

Complex Convexity of Orlicz–Lorentz Spaces and its Applications

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Summary. We give sufficient and necessary conditions for complex extreme points of the unit ball of Orlicz–Lorentz spaces, as well as we find criteria for the complex rotundity and uniform complex rotundity of these spaces. As an application we show that the set of norm-attaining operators is dense in the space of bounded linear operators from $d_*(w, 1)$ into $d(w, 1)$, where $d_*(w, 1)$ is a predual of a complex Lorentz sequence space $d(w, 1)$, if and only if $w \in c_0 \setminus \ell_2$.

1. Introduction. The notions of complex rotundity and uniform complex rotundity have been introduced in [22] and [10], respectively. The motivation came from vector-valued analytic function theory. In [22], it was proved that for any X -valued analytic function, the strong maximum modulus theorem holds true whenever X is a complex rotund Banach space. Furthermore, in [22] it was shown that L_1 is complex rotund, and later on in [10] that it is also uniformly complex rotund. Characterizations of complex (uniform) rotundity of Orlicz or Musielak–Orlicz spaces were given in [6], while complex analogues of various moduli of convexity of a normed space were studied in [7] and [9].

In this paper, we find criteria for complex extreme points of the unit ball and for the complex rotundity and uniform complex rotundity of Orlicz–Lorentz spaces. As corollaries we obtain the corresponding criteria for complex rotundity in Lorentz spaces. These results extend the already known criteria for real extreme points, rotundity and uniform rotundity in Lorentz or Orlicz–Lorentz spaces (cf. [2, 12, 15, 16, 17]). In the last section we apply

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complex rotundity of Lorentz sequence spaces to provide an answer to the problem raised in [1], in the case of complex spaces. In fact, employing the strong maximum modulus theorem in the complex Lorentz sequence space $d(w, 1)$, we show that the set of norm-attaining operators is dense in the space of bounded linear operators from $d_*(w, 1)$ into $d(w, 1)$, where $d_*(w, 1)$ is a predual of $d(w, 1)$, if and only if $w \in c_0 \setminus \ell_2$.

Let \mathbb{R} , \mathbb{R}_+ , \mathbb{C} and \mathbb{N} denote the sets of all real, nonnegative real, complex and natural numbers, respectively. We assume here that $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a *Young function*, that is, φ is convex, $\varphi(0) = 0$ and $\varphi(u) > 0$ for $u > 0$. A non-increasing function $w : [0, \gamma) \rightarrow (0, \infty)$ locally integrable with respect to the Lebesgue measure $|\cdot|$ on $[0, \gamma)$, where $\gamma \leq \infty$, is called a *weight function*. For $f : [0, \gamma) \rightarrow \mathbb{C}$ measurable with respect to $|\cdot|$, define the distribution function $d_f(\theta) = |\{t \in [0, \gamma) : |f(t)| > \theta\}|$ for $\theta \geq 0$, and its generalized inverse

$$f^*(t) = \inf\{\theta > 0 : d_f(\theta) \leq t\}, \quad t \in [0, \gamma),$$

called the *decreasing rearrangement* of f . The *Orlicz–Lorentz space* $\Lambda_{\varphi, w}$ is then defined as the set of all measurable functions $f : [0, \gamma) \rightarrow \mathbb{C}$ such that $\varrho_{\varphi}(\lambda f) < \infty$ for some $\lambda > 0$, where

$$\varrho_{\varphi}(f) = \int_0^{\gamma} \varphi(f^*(t))w(t) dt = \int_0^{\gamma} \varphi(f^*)w.$$

The space $\Lambda_{\varphi, w}$ equipped with the norm

$$\|f\| = \inf\{\lambda > 0 : \varrho_{\varphi}(f/\lambda) \leq 1\}$$

is a Banach space.

Now, let us define an Orlicz–Lorentz sequence space $\lambda_{\varphi, w}$. Given a complex-valued sequence $x = \{x(n)\} = \{x(n)\}_{n=1}^{\infty}$, consider the function $f(t) = \sum_{k=1}^{\infty} x(k)\chi_{[k-1, k)}(t)$, $t \geq 0$, and define a decreasing rearrangement of x as follows:

$$x^*(n) = f^*(n-1), \quad n \in \mathbb{N}.$$

Let further $w = \{w(n)\}$ be a weight sequence, that is, $w(n) > 0$ for all $n \in \mathbb{N}$, and $\{w(n)\}$ is non-increasing with $\sum_{n=1}^{\infty} w(n) = \infty$. Then analogously to the non-atomic case, the Orlicz–Lorentz sequence space $\lambda_{\varphi, w}$ consists of all sequences $x = \{x(n)\}$ such that for some $\lambda > 0$,

$$\varrho_{\varphi}(\lambda x) = \sum_{n=1}^{\infty} \varphi(\lambda x^*(n))w(n) < \infty,$$

and equipped with the norm $\|x\| = \inf\{\lambda > 0 : \varrho_{\varphi}(x/\lambda) \leq 1\}$, $\lambda_{\varphi, w}$ is a Banach space. Observe here that the condition $\sum_{n=1}^{\infty} w(n) = \infty$ implies that $\lambda_{\varphi, w} \hookrightarrow c_0$.

If $\varphi(u) = u^p$, $1 \leq p < \infty$, then $\Lambda_{\varphi,w}$ (resp., $\lambda_{\varphi,w}$) is a Lorentz space $\Lambda_{p,w}$ (resp., Lorentz sequence space $d(w,p)$) (cf. [12, 20]) with the norm

$$\|f\| = \left(\int_0^\gamma (f^*)^p w \right)^{1/p} \quad \left(\text{resp., } \|x\|_{w,p} = \left(\sum_{n=1}^\infty x^*(n)^p w(n) \right)^{1/p} \right).$$

The spaces $\Lambda_{\varphi,w}$ and $\lambda_{\varphi,w}$ coincide with the Orlicz space L_φ and the Orlicz sequence space ℓ_φ , respectively ([21]), whenever $w \equiv 1$.

The functional ϱ_φ is a *modular* [21]. In particular ϱ_φ is convex and orthogonally subadditive, that is, for any measurable functions f, g , we have $\varrho_\varphi(\alpha f + \beta g) \leq \alpha \varrho_\varphi(f) + \beta \varrho_\varphi(g)$ for $\alpha + \beta = 1$, $\alpha, \beta \geq 0$ and $\varrho_\varphi(f + g) \leq \varrho_\varphi(f) + \varrho_\varphi(g)$ if $\min\{|f|, |g|\} = 0$.

We shall further need several growth conditions on φ or w . We say that φ satisfies the Δ_2^w -condition if in the case of $\int_0^\gamma w < \infty$ (resp., $\int_0^\gamma w = \infty$) there exist $K \geq 1$ and $x_0 \geq 0$ (resp., there is $K \geq 1$) such that

$$\varphi(2x) \leq K\varphi(x) \quad \text{for all } x \geq x_0 \text{ (resp. for all } x \geq 0).$$

Recall also that φ satisfies the δ_2 -condition if there exist $x_0 > 0$ and $K > 0$ such that

$$\varphi(2x) \leq K\varphi(x) \quad \text{for all } 0 \leq x \leq x_0.$$

The weight function w is *regular* if

$$K = \inf_{0 < x < \gamma/2} \frac{\int_0^{2x} w}{\int_0^x w} > 1.$$

Most of the results collected in the next lemma are well known, and they correspond to the analogous results in Orlicz spaces [6]. We provide the proof here for the sake of completeness.

