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On Semicontinuity in Impulsive Dynamical Systems

by

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Summary. In the important paper on impulsive systems [K1] several notions are introduced and several properties of these systems are shown. In particular, the function ϕ which describes "the time of reaching impulse points" is considered; this function has many important applications. In [K1] the continuity of this function is investigated. However, contrary to the theorem stated there, the function ϕ need not be continuous under the assumptions given in the theorem. Suitable examples are shown in this paper. We characterize the function ϕ from the point of view of its semicontinuity. Also, we show the analogous properties for impulsive systems given by semidynamical systems. In the last section we investigate the continuity properties of the escape time function in impulsive systems.

1. Preliminaries. Let X be a metric space. A pair (X, π) is a *dynamical system* (or a *flow*) if $\pi : \mathbb{R} \times X \to X$ is a continuous function with $\pi(0, x) = x$ and $\pi(t, \pi(s, x)) = \pi(t + s, x)$ for every t, s, x; replacing $\mathbb{R}$ by $\mathbb{R}_+$ we get the definition of a *semidynamical system* (or a *semiflow*). For the elementary properties of dynamical and semidynamical systems, see [BH], [BS], [NS], [P1], [P2], [V]. We define the *positive trajectory* of x as $\pi^+(x) = \pi([0, +\infty) \times \{x\})$.

In a semidynamical system, for $t \geq 0$ and $y \in X$ by $F(t, y)$ we mean $\{z \in X : \pi(t, z) = y\}$. Analogously we define $F(\varDelta, D)$ for $\varDelta \subset [0, +\infty)$ and $D \subset X$. A point $x \in X$ is said to be *a start point* if $F(t, x) = \emptyset$ for $t > 0$.

An *impulsive system* (X, π, M, I) consists of a semidynamical system (X, π) (which may be a dynamical system) together with a nonempty closed subset M of X and a continuous function $I : M \to X$. We assume that for each $x \in M$ there is an $\varepsilon_x > 0$ such that $\pi((-\varepsilon_x, 0), x) \cap M = \emptyset$ and

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$\pi((0,\varepsilon_x),x)\cap M=\emptyset$ (for dynamical systems) or $F((0,\varepsilon_x),x)\cap M=\emptyset$ and $\pi((0,\varepsilon_x),x)\cap M=\emptyset$ (for semidynamical systems). These conditions mean that the points of M are isolated on every trajectory of the system (X,π). We call M the *impulse set* and I the *impulse function*.

By $M^+(x)$ we mean the set $(\pi^+(x)\cap M)\setminus\{x\}$.

1.1. Definition. We define a function $\phi : X\to(0,+\infty]$ by

$$\phi(x)=\begin{cases} s & \text{if } \pi(s,x)\in M \text{ and } \pi(t,x)\notin M \text{ for } t\in(0,s),\\ +\infty & \text{if } M^+(x)=\emptyset\end{cases}$$

(i.e. $\phi(x)$ is the smallest positive time for which the positive trajectory of x meets M).

1.2. Definition. For $x\in M$ we call the point $\pi(\phi(x),x)$ the *impulse point* of x.

1.3. Definition. The trajectory $\tilde\pi^+(x)$ of a point x is defined as follows. We start from x. If $M^+(x)=\emptyset$ then we put $\tilde\pi(s,x)=\pi(s,x)$ for any $s\ge 0$. If $M^+(x)\ne\emptyset$ then we put $\tilde\pi(s,x)=\pi(s,x)$ for $s<\phi(x)$ and $\tilde\pi(\phi(x),x)=I(\pi(\phi(x),x))$. Then we continue the above procedure starting at $\tilde\pi(s,x)$ and so on.

1.4. Notation. For $x\in X$, we denote by x^1 the point $\pi(\phi(x),x)$ and by x^{1+} the point $\tilde\pi(\phi(x),x)=I(\pi(\phi(x),x))=I(x^1)$. By x^2 we denote the point $\pi(\phi(x^{1+}),x^{1+})$ and by x^{2+} the point $\tilde\pi(\phi(x^{1+}),x^{1+})=I(\pi(\phi(x^{1+}),x^{1+}))=I(x^2)$ and so on.

