}(\tilde{x}) \in \tilde{K}_B\Big\}\Big) > \beta' \tag{9}$$

for some positive β' and infinitely many n's.

Let $n \in \mathbb{N}$ be such that (9) holds. Note that since (9) is satisfied for infinitely many n's, we can require n to be as large as we wish. So we assume additionally that n is so large that

$$K \exp(-(n-l)\delta) < \frac{\rho}{2} \tag{10}$$

where the constants δ, K come from Lemma 5. More conditions on n will appear below.

For every $\tilde{x} = (\ldots, x_{-n}, x_{-(n-1)}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots)$ satisfying (9) one can choose a preimage of $x_0 = \pi(\tilde{x})$ under f^l, lying in $\frac{1}{4}B$. We denote this preimage by x^l. We claim that there is a branch f_ν^{-n}, well defined in B, such that $f^{-n}(x_{n-l}) = x^l$. Indeed, let $f_\tau^{-(n-l)}$ be the (K, δ, M)-good branch sending $x_{n-l} = \pi(\tilde{f}^{n-l}(\tilde{x}))$ to $x_0 = \pi(\tilde{x})$. By definition of a good branch and by (10) we have

$$\operatorname{diam}(f^{-(n-l)}(B)) < K \exp(-(n-l)\delta)) < \frac{\rho}{2}.$$

Since $x_0 \notin \bigcup D_i$,

$$B\left(x_0, \frac{1}{2}\rho\right) \cap \bigcup \frac{1}{2} D_i = \emptyset.$$

Consequently,

$$x_0 \in f_\tau^{-(n-l)}(B) \subset B\left(x_0, \frac{1}{2}\rho\right) \subset \hat{\mathbb{C}} \setminus \bigcup \frac{1}{2} D_i. \tag{11}$$

Therefore, branches of f^{-l} are well defined in $f_\tau^{-(n-l)}(B)$. Let f_η^{-l} be the branch mapping x_0 to x^l. The required branch f_ν^{-n} is the composition $f_\eta^{-l} \circ f_\tau^{-(n-l)}$.

Set

$$S = \sup_{z\colon f^l(z) \notin \bigcup \frac{1}{2} D_i} \frac{1}{\|f^l)'(z)\|} < \infty.$$

Then $\|(f_\eta^{-l})'\| \leq S$, and using (11) we get

$$\begin{aligned} \operatorname{diam} f_\nu^{-n}(B) &\leq S \cdot \operatorname{diam} f_\tau^{-(n-l)}(B) \leq SK \exp(-(n-l)\delta)) \\ &< \frac{1}{16} r = \frac{1}{4} \cdot \operatorname{radius}(B) \end{aligned} \tag{12}$$

if n has been chosen large enough.

Since $x^l \in f_\eta^{-n}(B) \cap \frac{1}{4}B$ we conclude from (12) that $f_\nu^{-n}(B) \subset \frac{1}{2}B$. This implies that there exists a fixed point of f^n, i.e. a periodic point of f, in $\frac{1}{2}B$. Denote it by y.

We shall estimate the derivative $\|(f^n)'(y)\|$ from below:

$$\begin{aligned}(13)\qquad \|(f^n)'(y)\| &= \|(f^l)'(y)\| \cdot \|(f^{n-l})'(f^l(y))\| \\ &\geq \frac{1}{S} \inf_{w\in f_\tau^{-(n-l)}(B)} \|(f^{n-l})'(w)\| \geq \frac{1}{S}\,\frac{1}{D}\|(f^{n-l})'(x_0)\|\end{aligned}$$

Here, D stands for the distortion estimate, in the ball B, of a spherical derivative of the good branch $f_\tau^{-(n-l)}$; recall that this branch is defined on the twice larger ball $2B$. See e.g. [BKZ] or [PU] for a precise formulation of the Spherical Koebe Distortion Theorem.

Consequently, we have

$$\begin{aligned}(14)\qquad \log\|(f^n)'(y)\| &\geq -\log(SD) + \log\|(f^{n-l})'(x_0)\| \\ &\geq -\log(SD) + \frac{A\sigma\sqrt{n-l}}{\alpha} + \frac{(n-l)\log d}{\alpha}\end{aligned}$$

and

$$\begin{aligned}(15)\qquad \frac{1}{n}\log\|(f^n)'(y)\| &\geq -\frac{1}{n}\log(SD) + \frac{A\sigma\sqrt{n-l}}{n\alpha} + \frac{n-l}{n}\cdot\frac{\log d}{\alpha} \\ &> \frac{\log d}{\alpha}\end{aligned}$$

if n was chosen large enough. Applying (2) we see that

$$\chi_m(f) \geq \frac{1}{n}\log\|(f^n)'(y)\| > \frac{\log d}{\alpha} = \chi_a(f).$$

This ends the proof of Proposition 4. ■

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