LEMMA 1.1. *Suppose that φ satisfies the Δ_2^w -condition and let $f, f_n \in \Lambda_{\varphi,w}$. Then:*

- (1) $\varrho_\varphi(f_n) \rightarrow \infty$ if and only if $\|f_n\| \rightarrow \infty$.
- (2) For any $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that if $\varrho_\varphi(f) \leq 1 - \varepsilon$ then $\|f\| \leq 1 - \delta$.
- (3) For any $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that if $\varrho_\varphi(f) \geq 1 + \varepsilon$ then $\|f\| \geq 1 + \delta$.
- (4) $\varrho_\varphi(f_n) \rightarrow 1$ if and only if $\|f_n\| \rightarrow 1$.
- (5) Norm and modular convergence are equivalent, i.e., $\varrho_\varphi(f_n) \rightarrow 0$ if and only if $\|f_n\| \rightarrow 0$.

Proof. (1) Suppose first that $\int_0^\gamma w = \infty$. Then the Δ_2^w -condition is satisfied for all arguments. If $\{\|f_n\|\}$ is bounded, then there is $m \in \mathbb{N}$ such that $\|f_n\| \leq 2^m$ for all $n \in \mathbb{N}$. So $\varrho_\varphi(f_n/2^m) \leq 1$ for all $n \in \mathbb{N}$, and thus for every $n \in \mathbb{N}$,

$$\varrho_\varphi(f_n) \leq K^m \varrho_\varphi(f_n/2^m) \leq K^m,$$

which shows that $\{\varrho_\varphi(f_n)\}$ is bounded. Suppose now that $\int_0^\gamma w < \infty$. As above, if $\{\|f_n\|\}$ is bounded, then there is $m \in \mathbb{N}$ such that $\varrho_\varphi(f_n/2^m) \leq 1$ for all $n \in \mathbb{N}$. Then for x_0 from the Δ_2^w -condition and every $n \in \mathbb{N}$,

$$\begin{aligned} \varrho_\varphi(f_n) &\leq \varrho_\varphi(f_n \chi_{\{|f_n| \leq 2^m x_0\}}) + \varrho_\varphi(f_n \chi_{\{|f_n| > 2^m x_0\}}) \\ &\leq \varphi(2^m x_0) \int_0^\gamma w + K^m \varrho_\varphi(f_n/2^m) \\ &\leq \varphi(2^m x_0) \int_0^\gamma w + K^m < \infty. \end{aligned}$$

This proves that if $\varrho_\varphi(f_n) \rightarrow \infty$ then $\|f_n\| \rightarrow \infty$. The converse implication is obvious by definition of $\|\cdot\|$.

Suppose that (2) does not hold. Then there is $\varepsilon > 0$ and a sequence $\{f_n\}$ such that $\varrho_\varphi(f_n) < 1 - \varepsilon$ and $1/2 \leq \|f_n\| \uparrow 1$. Let $L = \sup_n \{\varrho_\varphi(2f_n)\}$ and $a_n = \|f_n\|^{-1} - 1$. By (1), $L < \infty$. Then for every $n \in \mathbb{N}$,

$$\begin{aligned} 1 = \varrho_\varphi(f_n/\|f_n\|) &= \varrho_\varphi(2a_n f_n + (1 - a_n)f_n) \leq a_n \varrho_\varphi(2f_n) + (1 - a_n) \varrho_\varphi(f_n) \\ &\leq a_n L + 1 - \varepsilon \rightarrow 1 - \varepsilon \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which is a contradiction.

Suppose now that (3) does not hold. Then there are $\varepsilon > 0$ and a sequence $\{f_n\}$ such that $\varrho_\varphi(f_n) > 1 + \varepsilon$, $\|f_n\| \downarrow 1$ and $1 \leq \|f_n\| \leq 2$, $n \in \mathbb{N}$. Setting now $b_n = \|f_n\|/(2\|f_n\| - 1)$, we obtain

$$\begin{aligned} 1 + \varepsilon < \varrho_\varphi(f_n) &= \varrho_\varphi\left(b_n \frac{f_n}{\|f_n\|} + (1 - b_n)2f_n\right) \\ &\leq b_n \varrho_\varphi\left(\frac{f_n}{\|f_n\|}\right) + (1 - b_n) \varrho_\varphi(2f_n) \\ &\leq b_n + (1 - b_n)L \rightarrow 1 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which is a contradiction again. The last two claims are easy consequences of (1)–(3) and properties of ϱ_φ and $\|\cdot\|$. ■

REMARK 1.2. Notice that Lemma 1.1 also holds in Orlicz–Lorentz sequence spaces if we replace the Δ_2^w -condition by the δ_2 -condition.

Finally, recall the following auxiliary results on geometric properties of Orlicz–Lorentz spaces.

THEOREM 1.3 ([5, 16]). *The following conditions are equivalent.*

(1) φ satisfies the Δ_2^w -condition and $\int_0^\gamma w = \infty$ if $\gamma = \infty$ (resp., φ satisfies the δ_2 -condition).

- (2) No isometric copy of ℓ_∞ is contained in $\Lambda_{\varphi,w}$ (resp., $\lambda_{\varphi,w}$).
- (3) $\Lambda_{\varphi,w}$ (resp., $\lambda_{\varphi,w}$) is separable.

Recall that a Banach space X contains ℓ_p^n 's λ -uniformly or contains λ -uniformly copies of all ℓ_p^n 's (for $\lambda > 1$ and $1 \leq p \leq \infty$) if for each $n \in \mathbb{N}$ there is an n -dimensional subspace E of X together with an isomorphism $U_n : \ell_p^n \rightarrow E$ such that $\|U_n\| \|U_n^{-1}\| < \lambda$ ([8]).

THEOREM 1.4 ([19]). *Suppose that the weight function w is not regular. Then $\Lambda_{\varphi,w}$ contains ℓ_∞^n 's λ -uniformly for all $\lambda > 1$.*

A measurable function $f : [0, \infty) \rightarrow \mathbb{C}$ satisfies condition (+) if there is $\theta_1 \geq 0$ such that $|f(t)| \geq \theta_1$ a.e. and $d_{|f|-\theta_1}(\theta) < \infty$ for all $\theta > 0$. Equivalently, f satisfies condition (+) if for all $\theta > 0$ either $d_f(\theta) < \infty$ or $|\{t : |f(t)| < \theta\}| = 0$.

LEMMA 1.5 ([15, 16]). *Suppose that f satisfies condition (+) and there exists $A \subset [0, \gamma)$ with positive measure such that $|f(t)| < |g(t)|$ for all $t \in A$ and $|f| \leq |g|$. Then there exists a set B with positive measure such that $f^*(t) < g^*(t)$ for $t \in B$.*

2. Complex rotundity of Orlicz–Lorentz spaces. An element x of the unit sphere S_X of a complex Banach space $(X, \|\cdot\|)$ is a *complex extreme point* of the unit ball B_X of X if $\{x + \zeta y : |\zeta| \leq 1, \zeta \in \mathbb{C}\} \subset B_X$ for some $y \in X$ implies that $y = 0$. A complex Banach space is *complex rotund* if every element of the unit sphere is a complex extreme point of its unit ball. It has been shown in [22] that L_∞ or ℓ_∞ are not complex rotund, while L_1 or ℓ_1 have this property. This clearly shows that complex rotundity and rotundity (in the usual real sense) are very different properties.