Thus, for any point $x\in X$ exactly one of the following three conditions holds:

(i) $M^+(x)=\emptyset$,
(ii) for some $n\ge 1$: x^{k+} is defined for $k=1,\ldots,n$ and $M^+(x^{n+})=\emptyset$,
(iii) for any $k\ge 1$: x^{k+} is defined and $M^+(x^{k+})\ne\emptyset$.

1.5. Definition. For any $x\in X$ we define the *escape time* $\widetilde\omega(x)$ of x as $\sup\{s:\widetilde\pi(s,x)\text{ is defined}\}$.

Clearly, if x satisfies (i) or (ii) then $\widetilde\omega(x)=+\infty$. If x satisfies (iii) then either $\widetilde\omega(x)=+\infty$ or $\widetilde\omega(x)\in(0,+\infty)$.

2. The continuity of the function ϕ in the case of systems generated by dynamical systems

2.1. Definition. For a dynamical system $(X,\mathbb{R},\pi)$ a set S containing x is called a *section* (a λ-*section*) through x $(\lambda>0)$ if the set $U_S=\pi((-\lambda,\lambda),S)$ is a neighbourhood (not necessarily open) of x and for every $y\in U$ there are a unique $z\in S$ and a unique $t\in(-\lambda,\lambda)$ with $\pi(t,z)=y$ (see for instance [BS], [NS], [C1]). The set U_S is then called a *tube* (given by the section S) *through* x.

For the following conditions, we assume that an impulse system with impulse set M is given.

2.2. Definition. A tube U_S given by a section S through x such that $S \subset M \cap U_S$ will be called a TC-*tube through* x. We will say that a point $x \in M$ satisfies (TC) (*Tube Condition*) if there exists a TC-tube U_S through x.

2.3. Definition. A tube U_S given by a section S through x such that $S = M \cap U_S$ will be called an STC-*tube through* x. We will say that a point $x \in M$ satisfies (STC) (*Strong Tube Condition*) if there exists an STC-tube U_S through x.

2.4. Remark. The condition described in Definition 2.2 was introduced in [K1]. In that paper, a set M such that each $x \in M$ satisfies (TC) is said to be *well placed in* X. It turns out that the stronger condition (STC) applied to the impulsive set M of an impulsive system has important applications, frequently more important than the original condition (TC).

2.5. Example. Consider the system on $\mathbb{R}^2$ given by $\pi(t,(x,y)) = (x+t,y)$ and $M = \{(x,y) : x = 0\} \cup \{(x,y) : x = y, x \geq 0\}$ (see Figure 1). It is easy to see that $(0,0)$ satisfies (TC) but not (STC).

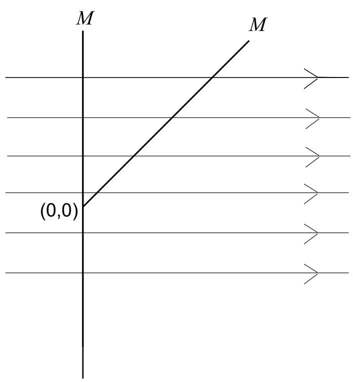

Fig. 1

Theorem 4 in [K1] is devoted to the continuity of the function $x \mapsto \phi(x)$. The theorem says that if in an impulsive system each $x \in M$ satisfies (TC) then ϕ is continuous on X. Upper semicontinuity of this function is proved and it is stated that lower semicontinuity also holds. However, ϕ need not be lower semicontinuous, which can be seen from the following example.

2.6. Example. Let (X,π,M,I) be an impulsive system with $M = \mathbb{N}$, $\pi(t,x) = t + x$, $I(n) = n + 1/2$. Then $\phi(x) = -x$ for $x < 0$ and $\phi(x) = 1 + E(x) - x$, where $E(x)$ is the integer part of x, for $x \geq 0$. The function

ϕ is not lower semicontinuous at any $x \in \mathbb{N}$. Of course, each $x \in M$ satisfies the condition (TC) (as well as (STC)).

