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MEASURE AND INTEGRATION

Relations between Shy Sets and Sets of ν_p-Measure Zero in Solovay's Model

by

G. PANTSULAIA

Presented by Andrzej LASOTA

Summary. An example of a non-zero non-atomic translation-invariant Borel measure ν_p on the Banach space ℓ_p ($1 \le p \le \infty$) is constructed in Solovay's model. It is established that, for $1 \le p < \infty$, the condition "ν_p-almost every element of ℓ_p has a property P" implies that "almost every" element of ℓ_p (in the sense of [4]) has the property P. It is also shown that the converse is not valid.

The problem of the existence of an analogue of Lebesgue measure on infinite-dimensional topological vector spaces is interesting in itself and it has been studied for more than half a century by many people (cf. e.g. [2], [8], [9]). Among others, the result of I. V. Girsanov and B. S. Mityagin [2] deserves a special mention. It asserts that a σ-finite quasi-invariant Borel measure defined on an infinite-dimensional locally convex topological vector space is identically zero. Using this result, we deduce that the properties of σ-finiteness and translation-invariance are not consistent for non-zero Borel measures in infinite-dimensional topological vector spaces. A. B. Kharazishvili [5], ignoring the property of translation-invariance, constructed an example of a non-zero σ-finite Borel measure in the Hilbert space ℓ_2 which is invariant with respect to an everywhere dense (in ℓ_2) linear manifold. In the present article we ignore the property of σ-finiteness and give a construction of a non-zero non-atomic translation-invariant Borel measure ν_p on the Banach space ℓ_p ($1 \le p \le \infty$) in the well-known Solovay model (cf. [7]), which is the following system of axioms:

(ZF) & (DC) & (every subset of $\mathbb{R}$ is measurable in the Lebesgue sense),

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where (ZF) denotes the Zermelo–Fraenkel set theory and (DC) denotes the axiom of Dependent Choices (cf. [1], [7]). Finally, we study relations between shy sets (cf. [4]) and sets of ν_p-measure zero in the above-mentioned model.

In what follows, we need some standard notions and auxiliary propositions. Let (E, S, μ) be a measure space. Following [1, p. 88], the measure μ is called *diffused* (or *continuous*) if

$$(\forall x \in E)(\{x\} \in S \ \& \ \mu(\{x\}) = 0).$$

LEMMA 1 ([1, Theorem 4.12, p. 106]). *Let E_1 and E_2 be any two Polish topological spaces. Let μ_1 and μ_2 be diffused Borel probability measures on E_1 and E_2 respectively. Then there exists a Borel isomorphism*

$$\varphi : (E_1, B(E_1)) \to (E_2, B(E_2))$$

such that

$$(\forall X \in B(E_1))(\mu_1(X) = \mu_2(\varphi(X))).$$

LEMMA 2. *Let E be a Polish topological space and let μ be a diffused Borel probability measure on E. Then in Solovay's model the completion $\overline{\mu}$ of μ is defined on the power set of E.*

Proof. Let b_1 be a probability Borel measure defined on $[0, 1]$ whose completion coincides with the standard Lebesgue measure on $[0, 1]$. According to Lemma 1, the measure b_1 is Borel isomorphic to the probability measure μ. Denote this isomorphism by φ and consider an arbitrary set $W \subset E$. It is clear that $\varphi^{-1}(W) = X \cup Y$, where

$$X \in B([0, 1]), \qquad (\exists Z)(Y \subset Z \in B([0, 1]) \ \& \ b_1(Z) = 0).$$

The isomorphism between the measures b_1 and μ implies

$$\mu(\varphi(Z \setminus X)) = 0.$$

On the one hand, we can write

$$W = \varphi(X) \cup \varphi(Y).$$

On the other hand, we have $\varphi(Y) \subset \varphi(Z)$. Clearly, $\overline{\mu}(\varphi(Y)) = 0$ since $\mu(\varphi(Z)) = 0$. ∎