Now, we characterize the complex extreme points of the unit ball of $\Lambda_{\varphi,w}$.

THEOREM 2.1. *A function $f \in \Lambda_{\varphi,w}$ is a complex extreme point of the unit ball of $\Lambda_{\varphi,w}$ if and only if the following conditions are satisfied:*

- (1) $\varrho_\varphi(f) = 1$,
- (2) f satisfies condition (+).

Proof. First recall [3] that for any measurable functions $h_i, i = 1, 2$, on $[0, \gamma)$ we have

$$\int_0^\gamma (h_1 + h_2)^* w \leq \int_0^\gamma h_1^* w + \int_0^\gamma h_2^* w.$$

Now suppose that f satisfies (1) and (2) and assume that $\|f + \zeta g\| \leq 1$ for

all $|\zeta| \leq 1$ for some $g \in \Lambda_{\varphi, w}$. So, in view of the above inequality,

$$\begin{aligned} 1 &= \int_0^\gamma \varphi(f^*) w \leq \int_0^\gamma \varphi(|f|)^* w \leq \int_0^\gamma \left(\frac{\varphi(|f + \zeta g|)}{2} + \frac{\varphi(|f - \zeta g|)}{2} \right)^* w \\ &\leq \int_0^\gamma \left(\frac{\varphi(|f + \zeta g|)^*}{2} + \frac{\varphi(|f - \zeta g|)^*}{2} \right) w \leq 1. \end{aligned}$$

Hence for any $|\zeta| \leq 1$,

$$\varphi(|f|)^* = \left(\frac{\varphi(|f + \zeta g|)}{2} + \frac{\varphi(|f - \zeta g|)}{2} \right)^*.$$

But by condition (+) and Lemma 1.5, the last equality implies that for $|\zeta| \leq 1$,

$$\varphi(|f|) = \frac{1}{2}(\varphi(|f + \zeta g|) + \varphi(|f - \zeta g|)) \quad \text{a.e.}$$

Without loss of generality we may assume that the equality above holds for all $t \in [0, \gamma)$. Using the convexity of φ , for each $t \in [0, \gamma)$, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$\varphi(|f(t)|) = \alpha|f(t)| + \beta,$$

and for all $u \in \mathbb{R}_+$,

$$\varphi(u) \geq \alpha u + \beta.$$

For fixed $t \in [0, \gamma)$, there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ so that for all $|\zeta| \leq 1$,

$$\begin{aligned} \varphi(|f(t)|) &= \alpha|f(t)| + \beta = \frac{1}{2}(\varphi(|f(t) + \zeta g(t)|) + \varphi(|f(t) - \zeta g(t)|)) \\ &\geq \frac{1}{2}(\alpha|f(t) + \zeta g(t)| + \beta + \alpha|f(t) - \zeta g(t)| + \beta). \end{aligned}$$

Hence for all $|\zeta| \leq 1$ and $t \in [0, \gamma)$,

$$2|f(t)| = |f(t) + \zeta g(t)| + |f(t) - \zeta g(t)|.$$

This implies that $g = 0$ and proves the sufficiency.

Assume now that $f \in S_{\Lambda_{\varphi, w}}$ is a complex extreme point of $B_{\Lambda_{\varphi, w}}$. To show (1) suppose that $\varrho_\varphi(f) < 1$ and $\|f\| = 1$. By the definition of the norm, $\varrho_\varphi(rf) = \infty$ for all $r > 1$. Setting $T = \text{supp } f = \{t : f(t) \neq 0\}$ and $a = 1 - \varrho_\varphi(f)$, we have $a > 0$. First we shall prove that there is $A_0 \subset T$ so that $|A_0| > 0$ and $\varrho_\varphi(f\chi_{A_0}) < a$, and $\varrho_\varphi(rf\chi_{A_0}) < \infty$ for all $r > 1$. In fact, take B_n so that $T = \bigcup_n B_n$ and $|B_n| < \infty$. We also have $T = \bigcup_n D_n$, where $D_n = \{t \in T : |f(t)| \leq n\}$. Clearly, $T = \bigcup_{n,m} B_n \cap D_m$. Since $|T| > 0$, there exist $n, m \in \mathbb{N}$ such that $|D_n \cap B_m| > 0$. Denoting by D the set $D_n \cap B_m$ we have $0 < |D| < \infty$, $D \subset T$, and

$$\varrho_\varphi(f\chi_D) \leq \int_0^\gamma \varphi(n\chi_D)^* w = \int_0^\gamma \varphi(n)\chi_{[0, |D|)} w = \varphi(n) \int_0^{|D|} w < \infty,$$

by the local integrability of w . By the absolute continuity of the integral, there exists $A_0 \subset D$ such that

$$\varrho_\varphi(f\chi_{A_0}) \leq \varphi(n) \int_0^{|A_0|} w < a.$$

For $r > 1$ we also have

$$\varrho_\varphi(rf\chi_{A_0}) \leq \varphi(rn) \int_0^{|A_0|} w < \infty.$$

Continuity of the function $r \mapsto \varrho_\varphi(rf\chi_{A_0})$, $r > 0$, yields $\varrho_\varphi(r_1f\chi_{A_0}) = a$ for some $r_1 > 1$. Choose $\delta > 0$ so that $|z + 1| \leq r_1$ for all $|z| \leq \delta$. Then setting $g = \delta f\chi_{A_0}$, we have for all $|\zeta| \leq 1$,

$$\begin{aligned} \varrho_\varphi(f + \zeta g) &= \varrho_\varphi(f\chi_{T \setminus A_0} + (1 + \zeta\delta)f\chi_{A_0}) \\ &\leq \varrho_\varphi(f) + \varrho_\varphi(r_1f\chi_{A_0}) = \varrho_\varphi(f) + a = 1. \end{aligned}$$

Hence $\|f + \zeta g\| \leq 1$ for all $|\zeta| \leq 1$, where $g \neq 0$ a.e., so f is not a complex extreme point.

In order to show (2) suppose that f does not satisfy condition (+). This clearly implies that $\gamma = \infty$. Then there is $\theta > 0$ such that $d_f(\theta) = \infty$ and $|\{t : |f(t)| < \theta\}| > 0$. Defining $A = \{t : |f(t)| < \theta\}$ we choose a set $A_1 \subset A$ with positive measure and $a > 0$ such that $|f(t)| \leq \theta - 2a$ for all $t \in A_1$. Letting $g(t) = a\chi_{A_1}(t)$, we have $(f + \zeta g)^* = f^*$ for all $|\zeta| \leq 1$. This shows that f is not a complex extreme point and completes the proof. ■

COROLLARY 2.2. *The complex Orlicz–Lorentz space $\Lambda_{\varphi,w}$ is complex rotund if and only if (i) φ satisfies the Δ_2^w -condition and (ii) $\int_0^\infty w = \infty$ if $\gamma = \infty$.*

Proof. Suppose that (i) or (ii) does not hold. Then by Theorem 1.3, there is an isometric copy of ℓ_∞ contained in $\Lambda_{\varphi,w}$. Since ℓ_∞ is not complex rotund, neither is $\Lambda_{\varphi,w}$.