However, we have

2.7. THEOREM. *Assume that (X, π, M, I) is an impulsive system given by a dynamical system (X, π). Then for any $x \notin M$ the function ϕ is lower semicontinuous at x.*

Proof. Let $x \notin M$ and $\phi(x) = c \in (0, +\infty)$. The function ϕ is lower semicontinuous at x if and only if for any sequence $x_n \to x$ with $\phi(x_n) \to t$ we have $t \geq c$. Suppose to the contrary that there exists a sequence $p_n \to x$ with $\phi(p_n) \to t < c$. For n large enough we have $p_n \notin M$ as M is closed. Thus $\pi(\phi(p_n), p_n) \in M = \overline{M}$; on the other hand, $\pi(\phi(p_n), p_n) \to \pi(t, x)$, so $\pi(t, x) \in M$, which means that $\phi(x) \leq x < c$ and finishes the proof.

Thus lower semicontinuity may not hold only at points in M.

In fact, we have the following

2.8. PROPOSITION. *Assume that (X, π, M, I) is an impulsive system given by a dynamical system (X, π). Then ϕ is not lower semicontinuous at x for any $x \in M$.*

Proof. Take $\varepsilon > 0$ and $y \in X$ such that $\pi(\varepsilon, y) = x$ and $\pi([0, \varepsilon), y) \cap M = \emptyset$ (this can be done as the points of M are isolated on the trajectory). Let (ε_n) be an increasing sequence with $\varepsilon_n \to 0$. Then $\pi(\varepsilon_n, y) \to x$ and $\phi(\pi(\varepsilon_n, y)) = \varepsilon - \varepsilon_n \to 0$. For a lower semicontinuous function ϕ we would have $\phi(x) \leq 0$, however, $\phi(x) \in (0, +\infty]$.

Even $\phi|_M$ need not be lower semicontinuous. Consider the following

2.9. EXAMPLE. Take a system $(\mathbb{R}, \pi)$ and $A = M$ as in Example 2.5. Let $I(x, y) = (|x|/2, y)$. As was noted above, each $x \in M$ satisfies (TC).

We see that for any $y > 0$ we have $\phi((0, y)) = y$; on the other hand, $\phi((0, 0)) = +\infty$. Thus $\phi|_M$ is not lower semicontinuous.

However, the following theorem holds.

2.10. THEOREM. *If in an impulsive system (X, π, M, I) given by a dynamical system (X, π) each x contained in the impulse set M satisfies* (STC), *then $\phi|_M$ is lower semicontinuous.*

This theorem is a simpler case of Theorem 3.7 which will be proved in the next section.

To summarize the results on semicontinuity on ϕ, using the upper semicontinuity proved in [K1] we have:

2.11. THEOREM. *Assume that (X, π, M, I) is an impulsive system given by a dynamical system (X, π). Then:*

(1) *if each element of M satisfies* (TC), *then ϕ is continuous at x if and only if $x \notin M$,*

(2) *ϕ is not lower semicontinuous at x if and only if $x \in M$,*

(3) *if each element of M satisfies* (STC) *then $\phi|M$ is lower semicontinuous.*

3. The continuity of the function ϕ in the case of systems generated by semidynamical systems. The concept of section is of fundamental importance in the theory of dynamical systems. For the semidynamical systems, the definition of sections which generalizes the definition from the case of dynamical systems was presented in [C1]. Also, an existence theorem was proved. Such sections give an opportunity of local presentation of semidynamical systems in a parallelizable way.

Recall

3.1. Definition. A closed set S containing x is called a *section* (a λ-*section*) through x if there exists a closed set L such that:

(a) $F(\lambda, L) = S$,
(b) $F([0, 2\lambda], L)$ is a neighbourhood of x,
(c) $F(\mu, L) \cap F(\nu, L) = \emptyset$ for $0 \le \mu < \nu \le 2\lambda$.

We will call the set $F([0, 2\lambda], L)$ a *tube* (a λ-*tube*), and the set L a *bar*.

3.2. Definition. We define the conditions (TC) and (STC) for semidynamical systems in an analogous way as for dynamical systems.

We need

3.3. Lemma. *Assume that in a semidynamical system a point x satisfies the condition* (TC) (*or* (STC)) *and a* TC-*tube* (*resp. an* STC-*tube*) U_λ *is given by a λ-section S. Then for any $\eta < \lambda$ the set S also gives an η-section with a* TC-*tube* (*resp. an* STC-*tube*).