COROLLARY 1. *Let $\mathbb{N}$ be the set of all natural numbers. For $k \in \mathbb{N}$, let S_k be the unit circle in the Euclidean plane $\mathbb{R}^2$. We may identify S_k with the compact group of all rotations of $\mathbb{R}^2$ about the origin. Let λ be the Haar probability measure on the compact group $\prod_{k\in\mathbb{N}} S_k$. Then in Solovay's model the completion $\overline{\lambda}$ of λ is defined on the power set of $\prod_{k\in\mathbb{N}} S_k$.*

REMARK 1. For $k \in \mathbb{N}$, define the function $f_k : \mathbb{R} \to S_k$ by

$$f_k(x) = \exp\{\pi x i\} \quad \text{for } x \in \mathbb{R}.$$

Then clearly

$$\Big(\prod_{k\in\mathbb{N}} f_k\Big)(z+w) = \Big(\prod_{k\in\mathbb{N}} f_k\Big)(z) \circ \Big(\prod_{k\in\mathbb{N}} f_k\Big)(w)$$

for all $z, w \in \mathbb{R}^{\mathbb{N}}$, where $\mathbb{R}^{\mathbb{N}}$ denotes the vector space of all real-valued sequences, $\prod_{k\in\mathbb{N}} f_k$ denotes the direct product of the family of functions $(f_k)_{k\in\mathbb{N}}$, and "$\circ$" denotes the group operation in the product group $\prod_{k\in\mathbb{N}} S_k$.

REMARK 2. For $E \subset \mathbb{R}^{\mathbb{N}}$ and $g \in \prod_{k\in\mathbb{N}} S_k$, put

$$f_E(g) = \begin{cases} \operatorname{card}((\prod_{k\in\mathbb{N}} f_k)^{-1}(g) \cap E) & \text{if this is finite,} \\ +\infty & \text{in all other cases.} \end{cases}$$

Then

$$f_{E+h}(g) = f_E(g \circ g_h) \quad \text{for all } h = (h_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}},$$

where

$$g_h = \Big(\prod_{k\in\mathbb{N}} f_k\Big)(-h) = (\exp\{-\pi h_k i\})_{k\in\mathbb{N}}.$$

LEMMA 3 (see [6]). *In Solovay's model there exists a translation-invariant measure μ on $\mathbb{R}^{\mathbb{N}}$ such that*:

(1) $X \in \operatorname{dom}(\mu)$ *for all* $X \subset \mathbb{R}^{\mathbb{N}}$;
(2) $\mu([-1,1]^{\mathbb{N}}) = 1$.

Proof. Define

$$\mu(E) = \int\limits_{\prod_{k\in\mathbb{N}} S_k} f_E(g)\, d\overline{\lambda}(g) \quad \text{for } E \subset \mathbb{R}^{\mathbb{N}}.$$

The functional μ is a measure since $f_{\emptyset}(g) = 0$ and $f_{\bigcup_{k\in\mathbb{N}} E_k}(g) = \sum_{k\in\mathbb{N}} f_{E_k}(g)$ for any family $(E_k)_{k\in\mathbb{N}}$ of disjoint subsets of $\mathbb{R}^{\mathbb{N}}$.

The measure μ is translation-invariant. Indeed, for every $h \in \mathbb{R}^{\mathbb{N}}$, by the invariance of the measure $\overline{\lambda}$ and by Remark 2, we have

$$\begin{aligned} \mu(E+h) &= \int\limits_{\prod_{k\in\mathbb{N}} S_k} f_{E+h}(g)\, d\overline{\lambda}(g) \\ &= \int\limits_{\prod_{k\in\mathbb{N}} S_k} f_E(\Phi(g))\, d\overline{\lambda}(g) = \int\limits_{\Phi^{-1}(\prod_{k\in\mathbb{N}} S_k)} f_E(\Phi(g))\, d\overline{\lambda}(g) \\ &= \int\limits_{\prod_{k\in\mathbb{N}} S_k} f_E(g)\, d\overline{\lambda}_{\Phi}(g) = \int\limits_{\prod_{k\in\mathbb{N}} S_k} f_E(g)\, d\overline{\lambda}(g) = \mu(E), \end{aligned}$$

where we use the transformation rule for the image $\overline{\lambda}_{\Phi}$ of $\overline{\lambda}$ with respect to the transformation $\Phi : x \mapsto x \circ \overline{h}$.