Suppose now that conditions (i) and (ii) are satisfied. Then by Lemma 1.1, $\|f\| = 1$ if and only if $\varrho_\varphi(f) = 1$. If $\gamma < \infty$ then clearly every element in $\Lambda_{\varphi,w}$ satisfies condition (+). Then by Theorem 2.1, $\Lambda_{\varphi,w}$ is complex rotund. If $\gamma = \infty$, then for every $\theta > 0$ and every $f \in \Lambda_{\varphi,w}$,

$$\int_0^\infty \varphi(\theta(\chi_{\{t: |f(t)| > \theta\}})^*)w = \varphi(\theta) \int_0^{|\{f|>\theta\}|} w \leq \int_0^\infty \varphi(f)^*w < \infty.$$

So $|\{t : |f(t)| > \theta\}| < \infty$. Therefore every element in $\Lambda_{\varphi,w}$ satisfies condition (+). Again Theorem 2.1 shows that $\Lambda_{\varphi,w}$ is complex rotund. This completes the proof. ■

COROLLARY 2.3. *An element of the unit sphere of a Lorentz space $\Lambda_{p,w}$, $1 \leq p < \infty$, is a complex extreme point if and only if it satisfies condition (+).*

COROLLARY 2.4. *The Lorentz space $\Lambda_{p,w}$, $1 \leq p < \infty$, is complex rotund if and only if $\int_0^\infty w = \infty$ when $\gamma = \infty$.*

We finish this section with a result on complex rotundity in the sequence Orlicz–Lorentz space.

THEOREM 2.5. *The Orlicz–Lorentz sequence space $\lambda_{\varphi,w}$ is complex rotund if and only if φ satisfies the δ_2 -condition.*

Proof. If φ does not satisfy the δ_2 -condition, then by Theorem 1.3 there is an isometric copy of ℓ_∞ in $\lambda_{\varphi,w}$, and so $\lambda_{\varphi,w}$ cannot be complex rotund. Conversely, suppose that φ satisfies the δ_2 -condition. Take $x \in S_{\lambda_{\varphi,w}}$ and $y \in \lambda_{\varphi,w}$ such that $\|x + \zeta y\| \leq 1$ for all $|\zeta| \leq 1$. Then by the δ_2 -condition, Lemma 1.1 and Remark 1.2, $\varrho_\varphi(x) = 1$. Moreover, there exists a permutation $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that $x^*(n) = |x(\sigma(n))|$, since $\lim_{n \rightarrow \infty} x(n) = 0$. Then, by the Hardy–Littlewood inequality [3] for each $|\zeta| \leq 1$,

$$\sum_{n=1}^{\infty} \varphi(|x(\sigma(n)) + \zeta y(\sigma(n))|)w(n) \leq \sum_{n=1}^{\infty} \varphi((x + \zeta y)^*(n))w(n) = \varrho_\varphi(x + \zeta y) \leq 1.$$

Hence for each $|\zeta| \leq 1$,

$$\begin{aligned} 1 = \varrho_\varphi(x) &= \sum_{n=1}^{\infty} \varphi(|x(\sigma(n))|)w(n) \\ &\leq \frac{1}{2} \sum_{n=1}^{\infty} (\varphi(|x(\sigma(n)) + \zeta y(\sigma(n))|) + \varphi(|x(\sigma(n)) - \zeta y(\sigma(n))|))w(n) \leq 1. \end{aligned}$$

This implies that for each $|\zeta| \leq 1$ and $n \in \mathbb{N}$,

$$\varphi(|x(\sigma(n))|) = \frac{1}{2}(\varphi(|x(\sigma(n)) + \zeta y(\sigma(n))|) + \varphi(|x(\sigma(n)) - \zeta y(\sigma(n))|)).$$

By the same argument as in the proof of Theorem 2.1, we infer that $y = 0$ and this completes the proof. ■

COROLLARY 2.6. *The Lorentz sequence space $d(w, p)$ is complex rotund for all $1 \leq p < \infty$.*

3. Uniform complex rotundity of Orlicz–Lorentz spaces. We say that a complex Banach space $(X, \|\cdot\|)$ is *uniformly complex rotund*, called further *uniformly c -convex*, if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$, $\|y\| > \varepsilon$ and $\|x + \zeta y\| \leq 1$ for all $|\zeta| \leq 1$ then $\|x\| \leq 1 - \delta$. Notice that every uniformly convex complex Banach space is

uniformly c -convex. But in general the converse is not true, since it is well known that L_1 is uniformly c -convex [10].

The following moduli of complex convexity for a Banach space X were introduced in [7]. For $0 < p < \infty$ and $0 \leq \varepsilon \leq 2$, we define

$$H_p^X(\varepsilon) = \inf \left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta}y\|^p d\theta \right)^{1/p} - 1 : \|x\| = 1, \|y\| = \varepsilon \right\},$$

$$H_\infty^X(\varepsilon) = \inf \{ \sup \{ \|x + e^{i\theta}y\| : 0 \leq \theta \leq 2\pi \} - 1 : \|x\| = 1, \|y\| = \varepsilon \}.$$

Let f and g be non-negative, non-decreasing functions on $[0, 2]$. We shall write $f \preceq g$ if there is $K \geq 1$ such that $f(\varepsilon/K) \leq Kg(\varepsilon)$ for all $0 < \varepsilon < 1/K$. We write $f \sim g$ (f and g are then said to be equivalent at zero) if $f \preceq g$ and $g \preceq f$. It has been showed in [7] that for $0 < p < \infty$, the moduli H_p^X are all equivalent at zero.

We shall say that a complex Banach space $(X, \| \cdot \|)$ is *uniformly PL-convex* (resp., *uniformly H_∞^X -convex*) if $H_1^X(\varepsilon) > 0$ for all $\varepsilon > 0$ (resp., $H_\infty^X(\varepsilon) > 0$ for all $\varepsilon > 0$). It is clear that X is uniformly c -convex if and only if it is H_∞^X -convex. Moreover, $H_\infty^X(\varepsilon) \geq H_1^X(\varepsilon)$, and in [9] it was showed that there exists $A > 0$ such that for every complex Banach space X , $A(H_\infty^X(\varepsilon))^2 \leq H_1^X(\varepsilon)$ for all $0 < \varepsilon \leq 1$. This implies that for complex Banach spaces the notions of uniform c -convexity, uniform H_∞^X -convexity and uniform PL -convexity coincide.

We will next investigate uniform c -convexity of Orlicz–Lorentz spaces. Recall the following useful facts about uniform c -convexity.

LEMMA 3.1 ([6]). *Let X be a complex Banach space. Then the following conditions are equivalent.*

- (1) X is uniformly c -convex.
- (2) For every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that $x, y \in X, \|y\| > \varepsilon, \|x + \zeta y\| \leq 1$ for all $\zeta = \pm 1, \pm i$ implies $\|x\| \leq 1 - \delta(\varepsilon)$.
- (3) For every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ such that for any $x, y \in X$,

$$\|y\| \geq \varepsilon \max \{ \|x + y\|, \|x - y\|, \|x + iy\|, \|x - iy\| \}$$

implies

$$\|x\| \leq \frac{1 - \delta(\varepsilon)}{4} (\|x + y\| + \|x - y\| + \|x + iy\| + \|x - iy\|).$$

Notice that \mathbb{C} with the absolute value as a norm is a uniformly c -convex space.

The following lemma will allow us to reduce the investigations of the norm in $A_{\varphi,w}$ to the modular ϱ_φ .