The proof is an immediate consequence of the definitions and Lemma 1.9 in [C1].

The following results, similar to that for dynamical systems, hold.

3.4. Theorem. *If in an impulsive system (X, π, M, I) given by a semidynamical system (X, π) each point contained in the impulse set M satisfies* (TC), *then ϕ is upper semicontinuous.*

Proof. It is enough to prove upper semicontinuity at a point x where $\phi(x) = u \in (0, +\infty)$. We have $\pi(u, x) = y \in M$, $\pi((0, u), x) \cap M = \emptyset$. Take an $\varepsilon < u$ such that $U = F([0, 2\varepsilon], L)$ is a TC-tube with an ε-section through y equal to $F(\varepsilon, L)$ (this can be done with the use of Lemma 3.3). There is a neighbourhood V of x such that $\pi(u, V) \subset U$. Thus $\pi(u, z) \in F([0, 2\varepsilon], L)$ for any $z \in V$. Therefore for any $z \in V$ there exists an $\eta_z \in [0, 2\varepsilon]$ such that $\pi(u+\eta_z, z) \in L$. Hence $\pi(u+\eta_z-\varepsilon, z) \in F(\varepsilon, L) = S \subset M$, as $u+\eta_z-\varepsilon > 0$. We have shown that $\phi(z) \le u+\eta_z-\varepsilon < u+\varepsilon$. This proves the upper semicontinuity of ϕ at x.

3.5. Theorem. *Assume that (X,π,M,I) is an impulsive system given by a semidynamical system (X,π). For any $x \notin M$ the function ϕ is lower semicontinuous at x.*

The proof is analogous to that of Theorem 2.7.

3.6. Proposition. *Assume that (X,π,M,I) is an impulsive system given by a dynamical system (X,π). Assume that $x \in M$ and x is not a start point. Then ϕ is not lower semicontinuous at x.*

The proof is analogous to the proof of Proposition 2.8. The only difference is that in the case of semidynamical systems the set $F(t,x)$ may be empty for $t > 0$. However, x is not a start point, so there are $\varepsilon > 0$ and $y \in X$ such that $\pi(\varepsilon, y) = x$ and $\pi([0,\varepsilon), y) \cap M = \emptyset$.

3.7. Theorem. *If in an impulsive system (X,π,M,I) given by a semidynamical system (X,π) each element of the impulse set M satisfies* (STC), *then $\phi|_M$ is lower semicontinuous.*

Proof. Take an η-section S through $x \in M$ with STC-tube U such that $S = M \cap U$. Let $\varepsilon > 0$ and $s < \phi(x)$. According to Lemma 3.3 we may assume that $\eta < \varepsilon$ and $\eta < s$.

The set $\pi([\eta/2, s], x)$ is disjoint from the closed set M, so we can find an open set V such that $\pi([\eta/2, s], x) \subset V$ and $V \cap M = \emptyset$. Let $p_n \to x$. Then for sufficiently large n we have $\pi([\eta/2, s], p_n) \subset V$, as $\pi([\eta/2, s], x)$ is compact. According to (STC), $\pi([0,\eta], p_n) \cap M = \emptyset$ for n large enough, so $\pi([0,s], p_n) \cap M = \emptyset$ and $\phi(p_n) \ge s$. We have shown that $\liminf_{y\to x} \phi(y) \ge \phi(x)$, so ϕ is lower semicontinuous at x.

This proof also applies to Theorem 2.10, as in the case of a dynamical system a section according to Definition 3.1 is a section according to Definition 2.1 (see [C1]).

Summarizing, we have

3.8. Theorem. *Assume that (X,π,M,I) is an impulsive system given by a semidynamical system (X,π) and no start point of the system belongs to M. Then*:

(1) *if each element of M satisfies* (TC) *then ϕ is continuous at x if and only if $x \notin M$,*

(2) *ϕ is not lower semicontinuous at x if and only if $x \in M$,*

(3) *if each element of M satisfies* (STC) *then $\phi|_M$ is lower semicontinuous.*

3.9. Remark. The above assumptions are, in particular, fulfilled for any semidynamical system on a manifold, because it is known (see [BH]) that such systems do not have start points.