Note that

$$f_{[-1,1[^{\mathbb{N}}}(g) = 1 \quad \text{for all } g \in \prod_{k\in\mathbb{N}} S_k.$$

This implies that $\mu([-1,1[^{\mathbb{N}}) = 1$. It is clear that

$$[-1,1]^{\mathbb{N}} \setminus [-1,1[^{\mathbb{N}} = \bigcup_{k\in\mathbb{N}} X_k,$$

where $X_k = \{1\}_k \times [-1,1]^{\mathbb{N}\setminus\{k\}}$ and $\mu(X_k) = 0$. ∎

Corollary 2. *In Solovay's model, for* $1 \le p < \infty$, *there exists a translation-invariant measure* ν_p *in* $(\ell_p, \|\cdot\|_p)$ *which gets the value* 1 *on the set* $\prod_{k\in\mathbb{N}}[-1/2^{k+1}, 1/2^{k+1}]$, *where*

$$\ell_p = \Big\{(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{k\in\mathbb{N}} |x_k|^p < \infty\Big\},$$

and the norm $\|\cdot\|_p$ *is defined by*

$$\|(x_k)_{k\in\mathbb{N}}\|_p = \Big(\sum_{k\in\mathbb{N}} |x_k|^p\Big)^{1/p}.$$

Proof. We define

$$\nu_p(X) = \mu(A^{-1}(X)) \quad \text{for } X \in B(\ell_p),$$

where $B(\ell_p)$ denotes the Borel σ-algebra of subsets of ℓ^p and the operator $A : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is given by

$$A((x_k)_{k\in\mathbb{N}}) = \left(\frac{1}{2^{k+1}}\, x_k\right)_{k\in\mathbb{N}} \quad \text{for } (x_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}.$$ ∎

Corollary 3. *Let* ℓ_∞ *be the space of all bounded real-valued sequences* $(x_k)_{k\in\mathbb{N}}$ *with the standard norm* $\|\cdot\|_\infty$ *defined by*

$$\|(x_k)_{k\in\mathbb{N}}\|_\infty = \sup_{k\in\mathbb{N}} |x_k|.$$

Then, in Solovay's model, there exists a translation-invariant Borel measure ν_∞ *such that the closed unit ball has* ν_∞*-measure* 1.

Proof. Put

$$\nu_\infty(X) = \mu(X) \quad \text{for } X \in B(\ell_\infty).$$ ∎

Let us recall some notions due to Brian R. Hurt, Tim Sauer and James A. Yorke (cf. [4]). Let V be a complete metric linear space, by which we mean a vector space with a complete metric for which addition and scalar multiplication are continuous. Let μ be a Borel measure defined on V. A measure μ is said to be *transverse* to a Borel set $S \subset V$ (cf. [4, Definition 1, p. 221]) if:

(i) there exists a compact set $U \subset V$ for which $0 < \mu(U) < \infty$;

(ii) $\mu(S + v) = 0$ for every $v \in V$.

A Borel set $S \subset V$ is called *shy* (cf. [4, Definition 2, p. 222]) if there exists a measure transverse to S. More generally, a subset of V is called *shy* if it is contained in a shy Borel set. The complement of a shy set is called a *prevalent* set. We say that "almost every" element of V satisfies a given property (cf. [4, p. 226]) if the subset of V on which the property holds is prevalent.

Note that if V is infinite-dimensional, then all compact subsets of V are shy sets (cf. [4, Fact 8, p. 215]).

Let P be any sentence formulated for elements in ℓ_p.

THEOREM 1. *In Solovay's model, for* $1 \leq p < \infty$, *the condition*

(j) *ν_p-almost every element of ℓ_p has the property P*

implies that

(jj) *"almost every" element of ℓ_p has the property P.*

Proof. Assume that ν_p-almost every element of ℓ_p has the property P. This means that $\nu_p(S) = 0$, where

$$S = \{x \in \ell_p : P(x) \text{ is false}\}.$$

To show that S is shy, i.e., the measure ν_p is transverse to S, note that $\prod_{k\in\mathbb{N}}[-1/2^{k+1}, 1/2^{k+1}]$ is a compact set in ℓ_p and

(i*) $0 < \nu_p(\prod_{k\in\mathbb{N}}[-1/2^{k+1}, 1/2^{k+1}]) = 1 < \infty$;
(ii*) $\nu_p(S+h) = \nu_p(S)$ for all $h \in \ell_p$. ∎

REMARK 3. An analogue of Theorem 1 is also valid for an arbitrary translation-invariant Borel measure ν on $\mathbb{R}^{\mathbb{N}}$ for which there exists a compact set $F \subset \mathbb{R}^{\mathbb{N}}$ with $0 < \nu(F) < \infty$.