LEMMA 3.2. *Suppose φ satisfies the Δ_2^w -condition. Then the Orlicz–Lorentz space $A_{\varphi,w}$ is uniformly c -convex if and only if ϱ_φ is uniformly*

c -convex, i.e. for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $f, g \in \Lambda_{\varphi, w}$ and

$$\varrho_{\varphi}(g) > \varepsilon, \quad \varrho_{\varphi}(f + \zeta g) \leq 1 \quad \text{for all } |\zeta| \leq 1$$

(or equivalently for all $\zeta = \pm 1, \pm i$), then $\varrho_{\varphi}(f) \leq 1 - \delta$.

Proof. Suppose φ satisfies the Δ_2^w -condition and assume $\Lambda_{\varphi, w}$ is not uniformly c -convex. Then there are $\varepsilon_0 > 0$ and sequences $\{f_n\}, \{g_n\}$ in $\Lambda_{\varphi, w}$ so that for every $n \in \mathbb{N}$ and every $|\zeta| \leq 1$,

$$\|f_n + \zeta g_n\| \leq 1, \quad \|g_n\| \geq \varepsilon_0, \quad \lim_n \|f_n\| = 1.$$

By definition of $\|\cdot\|$ it follows that $\varrho_{\varphi}(f_n + \zeta g_n) \leq 1$. Moreover, by Lemma 1.1(4), $\lim_n \varrho_{\varphi}(f_n) = 1$, and by Lemma 1.1(5), there exists $\varepsilon_1 > 0$ such that $\varrho_{\varphi}(g_n) \geq \varepsilon_1$ for every $n \in \mathbb{N}$. So ϱ_{φ} is not uniformly c -convex. The converse can be proved by a similar argument. ■

For a regular weight w we obtain the following lemma (cf. [12]).

LEMMA 3.3. *Suppose that the weight function w is regular, so that*

$$(3.1) \quad K = \inf_{0 < x < \gamma/2} \frac{\int_0^{2x} w}{\int_0^x w} > 1.$$

Then for fixed $m \in \mathbb{N}$, there is $\theta = \theta(m) > 0$ such that for every $\sigma \in (0, \gamma/(2^m + 1))$,

$$(3.2) \quad \int_{2^m \sigma}^{(2^m+1)\sigma} w \geq \theta \int_0^{\sigma} w.$$

If $\gamma < \infty$ then for each $0 < a < 1$, there is $\theta = \theta(a) > 0$ such that for every $\sigma \in [a\gamma, \gamma]$,

$$(3.3) \quad \int_{\gamma-\sigma}^{\gamma} w \geq \theta \int_0^{\sigma} w.$$

Proof. Let $0 < \sigma < \gamma/2^{m+1}$. Since w is non-increasing,

$$\int_{2^m \sigma}^{(2^m+1)\sigma} w \geq \frac{1}{2^m} \int_{2^m \sigma}^{2^{m+1}\sigma} w.$$

Then

$$\begin{aligned} \frac{\int_{2^m \sigma}^{(2^m+1)\sigma} w}{\int_0^{\sigma} w} &= \frac{\int_{2^m \sigma}^{(2^m+1)\sigma} w}{\int_0^{2^m \sigma} w} \frac{\int_0^{2^m \sigma} w}{\int_0^{\sigma} w} \geq K^m \frac{\int_{2^m \sigma}^{(2^m+1)\sigma} w}{\int_0^{2^m \sigma} w} \\ &\geq \left(\frac{K}{2}\right)^m \frac{\int_{2^m \sigma}^{2^{m+1}\sigma} w}{\int_0^{2^m \sigma} w} = \left(\frac{K}{2}\right)^m \frac{\int_0^{2^{m+1}\sigma} w - \int_0^{2^m \sigma} w}{\int_0^{2^m \sigma} w} \\ &\geq \left(\frac{K}{2}\right)^m (K - 1) > 0. \end{aligned}$$

So $\theta = (K/2)^m(K - 1) > 0$ satisfies (3.2) if $\gamma = \infty$. Now if $\gamma < \infty$ and $\gamma/2^{m+1} \leq \sigma < \gamma/(2^m + 1)$, then $\theta = \min\{(K/2)^m(K - 1), \int_{(1-2^{-m-1})\gamma}^\gamma w / \int_0^\gamma w\} > 0$ satisfies (3.2) if $\gamma < \infty$. For the last claim, if $\gamma < \infty$ then notice that the function $\sigma \mapsto \int_{\gamma-\sigma}^\gamma w / \int_0^\sigma w$ is positive and continuous on $[a\gamma, \gamma]$. So there is $\theta = \theta(a) > 0$ such that for every $a\gamma \leq \sigma \leq \gamma$,

$$\int_{\gamma-\sigma}^\gamma w \geq \theta \int_0^\sigma w,$$

which completes the proof. ■

THEOREM 3.4. *The Orlicz–Lorentz space $\Lambda_{\varphi,w}$ is uniformly c -convex if and only if the following conditions are fulfilled:*

- (1) φ satisfies the Δ_2^w -condition;
- (2) the weight function w is regular.

Proof. Suppose that φ satisfies the Δ_2^w -condition and w is regular, but $\Lambda_{\varphi,w}$ is not uniformly c -convex. By Lemma 3.2, ϱ_φ is not uniformly c -convex. So there are $\varepsilon > 0$ and sequences $\{f_n\}, \{g_n\}$ in $\Lambda_{\varphi,w}$ so that for every $n \in \mathbb{N}$ and every $|\zeta| \leq 1$,

$$\varrho_\varphi(f_n + \zeta g_n) \leq 1, \quad \varrho_\varphi(g_n) \geq 2\varepsilon, \quad \lim_n \varrho_\varphi(f_n) = 1.$$

By Theorem 1.3, $\Lambda_{\varphi,w}$ is separable. Then the family of all simple functions with supports of finite measure is dense in $\Lambda_{\varphi,w}$. So we may assume that for each $n \in \mathbb{N}$ there is $\xi_n > 0$ so that both f_n and g_n are constant on each interval of the form $[(k - 1)\xi_n, k\xi_n) \in [0, \gamma)$, $k \in \mathbb{N}$, and both f_n and g_n have supports of finite measure.

For $n \in \mathbb{N}$, let

$$E_n = \left\{ t \in [0, \gamma) : |g_n(t)| \geq \frac{\varepsilon}{4} \max\{|f_n(t) + \zeta g_n(t)| : \zeta = \pm 1, \pm i\} \right\}.$$

Then

$$\varrho_\varphi(g_n \chi_{E_n^c}) \leq \frac{\varepsilon}{4} \sum_{\zeta = \pm 1, \pm i} \varrho_\varphi(f_n + \zeta g_n) \leq \varepsilon,$$

where $E_n^c = [0, \gamma) \setminus E_n$. Hence $2\varepsilon \leq \varrho_\varphi(g_n) \leq \varrho_\varphi(g_n \chi_{E_n}) + \varrho_\varphi(g_n \chi_{E_n^c})$ implies that

$$(3.4) \quad \varrho_\varphi(g_n \chi_{E_n}) \geq \varepsilon.$$

Setting

$$s_n = \frac{1}{4} \sum_{\zeta = \pm 1, \pm i} \varphi(|f_n + \zeta g_n|) = \frac{1}{4} \sum_{\zeta = \pm 1, \pm i} \varphi(|g_n + \zeta f_n|),$$

by the uniform convexity of \mathbb{C} there is $\delta_1 > 0$ such that for all $n \in \mathbb{N}$ and all $t \in E_n$,

$$\varphi(|f_n(t)|) \leq \frac{1-\delta_1}{4} \sum_{\zeta=\pm 1, \pm i} \varphi(|f_n(t) + \zeta g_n(t)|) = (1-\delta_1)s_n(t).$$