3.10. Remark. The results presented in Sections 1 and 2 for the impulse set M and function ϕ remain true for the set M in dynamical or semidynamical systems without impulsive structure. The definition of ϕ and the conditions (TC) and (STC) may be formulated in an analogous way for the set M which has nothing to do with the function I and impulses. The reason for our choice of presentation is that our results relate to those of [K1]. Moreover, the function ϕ is used for the investigation of limit sets and systems of characteristic 0 in impulsive systems.

4. The continuity of the function $\widetilde{\omega}$ and Kamke's Axiom. In a local semidynamical system, $\pi(t,x)$ is defined for $t \in [0,\omega(x))$ where $\omega(x) \in (0,+\infty]$. In the definition of local systems the openness of the domain is required, i.e. the set $\operatorname{dom}\widetilde{\pi} = \bigcup\{[0,\omega(x)) \times \{x\} : x \in X\}$ must be open in $\mathbb{R}_+ \times X$. This is called Kamke's Axiom (see [BH]). In impulsive systems, the domain of the positive impulsive trajectory $\widetilde{\pi}(x)$ is also an interval which need not be equal to $[0,+\infty)$. It is natural to ask about the openness of the domain here, especially because in some cases it is useful to transform an impulsive system to a classical system (this will be of importance in particular in [C2]).

4.1. Lemma. *The set* $\operatorname{dom}\widetilde{\pi}$ *is open in* $\mathbb{R}_+ \times X$ *if and only if the function* $\widetilde{\omega} : X \ni x \mapsto \widetilde{\omega}(x) \in (0,+\infty]$ *is lower semicontinuous.*

The proof is analogous to the proof of the same property for semidynamical systems ([P1, Lemma 1.3.1, Remark 1.3.4]; for the necessity, see also [BH, Lemma 1.8]).

However, the domain of $\widetilde{\pi}$ need not be open. Consider the following examples:

4.2. Example. Let $X = \mathbb{R}$, $M = \mathbb{N}_1$, $\pi(t,x) = t + x$ and $I(n) = n + 1 - 1/n(n+1)$. For any $k \in \mathbb{N}_1$ we have

$$\widetilde{\omega}(p) = k - p + \frac{1}{k} \quad \text{if } p \in (k-1,k], \quad \widetilde{\omega}(k) = 1 + \frac{1}{k+1}.$$

We see that $\widetilde{\omega}$ is not lower semicontinuous at any $k \in \mathbb{N}_1$.

4.3. Example. Let $X = \mathbb{R}^2$ and $\pi(t,x) = t + x$. Set $M_1 = \{(n,y) : y \le 0\}$ and $M_n = \{(n,y) : y \in \mathbb{R}\}$ for $n \in \mathbb{N}_2$; $M = \bigcup\{M_n : n \ge 1\}$. We define $I(1,y) = (0,y)$ and

$$I(n,y) = \left(n + 1 - \frac{1}{n(n+1)}, y\right) \quad \text{for } n \ge 2$$

(see Figure 2).

Note that the point $(1,0)$ does not satisfy the condition (TC). As in the previous example, $\widetilde{\omega}$ is not lower semicontinuous at any $(x,y)\in M$.

Let $x<2$, $y>0$. We have $\widetilde{\omega}(x,y)=2-x+1/2$. However, for any $x<1$ and $y\le 0$ we have $\widetilde{\omega}(x,y)=+\infty$. Thus $\widetilde{\omega}$ is also not lower semicontinuous at (x,y) for any $(x,y)\in L=(-\infty,1]\times\{0\}$.

It can be easily checked that for any $(x,y)\in\mathbb{R}^2\setminus(M\cup L)$ the function $\widetilde{\omega}$ is lower semicontinuous at (x,y).

