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A Note on an Application of the Lasota–York Fixed Point Theorem in the Turbulent Transport Problem

by

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Summary. We study a model of motion of a passive tracer particle in a turbulent flow that is strongly mixing in time variable. In [8] we have shown that there exists a probability measure equivalent to the underlying physical probability under which the quasi-Lagrangian velocity process, i.e. the velocity of the flow observed from the vintage point of the moving particle, is stationary and ergodic. As a consequence, we proved the existence of the mean of the quasi-Lagrangian velocity, the so-called Stokes drift of the flow. The main step in the proof was an application of the Lasota–York theorem on the existence of an invariant density for Markov operators that satisfy a lower bound condition. However, we also needed some technical condition on the statistics of the velocity field that allowed us to use the factoring property of filtrations of σ-algebras proven by Skorokhod. The main purpose of the present note is to remove that assumption (see Theorem 2.1). In addition, we prove the existence of an invariant density for the semigroup of transition probabilities associated with the abstract environment process corresponding to the passive tracer dynamics (Theorem 2.7). In Remark 2.8 we compare the situation considered here with the case of steady (time independent) flow where the invariant measure need not be absolutely continuous (see [9]).

1. Introduction. A simple model of transport of a *passive tracer* in a complicated medium is provided by the Itô stochastic equation

$$(1.1)\qquad \begin{cases} d\mathbf{x}(t) = \mathbf{u}(t,\mathbf{x}(t);\omega)\,dt + \sqrt{2\kappa}\,d\mathbf{w}(t), \\ \mathbf{x}(0) = \mathbf{0}, \end{cases}$$

where $\mathbf{u} : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is a d-dimensional random field, independent

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of the d-dimensional standard Brownian motion $\mathbf{w}(\cdot)$. The random field $\mathbf{u}(\cdot)$ describes the velocity field of a turbulent flow, and the parameter $\kappa > 0$ determines the strength of the molecular diffusivity of the medium. It is also usually assumed (see e.g. [7]) that in the case of fully developed turbulence the velocity field $\mathbf{u}(\cdot)$ is time-space stationary. There exists an extensive physics literature dedicated to the passive tracer model, and the reader is referred to e.g. [4] or [12] to gain more insight into the physical aspect of the subject.

A fundamental question pertaining to the passive tracer motion is to determine its long time, large scale properties from the statistics of the drift. For example, one may ask whether the trajectory process $\mathbf{x}(\cdot)$ obeys the law of large numbers, i.e. whether the limit

$$\mathbf{v}_* := \lim_{t\to+\infty} \frac{\mathbf{x}(t)}{t} \tag{1.2}$$

exists almost surely. This limit is sometimes called the *Stokes drift of the flow.*

It has been well known since the work of Lumley (see [11] and also [14]) that the assumption of stationarity and incompressibility of the drift, i.e. $\nabla_{\mathbf{x}} \cdot \mathbf{u}(t,\mathbf{x}) \equiv 0$, implies that the quasi-Lagrangian velocity process $\eta(t) := \mathbf{u}(t,\mathbf{x}(t))$, $t \ge 0$, is also stationary. Assuming in addition that $\kappa > 0$ it can be concluded that the process $\eta(\cdot)$ is also ergodic (see e.g. [5, Proposition 1, p. 758]). In our previous paper [8, Theorem 2.1, p. 639], we have proved that whenever the field $\mathbf{u}(\cdot,\cdot)$ is sufficiently regular, has finite dependency range in temporal variable (see condition (FDT) below) and satisfies a certain regularity condition (see (ACFDD) below), then there exists an absolutely continuous change of probability measure such that the quasi-Lagrangian process is both stationary and ergodic under the new measure. The key step in the proof was an application of the Lasota–York theorem on the existence of invariant densities for Markov operators that satisfy a certain lower bound condition (see [10, Theorem 5.6.2]).

The condition (ACFDD) used in our previous paper is of purely technical nature and we needed it only to apply a certain result on σ-algebra factorization due to Skorokhod [16]. In the main result of the present paper (Theorem 2.1), we are able to prove the conclusions of [8] without assuming (ACFDD).

Additionally, in Theorem 2.7 we prove the existence of an invariant density for the semigroup of transition probabilities corresponding to an abstract environment process.