Notice that for every $t \in [0, \gamma)$,

$$\varphi(|f_n(t)|) \leq s_n(t), \quad \varphi(|g_n(t)|) \leq s_n(t).$$

Let

$$h_n = (1-\delta_1)s_n\chi_E + \varphi(|f_n|)\chi_{E^c}.$$

We use here the method invented in [12]. For convenience, we further omit the index n . Thus, until the end of the proof, f, g, h, s, E and ξ will stand for f_n, g_n, h_n, s_n, E_n and ξ_n , respectively. Since f and g are constant on each interval $[(k-1)\xi, k\xi)$ and have supports of finite measure, it follows that h as well as $h\chi_E$ are constant on each interval $[(k-1)\xi, k\xi)$ and have a finite number of values. Thus, for some $j, r \in \mathbb{N}$,

$$\begin{aligned} h^* &= \sum_{i=1}^j h^i \chi_{[(i-1)\xi, i\xi)}, & (h\chi_E)^* &= \sum_{k=1}^r h^{i_k} \chi_{[(k-1)\xi, k\xi)}, \\ (s\chi_E)^* &= \frac{1}{1-\delta_1} \sum_{k=1}^r h^{i_k} \chi_{[(k-1)\xi, k\xi)} = \sum_{k=1}^r s^k \chi_{[(k-1)\xi, k\xi)}, \end{aligned}$$

where $h^1 \geq \dots \geq h^j > 0$, $h^{i_1} \geq \dots \geq h^{i_r} > 0$ and $s^k = \frac{1}{1-\delta_1} h^{i_k}$. Notice that for each $t \in [(i_k-1)\xi, i_k\xi)$,

$$h^*(t) + \frac{\delta_1}{1-\delta_1} h^*(t) = h^{i_k} + \frac{\delta_1}{1-\delta_1} h^{i_k} = s^k.$$

Hence by the Hardy–Littlewood inequality [3],

$$\begin{aligned} &\int_0^\gamma h^* w + \frac{\delta_1}{1-\delta_1} \sum_{k=1}^r \int_{(i_k-1)\xi}^{i_k\xi} h^* w \\ &= \sum_{k=1}^r \int_{(i_k-1)\xi}^{i_k\xi} \left(h^* w + \frac{\delta_1}{1-\delta_1} h^* w \right) + \sum_{\substack{j \in \{1, \dots, n\} \\ j \notin \{i_1, \dots, i_r\}}} \int_{(j-1)\xi}^{j\xi} h^* w \\ &\leq \sum_{k=1}^r s^k \int_{(i_k-1)\xi}^{i_k\xi} w + \sum_{\substack{j \in \{1, \dots, n\} \\ j \notin \{i_1, \dots, i_r\}}} \int_{(j-1)\xi}^{j\xi} s^* w \\ &\leq \int_0^\gamma s^* w = \int_0^\gamma \frac{1}{4} \left(\sum_{\zeta=\pm 1, \pm i} \varphi(|f + \zeta g|)^* w \right) \\ &= \frac{1}{4} \sum_{\zeta=\pm 1, \pm i} \varrho_\varphi(|f + \zeta g|) \leq 1. \end{aligned}$$

Since $h^* \geq \varphi(|f|)^*$,

$$(3.5) \quad \int_0^\gamma h^* w \geq \varrho_\varphi(f) \rightarrow 1.$$

Thus

$$(3.6) \quad \sum_{k=1}^r \int_{(i_k-1)\xi}^{i_k\xi} h^* w \rightarrow 0.$$

Moreover, in view of (3.4) we get

$$(3.7) \quad \begin{aligned} \int_0^\gamma (h\chi_E)^* w &= (1 - \delta_1) \int_0^\gamma (s\chi_E)^* w \\ &\geq (1 - \delta_1) \int_0^\gamma \varphi(|g|\chi_E)^* w = (1 - \delta_1) \varrho_\varphi(g\chi_E) \geq (1 - \delta_1)\varepsilon. \end{aligned}$$

By regularity of w , suppose further that $K > 1$ is a constant in (3.1).

Applying the definition of h , in view of (3.6) and (3.7), and letting $\varepsilon_3 = (1 - \delta_1)\varepsilon > 0$, $\alpha^k = h^{i_k}$, $\beta^r = \alpha^r$, and $\beta^k = \alpha^k - \alpha^{k+1}$ for $k < r$, we obtain the following formulas:

$$(3.8) \quad \begin{aligned} \beta^1 \int_0^{i_1\xi} w + \beta^2 \int_0^{i_2\xi} w + \dots + \beta^r \int_0^{i_r\xi} w \\ = \int_0^{i_1\xi} h^{i_1} w + \int_{i_1\xi}^{i_2\xi} h^{i_2} w + \dots + \int_{i_{r-1}\xi}^{i_r\xi} h^{i_r} w \leq \int_0^\gamma h^* w \leq 1, \end{aligned}$$

$$(3.9) \quad \beta^1 \int_0^\xi w + \beta^2 \int_0^{2\xi} w + \dots + \beta^r \int_0^{r\xi} w = \int_0^\gamma (h\chi_E)^* w \geq \varepsilon_3 > 0,$$

$$(3.10) \quad \beta^1 \int_{(i_1-1)\xi}^{i_1\xi} w + \dots + \beta^r \left(\int_{(i_1-1)\xi}^{i_1\xi} w + \dots + \int_{(i_r-1)\xi}^{i_r\xi} w \right) = \sum_{k=1}^r \int_{(i_k-1)\xi}^{i_k\xi} h^* w \rightarrow 0.$$

Choose m so that $K^m \varepsilon_3 / 2 > 1$. Now in view of (3.8) and regularity of w ,

$$1 \geq \sum_{i_k > 2^m k} \beta^k \int_0^{i_k\xi} w \geq \sum_{i_k > 2^m k} \beta^k \int_0^{2^m k\xi} w \geq K^m \sum_{i_k > 2^m k} \beta^k \int_0^{k\xi} w,$$

where $k = 1, \dots, r$. This implies that

$$\sum_{i_k > 2^m k} \beta^k \int_0^{k\xi} w \leq \frac{1}{K^m} < \frac{\varepsilon_3}{2}.$$

Hence by (3.9),

$$\sum_{i_k \leq 2^m k} \beta^k \int_0^{k\xi} w \geq \frac{\varepsilon_3}{2}.$$

Assume first that $\gamma = \infty$. In view of monotonicity of w , by Lemma 3.3, there is $\theta > 0$ depending only on m , such that

$$\begin{aligned} & \sum_{i_k \leq 2^m k} \beta^k \left(\int_{(i_1-1)\xi}^{i_1\xi} w + \dots + \int_{(i_k-1)\xi}^{i_k\xi} w \right) \\ & \geq \sum_{i_k \leq 2^m k} k\beta^k \int_{(i_k-1)\xi}^{i_k\xi} w \geq \sum_{i_k \leq 2^m k} k\beta^k \int_{2^m k\xi}^{(2^m k+1)\xi} w \\ & \geq \sum_{i_k \leq 2^m k} \beta^k \int_{2^m k\xi}^{(2^m+1)k\xi} w \geq \sum_{i_k \leq 2^m k} \theta\beta^k \int_0^{k\xi} w \geq \frac{\theta\varepsilon_3}{2}. \end{aligned}$$