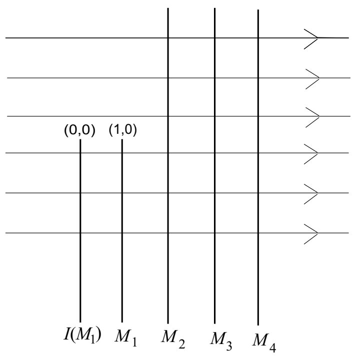

Fig. 2

The examples suggest that the lower semicontinuity of $\widetilde{\omega}$ may be destroyed by elements of M and the points which meet on their way the points not fulfilling (TC). Indeed, we have the following

4.4. Theorem. *Assume that (X,π,M,I) is an impulsive system given by a semidynamical system (X,π). Assume also that $M\cap I(M)=\emptyset$. Let $x\in X\setminus M$ be such that any $y\in M\cap\widetilde{\pi}^+(x)$ fulfills* (TC). *Then $\widetilde{\omega}$ is lower semicontinuous at x.*

Proof. As in the proof of 2.7, suppose to the contrary that there exists a sequence $y_n\to x$ such that $\widetilde{\omega}(y_n)\to\alpha_0<\widetilde{\omega}(x)$. Consider $\widetilde{\pi}^+(x)$. We will use the notation introduced in 1.4. There exists an $\alpha\in(\alpha_0,\widetilde{\omega}(x))$ such that $\widetilde{\pi}(\alpha,x)\ne x^{i+}$ for any $i\ge 1$. According to 1.3, we can find $k\in\mathbb{N}$ and $\beta>0$ such that $\widetilde{\pi}(\alpha,x)=\pi(\beta,x^{k+})$, i.e. there are only a finite number of impulse points of the trajectory $\widetilde{\pi}(x)$ before reaching $\widetilde{\pi}(\alpha,x)$, with x^k the last one. We have $\pi([0,\beta],x^{k+})\cap M=\emptyset$ and $\alpha=\phi(x)+\phi(x^{1+})+\ldots+\phi(x^{(k-1)+})+\beta$.

The point x^1 satisfies (TC) so we can find a γ_1-section M_1 with a TC-tube U_{γ_1} and bar L_{γ_1}; we have $U_1=F([0,2\gamma_1],L_{\gamma_1}),F(\gamma_1,L_{\gamma_1})=M_1$ and $M_1\subset U_\gamma\cap M$. According to Lemma 3.3, $U_\gamma=F([0,2\gamma],L_\gamma)$ (where $L_\gamma=F(\gamma_1-\gamma,L_\gamma)$) is a TC-tube with section M_1 for any $\gamma<\gamma_1$. For any $\gamma<\gamma_1$ the tube U_γ is a neighbourhood of x^1.

We prove that $\phi(y_n)\to\phi(x)$. For any $\gamma\in(0,\phi(x))$ there exists an N_γ such that $\pi(\phi(x),y_n)\in U_\gamma$ for $n\ge N_\gamma$, as $\pi(\phi(x),y_n)\to x^1$. Thus according to the definition of section either $\pi([0,\phi(x)+\gamma),y_n)\cap M_1\neq\emptyset$ or $\pi((\phi(x)-\gamma,0],y_n)\cap M_1\neq\emptyset$. This shows that $\phi(y_n)\le\phi(x)+\gamma$ for $n\ge N_\gamma$. The above reasoning is applicable to any $\gamma>0$, so we conclude that $\limsup_{n\to\infty}\phi(y_n)\le\phi(x)$. On the other hand, according to Theorem 3.5, $\liminf_{n\to\infty}\phi(y_n)\ge\phi(x)$ as $x\notin M$. We conclude that $\phi(y_n)\to\phi(x)$.

It follows that for n large enough there exists a δ_n^1 such that $\phi(y_n)=\phi(x)+\delta_n^1$ and $\delta_n^1\to 0$ as $n\to\infty$. We have $y_n^1=\pi(\phi(y_n),y_n)\in M$ and $\pi(\phi(y_n),y_n)\to\pi(\phi(x),x)=x^1$.