2. Notation and formulation of the main results. We let $\mathbf{w}(\cdot)$ be a standard d-dimensional Brownian motion over a probability space $\mathcal{T}_1 :=$

$(\Sigma, \mathcal{A}, W)$. Suppose also that (Ω, d) is a Polish metric space, and $\mathcal{B}$ its Borel σ-algebra. Let $\mathcal{T}_0 := (\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and $\mathbf{u} : \mathbb{R}\times\mathbb{R}^d\times\Omega \to \mathbb{R}^d$ a d-dimensional random vector field, i.e. $\mathbf{u}(t, \mathbf{x}; \omega)$ is a random vector for each $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. In most of the following notation we omit the argument ω of random elements; $\mathbb{E}$, $\mathbf{M}$ denote the expectation operators corresponding to $\mathbb{P}$, W respectively. We write $L^p := L^p(\mathcal{T}_0)$, and let $B(\Omega)$ stand for the space of bounded, Borel measurable functions on Ω.

2.1. *Assumptions on the Eulerian field and the quasi-Lagrangian process.* We assume that the random field satisfies the following hypotheses.

(S) The field is *time-space stationary*, i.e. for any $N \geq 1$, $((t_1, \mathbf{x}_1), \ldots, (t_N, \mathbf{x}_N)) \in \mathbb{R} \times \mathbb{R}^d$ and $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ the laws of $(\mathbf{u}(t_1, \mathbf{x}_1), \ldots, \mathbf{u}(t_N, \mathbf{x}_N))$ and of $(\mathbf{u}(t_1+t, \mathbf{x}_1+\mathbf{x}), \ldots, \mathbf{u}(t_N+t, \mathbf{x}_N+\mathbf{x}))$ coincide.

(RH) $\mathbf{u}(\cdot,\cdot)$ and $\nabla_{\mathbf{x}}\mathbf{u}(\cdot,\cdot)$ are jointly locally Hölder $\mathbb{P}$-a.s. Moreover, $\mathbf{u}(\cdot,\cdot)$ is *deterministically bounded*, i.e.

$$V_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^d} |\mathbf{u}(t, \mathbf{x})| < \infty. \tag{2.1}$$

(C) The field is *centered*, i.e.

$$\mathbb{E}\mathbf{u}(0, \mathbf{0}) = \mathbf{0}. \tag{2.2}$$

(FDT) (*Finite dependence range in the temporal variable*) If $\mathcal{U}_a^b$, $-\infty \leq a \leq b \leq \infty$, denotes the σ-algebra generated by $\mathbf{u}(t, \mathbf{x})$, $a \leq t \leq b$, $\mathbf{x} \in \mathbb{R}^d$ then there exists $T > 0$ for which the σ-algebras $\mathcal{U}_{-\infty}^t$ and $\mathcal{U}_{t+T}^\infty$ are independent for any $t \in \mathbb{R}$.

We set $\mathcal{T}_2 := (\Omega, \mathcal{U}_{-\infty}^0, \mathbb{P})$.

The solution $\mathbf{x}(\cdot)$ of equation (1.1) defines a stochastic process over the probability space $\mathcal{T}_0\otimes\mathcal{T}_1 := (\Omega\times\Sigma, \mathcal{B}\otimes\mathcal{A}, \mathbb{P}\otimes W)$. Our first principal result is

THEOREM 2.1. *Suppose that* $\mathbf{u}(\cdot,\cdot)$ *satisfies the hypotheses* (S), (RH), (C), (FDT) *and* $\kappa > 0$. *Then there exists a probability measure* $\mathbb{P}_*$ *on* $(\Omega, \mathcal{B})$, *equivalent to* $\mathbb{P}$, *such that the quasi-Lagrangian process* $\eta(t) := \mathbf{u}(t, \mathbf{x}(t))$, $t \geq 0$, *is stationary and ergodic over the probability space* $(\Omega \times \Sigma, \mathcal{B} \otimes \mathcal{A}, \mathbb{P}_* \otimes W)$.

REMARK 2.2. Ergodicity here is understood as the absence of non-trivial (w.r.t. the law of $\eta(\cdot)$) shift invariant sets, i.e.

(E) for any Borel subset A of $C([0,\infty); \mathbb{R}^d)$ satisfying

$$\mathbb{P}_* \otimes W[[\theta_h(\eta(\cdot)) \in A] \triangle [\eta(\cdot) \in A]] = 0 \tag{2.3}$$

for all $h \geq 0$ we have $\mathbb{P}_* \otimes W[\eta(\cdot) \in A] = 0$ or 1.

Here $\triangle$ denotes the symmetric difference of events and θ_h is the canonical shift on $C([0,\infty); \mathbb{R}^d)$, i.e. $\theta_h(\mathbf{x}(\cdot)) := \mathbf{x}(h + \cdot)$.