REMARK 4. We do not know whether the assertion of the theorem above remains true for ν_∞ because the problem of the existence of a compact subset with a non-zero finite ν_∞-measure in ℓ_∞ remains open in Solovay's model.

REMARK 5. A converse to Theorem 1 is not valid in Solovay's model. Indeed, denote by $P_p((a_i)_{i\in\mathbb{N}})$ $(1 < p < \infty)$ the following sentence: "$(a_i)_{i\in\mathbb{N}} \in \ell_p$ has the property that $\sum_{i\in\mathbb{N}} a_i$ diverges".

By [4, Proposition 2, p. 226], for $1 < p \leq \infty$ "almost every" sequence $(a_i)_{i\in\mathbb{N}}$ in ℓ_p has the property that $\sum_{i\in\mathbb{N}} a_i$ diverges. Since $\nu_p(\prod_{k\in\mathbb{N}}[-1/2^{k+1}, 1/2^{k+1}]) = 1 > 0$, an analogous result formulated in terms of the measure ν_p is not valid for $1 < p < \infty$.

THEOREM 2. *In Solovay's model μ-almost every sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{N}}$ has the property that $\sum_{i\in\mathbb{N}} a_i$ diverges.*

Proof. We set

$$T = \Big\{(a_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} a_i \text{ converges}\Big\}.$$

Note that $T = \bigcup_{n\in\mathbb{N}} T_n$, where

$$T_n = \Big\{(a_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{i\in\mathbb{N}} a_i \text{ converges } \& \ |a_i| \le 1/2 \text{ for } i \ge n\Big\}.$$

Hence, we must show that $\mu(T_n) = 0$ for $n \in \mathbb{N}$. Clearly, for $n \in \mathbb{N}$, we have

$$T_n = \bigcup_{g\in\{2\mathbb{Z}\}^{n-1}} ((([-1,1[^{n-1}+g) \times [-1/2,1/2]^{\mathbb{N}\setminus\{1,\dots,n-1\}}) \cap T_n).$$

As

$$\mu(([-1,1[^{n-1}+g) \times [-1/2,1/2]^{\mathbb{N}\setminus\{1,\dots,n-1\}}) = 0,$$

we conclude that $\mu(T_n) = 0$. ∎

According to Remark 3, we deduce that

Corollary 4. *In Solovay's model "almost every" sequence* $(a_i)_{i\in\mathbb{N}}$ *in* $\mathbb{R}^{\mathbb{N}}$ *has the property that* $\sum_{i\in\mathbb{N}} a_i$ *diverges.*

Note that this result can be obtained by the scheme presented in [4, Proposition 2, p. 226].

Finally, let $\mathcal{K}$ be the class of all translation-invariant Borel measures ν in ℓ_p $(1 \le p \le \infty)$ such that there exists a compact set $X_\nu \subset \ell_p$ with $0 < \nu(X_\mu) < \infty$. Let $\mathcal{L}_\mu$ be the σ-ideal of subsets of ℓ_p generated by the class of ν-measure zero Borel subsets in ℓ_p. Then

$$\bigcup_{\nu\in\mathcal{K}} \mathcal{L}_\mu \subseteq \text{S.S.}(\ell_p),$$

where S.S.(ℓ_p) denotes the class of all shy sets in ℓ_p. In this context, we state the following

Problem. Is S.S.$(\ell_p) \setminus \bigcup_{\nu\in\mathcal{K}} \mathcal{L}_\mu$ non-empty in Solovay's model?

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Department of Mathematics
Georgian Technical University
Kostava Street 77
01075 Tbilisi 75, Georgia
E-mail: gogi_pantsulaia@hotmail.com

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