But this contradicts (3.10), proving the sufficiency for $\gamma = \infty$. Now suppose that $\gamma < \infty$. Since w is non-increasing and $i_1 < \dots < i_r$, we have the following inequalities:

$$\begin{aligned} & \sum_{i_k \leq 2^m k} \beta^k \left(\int_{(i_1-1)\xi}^{i_1\xi} w + \dots + \int_{(i_k-1)\xi}^{i_k\xi} w \right) \\ & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} \beta^k \left(\int_{2^m k\xi}^{(2^m k+1)\xi} w + \dots + \int_{2^m k\xi}^{(2^m k+1)\xi} w \right) \\ & \quad + \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi \geq \gamma}} \beta^k \left(\int_{(i_1-1)\xi}^{i_1\xi} w + \dots + \int_{(i_k-1)\xi}^{i_k\xi} w \right) \\ & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} k\beta^k \int_{2^m k\xi}^{(2^m k+1)\xi} w + \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi \geq \gamma}} \beta^k \left(\int_{\gamma-k\xi}^{\gamma-(k-1)\xi} w + \dots + \int_{\gamma-\xi}^{\gamma} w \right) \\ & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} k\beta^k \int_{2^m k\xi}^{(2^m k+1)\xi} w + \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi \geq \gamma}} \beta^k \int_{\gamma-k\xi}^{\gamma} w. \end{aligned}$$

Then by (3.2) there is $\theta > 0$ depending only on m and such that

$$\begin{aligned}
 & \sum_{i_k \leq 2^m k} \beta^k \left(\int_{(i_1-1)\xi}^{i_1\xi} w + \dots + \int_{(i_k-1)\xi}^{i_k\xi} w \right) \\
 & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} k\beta^k \int_{2^m k\xi}^{(2^m k+1)\xi} w + \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi \geq \gamma}} \beta^k \int_{\gamma-k\xi}^{\gamma} w \\
 & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} \beta^k \int_{2^m k\xi}^{(2^m+1)k\xi} w + \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi \geq \gamma}} \beta^k \int_{\gamma-k\xi}^{\gamma} w \\
 & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} \theta\beta^k \int_0^{k\xi} w + \sum_{\substack{i_k \leq 2^m k \\ \gamma/(2^m+1) \leq k\xi \leq \gamma}} \beta^k \int_{\gamma-k\xi}^{\gamma} w,
 \end{aligned}$$

and by (3.3), we find $\theta_1 > 0$ depending only on m so that

$$\begin{aligned}
 & \sum_{i_k \leq 2^m k} \beta^k \left(\int_{(i_1-1)\xi}^{i_1\xi} w + \dots + \int_{(i_k-1)\xi}^{i_k\xi} w \right) \\
 & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} \theta\beta^k \int_0^{k\xi} w + \sum_{\substack{i_k \leq 2^m k \\ \gamma/(2^m+1) \leq k\xi \leq \gamma}} \beta^k \int_{\gamma-k\xi}^{\gamma} w \\
 & \geq \sum_{\substack{i_k \leq 2^m k \\ (2^m+1)k\xi < \gamma}} \theta\beta^k \int_0^{k\xi} w + \sum_{\substack{i_k \leq 2^m k \\ \gamma/(2^m+1) \leq k\xi \leq \gamma}} \theta_1\beta^k \int_0^{k\xi} w \\
 & \geq \min\{\theta, \theta_1\} \sum_{i_k \leq 2^m k} \beta^k \int_0^{k\xi} w \geq \frac{\min\{\theta, \theta_1\}\varepsilon_3}{2}.
 \end{aligned}$$

Again this contradicts (3.10) and the sufficiency is proved.

Conversely, if $\Lambda_{\varphi,w}$ is uniformly c -convex, then it is clearly complex rotund, and by Corollary 2.2, φ satisfies the Δ_2^w -condition. Notice also that a Banach space which contains ℓ_∞^n 's λ -uniformly for all $\lambda > 1$ cannot be uniformly c -convex. Hence, in view of Theorem 1.4, w is regular. The proof is complete. ■

THEOREM 3.5. *Let $\Lambda_{\varphi,w}$ be a complex Orlicz–Lorentz space. The following conditions are equivalent.*

- (1) $\Lambda_{\varphi,w}$ is uniformly c -convex.
- (2) φ satisfies the condition Δ_2^w and the weight function w is regular.

- (3) $\Lambda_{\varphi,w}$ is of cotype q for some $2 \leq q < \infty$.
 (4) $\Lambda_{\varphi,w}$ has an equivalent q -uniformly PL-convex norm for some $2 \leq q < \infty$.
 (5) $\Lambda_{\varphi,w}$ is uniformly PL-convex.

Proof. By Theorem 3.4 we know that (1) is equivalent to (2). The equivalences (3) \Leftrightarrow (4) and (1) \Leftrightarrow (5) have been proved in Corollaries 7.1 and 7.4 of [7] and Theorem 2.1 of [9], respectively. If (1) is satisfied then $\Lambda_{\varphi,w}$ cannot contain ℓ_∞^n 's λ -uniformly for all $\lambda \geq 1$, and by the well known result of Pisier (Theorem 14.1 in [8]), $\Lambda_{\varphi,w}$ has finite cotype, and so (3) is satisfied.

Finally, we will show that (3) \Rightarrow (2). Suppose that (2) is not true. If φ does not satisfy the Δ_2^w -condition, then by Theorem 1.3, $\Lambda_{\varphi,w}$ contains an isometric copy of ℓ_∞ , and so it cannot be of finite cotype. If w is not regular, then by Theorem 1.4, $\Lambda_{\varphi,w}$ contains ℓ_∞^n 's λ -uniformly for all $\lambda > 1$, and again by Pisier's result, it cannot have any finite cotype. ■

The next result on uniform c -convexity in Lorentz spaces is an immediate corollary of the previous theorem.

COROLLARY 3.6. *Let $\Lambda_{p,w}$, $1 \leq p < \infty$, be a complex Lorentz space. The following conditions are equivalent.*

- (1) $\Lambda_{p,w}$ is uniformly c -convex.
 (2) The weight function w is regular.
 (3) $\Lambda_{p,w}$ is of cotype q for some $2 \leq q < \infty$.
 (4) $\Lambda_{p,w}$ has an equivalent q -uniformly PL-convex norm for some $2 \leq q < \infty$.
 (5) $\Lambda_{p,w}$ is uniformly PL-convex.

4. Applications to norm-attaining operators. Let $\{w(n)\} \in c_0 \setminus \ell_1$ be a decreasing sequence of positive numbers, and let $\Psi(n) = \sum_{k=1}^n w(k)$. For a complex sequence $x = \{x(n)\}$, define the functional

$$\|x\|_\Psi = \sup_{n \geq 1} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)},$$

where $\{x^*(n)\}$ is a decreasing rearrangement of $x = \{x(n)\}$. The Marcinkiewicz sequence space m_Ψ is the space of all sequences x with $\|x\|_\Psi < \infty$. Its order continuous subspace $d_*(w, 1) = m_\Psi^0$ is the subspace of m_Ψ consisting of all $x = \{x(n)\}$ with

$$\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n x^*(k)}{\Psi(n)} = 0.$$

It is well known [14, 18, 20] that $d_*(w, 1)$ is a predual of the Lorentz sequence space $d(w, 1)$ equipped with the norm $\| \cdot \|_{w,1}$.