REMARK 2.3. An obvious consequence of the above theorem and the individual ergodic theorem is the existence of the Stokes drift defined by (1.2) (cf. also [8, Corollary 2.2]).

2.2. *The abstract environment process.* In this section we introduce an abstract process, closely related to the passive tracer problem. It is Markovian in the sense of Proposition 2.6 below. Our second principal result, Theorem 2.7, asserts the existence of an invariant density for this process.

Suppose that on Ω we are given an additive group of transformations $T_{t,\mathbf{x}} : \Omega \to \Omega$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. We assume that it satisfies the following.

(MP) The group preserves the measure $\mathbb{P}$, i.e. $\mathbb{P}(T_{t,\mathbf{x}}(A)) = \mathbb{P}(A)$ for all $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$, $A \in \mathcal{B}$.

(M) (*Measurability*) The mapping $(t, \mathbf{x}, \omega) \mapsto \mathbf{1}_A(T_{t,\mathbf{x}}\omega)$, $(t, \mathbf{x}, \omega) \in \mathbb{R} \times \mathbb{R}^d \times \Omega$, is jointly measurable for any $A \in \mathcal{B}$.

(SC) (*Stochastic continuity*) We have

$$\lim_{|t|+|\mathbf{x}|\to 0} \mathbb{P}[|\mathbf{1}_A(T_{t,\mathbf{x}}\omega) - \mathbf{1}_A(\omega)| \geq \eta] = 0, \quad \forall \eta > 0,\ A \in \mathcal{B}.$$

Suppose that $\widetilde{\mathbf{u}} : \Omega \to \mathbb{R}^d$ is a centered random vector. By (MP) the random field

$$\mathbf{u}(t, \mathbf{x}; \omega) = \widetilde{\mathbf{u}}(T_{t,\mathbf{x}}\omega), \quad \forall (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d, \tag{2.4}$$

is time-space stationary and centered. We assume further that it satisfies conditions (RH) and (FDT).

DEFINITION 2.4. We say that the group $T_{\cdot,\cdot}$ is *temporally ergodic* if

(ET) any $A \in \mathcal{B}$ satisfying $\mathbb{P}[T_{t,\mathbf{0}}(A) \triangle A] = 0$ for some $t > 0$ is trivial, i.e. $\mathbb{P}[A] = 0$ or 1.

REMARK 2.5. Typically, as in the previous section, we are given a time-space homogeneous random field $\mathbf{u}$ as the starting point. Then Ω is identified with a suitable space containing all possible realizations of $\mathbf{u}$, for example $\Omega := C(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d)$. The probability $\mathbb{P}$ can be taken to be the law of $\mathbf{u}$ on Ω. It is invariant under the measurable group $T_{t,\mathbf{x}}\omega(\cdot, \cdot) := \omega(\cdot + t, \cdot + \mathbf{x})$, $\omega \in \Omega$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. We let $\widetilde{\mathbf{u}}(\omega) := \omega(0, \mathbf{0})$. One can easily show that $T_{\cdot,\cdot}$ fulfills all the assumptions specified above, provided that $\mathbf{u}$ satisfies the assumptions of Section 2.1.

For each fixed $\omega \in \Omega$ we define an Ω-valued *environment process* $\xi(t) := T_{t,\mathbf{x}(t)}\omega$, where $\mathbf{x}(\cdot)$ is a solution to (1.1), with $\mathbf{u}$ given by (2.4). It is an object of interest in the homogenization theory of random media (see e.g. [13] and the references therein for more insight into the applications of this process).

Let $(\mathcal{Z}_t)_{t\geq 0}$ denote the natural filtration for $\xi(\cdot)$. Let $p^{\omega}(s,\mathbf{y};t,\mathbf{x})$, $s < t$, $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$, be the transition probability densities corresponding to the diffusion (1.1). For any $F\in B(\Omega)$ and $t\geq 0$ we define

$$P^tF(\omega) := \int_{\mathbb{R}^d} p^{\omega}(0,\mathbf{0};t,\mathbf{x})F(T_{t,\mathbf{x}}\omega)\,d\mathbf{x}. \tag{2.5}$$

An important property of the environment process is its Markovianity understood as follows.