Now we have $y_n^{1+}=I(y_n^1)\to I(x^1)=x^{1+}$ and $y_n^{1+}=\widetilde{\pi}(\phi(x)+\delta_n^1,y_n)$. We apply the above reasoning to the sequence (y_n^{1+}) instead of (y_n) and x^{1+} instead of x (we may do it according to the assumption that $I(M)\cap M=\emptyset$, so $x^{1+}\notin M$). Repeating this procedure finitely many times, we get a sequence (y_n^{k+}) such that $y_n^{k+}\to x^{k+}$ and

$$y_n^{k+}=\widetilde{\pi}(\phi(x)+\delta_n^1+\phi(x^{1+})+\delta_n^2+\ldots+\phi(x^{(k-1)+})+\delta_n^k,y_n)$$

for sufficiently large n. There exists an open set V such that $\pi([0,\beta],x^{k+})\subset V$ and $V\cap M=\emptyset$ as $\pi([0,\beta],x^{k+})\cap M=\emptyset$ (β was taken at the beginning of the proof). For n large enough we have $y_n^{k+}\in V$ and $\pi([0,\beta],y_n^{k+})\subset V$, so $\pi([0,\beta],y_n^{k+})\cap M=\emptyset$ and

$$\pi(\beta,y_n^{k+})=\widetilde{\pi}(\phi(x)+\delta_n^1+\phi(x^{1+})+\delta_n^2+\ldots+\phi(x^{(k-1)+})+\delta_n^k+\beta,y_n)$$

for those n. Put

$$u_n=\phi(x)+\delta_n^1+\phi(x^{1+})+\delta_n^2+\ldots+\phi(x^{(k-1)+})+\delta_n^k+\beta.$$

Letting $n\to\infty$ yields $u_n\to\phi(x)+\phi(x^{1+})+\ldots+\phi(x^{(k-1)+})+\beta$, as $\delta_n^j\to 0$ for all j. On the other hand,

$$\phi(x)+\phi(x^{1+})+\ldots+\phi(x^{(k-1)+})+\beta=\alpha,$$

so $u_n\to\alpha$. There exists an $\eta>0$ such that $u_n>\alpha_0+\eta$ for large n, because $\alpha_0<\alpha$. We have $\pi(\beta,y_n^{k+})\to\pi(\beta,x^{k+})$ and $\pi(\beta,y_n^{k+})=\widetilde{\pi}(u_n,y_n)$ so $u_n<\widetilde{\omega}(y_n)$ and, consequently, $\alpha_0<\alpha_0+\eta\le\widetilde{\omega}(y_n)$. This is a contradiction, as we assumed that $\widetilde{\omega}(y_n)\to\alpha_0$.

Under the assumption that $M\cap I(M)=\emptyset$ we may consider the impulsive system on the space $\widehat{X}=X\setminus M$ with the metric induced from X and the movement $\widehat{\pi}$ given by $\widetilde{\pi}$ defined as above (we use points of M to define the movement on $X\setminus M$ but we do not consider them as points of the space under investigation).

Then as an immediate corollary from the previous theorem we have

4.5. Theorem. *Assume that (X,π,M,I) is an impulsive system given by a semidynamical system (X,π) and $M\cap I(M)=\emptyset$. If each $x\in M$ fulfills* (TC) *then the domain of $\widehat{\pi}$ is open in $\mathbb{R}_+\times\widehat{X}$.*

References

[BH] N. P. Bhatia and O. Hajek, *Local Semi-Dynamical Systems*, Lecture Notes in Math. 90, Springer, 1970.

[BS] N. P. Bhatia and G. P. Szegő, *Stability Theory of Dynamical Systems*, Springer, 2002.

[C1] K. Ciesielski, *Sections in semidynamical systems*, Bull. Polish Acad. Sci. Math. 40 (1992), 297–307.

[C2] —, *On isomorphisms of impulsive dynamical systems*, to be published.

[K1] S. Kaul, *On impulsive semidynamical systems*, J. Math. Anal. Appl. 150 (1990), 120–128.

[K2] —, *On impulsive semidynamical systems II*, *Recursive properties*, Nonlinear Anal. 16 (1991), 635–645.

[LBS] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Sci., 1989.

[NS] V. V. Nemytskiĭ and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton Univ. Press, 1960.

[P1] A. Pelczar, *General Dynamical Systems*, Jagiellonian Univ. Lecture Notes 293, 1978 (in Polish).

[P2] —, *Introduction to the Theory of Differential Equations*, *Part II* (in Polish), Bibl. Mat. 67, PWN–Polish Sci. Publ., 1989.

[V] J. de Vries, *Elements of Topological Dynamics*, Math. Appl. 257, Kluwer, 1993.

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