Let X and Y be Banach spaces, and let $L(X, Y)$ be the space of all bounded linear operators from X to Y , and $NA(X, Y) \subset L(X, Y)$ be the set of all norm-attaining operators. Recall that a bounded operator $T : X \rightarrow Y$ is *norm-attaining* if $\|Tx\| = \|T\|$ for some $x \in B_X$. When $Y = \mathbb{R}$, Bishop and Phelps [4] showed that $NA(X, Y)$ is dense in $L(X, Y)$. Since then there have been many attempts to prove or disprove for what X or Y , $NA(X, Y)$ is dense in $L(X, Y)$. In [11], Gowers showed that $NA(G, \ell_p)$, $1 < p < \infty$, is not dense in $L(G, \ell_p)$, where $G = d_*(w, 1)$ and $w = \{n^{-1}\}_{n=1}^\infty$. Using the same space G , the authors of [1] showed that $NA(G, G^*)$ is not dense in $L(G, G^*)$ if G is equipped with a proper equivalent norm. They also raised the question whether this is also true for G with its original norm. Here we answer their question in the affirmative for complex spaces G . Let us start with the following elementary lemma.

LEMMA 4.1. *The identity operator $I : d_*(w, 1) \rightarrow d(w, 1)$ is a bounded operator if and only if the weight sequence $\{w(n)\}$ is in ℓ_2 .*

Proof. Suppose that the identity operator $I : d_*(w, 1) \rightarrow d(w, 1)$ is bounded. Recall that $d(w, 1)$ is a Köthe dual of $d_*(w, 1)$ ([20, 18]), that is,

$$\|x\|_{w,1} = \sup \left\{ \langle x, y \rangle = \sum_{n=1}^{\infty} x(n)y(n) : \|y\|_{\Psi} \leq 1 \right\}.$$

Hence for each $n \geq 1$, taking $x_n = \sum_{k=1}^n w(k)e_k$, where e_k are the standard unit vectors, we have $\|x_n\|_{\Psi} \leq 1$ and so,

$$\langle Ix_n, x_n \rangle = \sum_{k=1}^n w(k)^2 \leq \|I\|.$$

Thus $\{w(n)\} \in \ell_2$.

For the converse, suppose that $\{w(n)\} \in \ell_2$. By definition of $\|\cdot\|_{\Psi}$, for any $x \in d_*(w, 1)$ and each $n \in \mathbb{N}$,

$$\sum_{k=1}^n x^*(k) \leq \|x\|_{\Psi} \Psi(n).$$

Now, by summation by parts, if we set $\Psi(0) = 0$ and $S(n) = \sum_{k=1}^n x^*(k)$ for each $n \in \mathbb{N}$ and $S(0) = 0$, then

$$\begin{aligned} \sum_{k=1}^n x^*(k)w(k) &= \sum_{k=1}^n (S(k) - S(k-1))w(k) \\ &= \sum_{k=1}^{n-1} S(k)(w(k) - w(k+1)) + S(n)w(n) \end{aligned}$$

$$\begin{aligned}
&\leq \|x\|_\Psi \sum_{k=1}^{n-1} \Psi(k)(w(k) - w(k+1)) + \|x\|_\Psi \Psi(n)w(n) \\
&\leq \|x\|_\Psi \sum_{k=1}^n (\Psi(k) - \Psi(k-1))w(k) = \|x\|_\Psi \sum_{k=1}^n w(k)^2 \leq \|x\|_\Psi \sum_{k=1}^{\infty} w(k)^2.
\end{aligned}$$

Therefore the identity operator I is well defined and bounded. This completes the proof. ■

Recall the result ([13, 18, 14]) characterizing the geometric structure of the ball in Marcinkiewicz spaces.

LEMMA 4.2. *For each $x \in B_{d_*(w,1)}$, there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $\|x + \lambda y\| \leq 1$ for each $y \in B_{d_*(w,1)}$ with $y = (0, \dots, 0, y(n+1), \dots)$, and for each $|\lambda| \leq \varepsilon$.*

THEOREM 4.3. *Let $d(w, 1)$ and $d_*(w, 1)$ be complex spaces. Then the set $NA(d_*(w, 1), d(w, 1))$ is not dense in $L(d_*(w, 1), d(w, 1))$ if $\{w(n)\} \in \ell_2$.*

Proof. First we shall show that every norm-attaining operator from $d_*(w, 1)$ into $d(w, 1)$ has a finite rank. Suppose $T \in NA(d_*(w, 1), d(w, 1))$ and $\|T\| = \|Tx_0\|_{w,1}$ for $x_0 \in B_{d_*(w,1)}$. By Lemma 4.2, there are $n \geq 1$ and $\varepsilon > 0$ such that for each $y = (0, \dots, 0, y(n+1), \dots) \in B_{d_*(w,1)}$ and for each $|\zeta| < \varepsilon$,

$$\|x_0 + \zeta y\|_\Psi \leq 1.$$

Hence

$$\|T\| = \|Tx_0\|_{w,1} \geq \|T(x_0 + \zeta y)\|_{w,1} = \|Tx_0 + \zeta Ty\|_{w,1}.$$

Notice that the complex Lorentz space $d(w, 1)$ is complex rotund by Corollary 2.6. Therefore the strong maximum modulus theorem holds in $d(w, 1)$ (see [22]). This implies that $\|Tx_0 + \zeta Ty\|_{w,1}$ is constant for every $|\zeta| < \varepsilon$, and so $Ty = 0$. Thus every norm-attaining operator has a finite rank. Now, for any $T \in NA(d_*(w, 1), d(w, 1))$ there exists $n \in \mathbb{N}$ such that $Te_m = 0$ for all $m > n$. Hence

$$\|I - T\| \geq w(1)\|e_m\|_{w,1} = w(1)^2 > 0.$$

Thus $NA(d_*(w, 1), d(w, 1))$ is not dense in $L(d_*(w, 1), d(w, 1))$, and this completes the proof. ■

In fact we can say more. In [14], it is proved that the condition $w \in \ell_2$ holds if and only if the space $AL^2(d_*(w, 1))$ of norm-attaining continuous bilinear forms is not dense in the space $L^2(d_*(w, 1))$ of continuous bilinear forms on $d_*(w, 1)$. If we identify, in a natural way, the space $L^2(d_*(w, 1))$ with $L(d_*(w, 1), d(w, 1))$, then $AL^2(d_*(w, 1)) \subset NA(d_*(w, 1), d(w, 1))$ (see [1]).

This implies that if $w \in c_0 \setminus \ell_2$, then $NA(d_*(w, 1), d(w, 1))$ is dense in $L(d_*(w, 1), d(w, 1))$. Summarizing these facts we obtain the following result.

COROLLARY 4.4. *Let $d(w, 1)$ and $d_*(w, 1)$ be complex spaces. Then the set $NA(d_*(w, 1), d(w, 1))$ is dense in $L(d_*(w, 1), d(w, 1))$ if and only if $\{w(n)\} \notin \ell_2$.*

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