Proposition 2.6 ([5, Proposition 1, p. 758]). *The process $\xi(\cdot)$ is Markovian, with (P^t) being its semigroup of transition probabilities, i.e.*

$$\mathbb{E}[F(\xi(t+s))\,|\,\mathcal{Z}_t] = P^sF(\xi(t)), \quad \forall t,s\geq 0,$$

where $F\in B(\Omega)$. Moreover, if the field is incompressible, i.e. $\nabla_{\mathbf{x}}\cdot\mathbf{u}(t,\mathbf{x})\equiv 0$, we have

$$\int P^tF\,d\mathbb{P} = \int F\,d\mathbb{P} \quad \text{for all } t\geq 0,\ F\in B(\Omega),$$

i.e. $\mathbb{P}$ is an invariant measure for $(P^t)_{t\geq 0}$.

To prove the existence of an invariant measure for an environment process that is absolutely continuous w.r.t. $\mathbb{P}$, the underlying probability of the Eulerian flow, is a challenging problem. This is due to the fact that in general $\xi(\cdot)$ takes values in an infinite-dimensional space Ω. Our result concerning the environment process can be stated as follows.

Theorem 2.7. *Suppose that the group $T_{\cdot,\cdot}$ satisfies assumptions* (MP), (M), (SC) *and the random field $\mathbf{u}$ given by* (2.4) *satisfies conditions* (RH) *and* (FDT). *Then there exists an invariant positive density for the semigroup $(P^t)_{t\geq 0}$, i.e. an element $h_*\in L^1$ such that*

$$h_* > 0, \quad \int h_*\,d\mathbb{P} = 1, \tag{2.6}$$

$$\int h_*P^tF\,d\mathbb{P} = \int h_*F\,d\mathbb{P} \quad \text{for all } t\geq 0,\ F \text{ bounded } \mathcal{U}_0^{\infty}\text{-measurable.} \tag{2.7}$$

In addition, if the group $T_{\cdot,\cdot}$ satisfies the temporal ergodicity condition (ET), *then the semigroup (P^t) is ergodic, i.e. any F satisfying $P^tF = F$ $\mathbb{P}$-a.s. for some $t>0$ must be constant $\mathbb{P}$-a.s.*

Remark 2.8. At the end of this section we compare the above results with those of [9]. There we considered motions described by equation (1.1) with a time independent drift of the form $\mathbf{u}(\mathbf{x};\omega) := \mathbf{v} + \widetilde{\mathbf{u}}(\mathbf{x};\omega)$, where $\mathbf{v}\in\mathbb{R}^d$ is a constant vector and $\widetilde{\mathbf{u}}$ is a zero mean, stationary random field over $\mathcal{T}_0$ whose L^{∞}-norm is bounded by $|\mathbf{v}|$. Assuming some regularity of the field we have shown (see [8, Theorem 2.2]) that in this case there exists an invariant measure μ for the Lagrangian velocity $\mathbf{u}(\mathbf{x}(t))$, $t\geq 0$. This measure is absolutely continuous w.r.t. $\mathbb{P}\otimes W$. We can also consider the corresponding abstract environment process $\xi(t) = T_{\mathbf{x}(t)}\omega$, $t \ge 0$, defined on a suitable function space Ω. One can show that in this case there also exists an invariant measure on $\mathbb{P}_*$ defined over $(\Omega, \mathcal{B}(\Omega))$. It can be easily concluded from the argument in [9] that the restriction of $\mathbb{P}_*$ to any sub-σ-algebra $\mathcal{V}_R := \sigma\{\mathbf{u}(\mathbf{x}) : \mathbf{x}\cdot\mathbf{v} \le -R\}$ for $R > 0$ is absolutely continuous w.r.t. the corresponding restriction of $\mathbb{P}$. An example given in [3] for i.i.d. random walks in random environments suggests, however, that $\mathbb{P}_*$ need not be absolutely continuous w.r.t. $\mathbb{P}$.

3. Construction of the invariant measure

3.1. *Fields with absolutely continuous finite-dimensional distributions.* Suppose that the field $\mathbf{u}$ has absolutely continuous finite-dimensional distributions (condition (ACFDD)), i.e. all random vectors $(\mathbf{u}(t_1, \mathbf{x}_1), \ldots, \mathbf{u}(t_N, \mathbf{x}_N))$, where $N \ge 1$, $(t_i, \mathbf{x}_i) \ne (t_j, \mathbf{x}_j)$, $i \ne j \in \{1, \ldots, N\}$, are absolutely continuous with respect to the $N \cdot d$-dimensional Lebesgue measure. In Section 3.1 of [8] we defined for such fields a density preserving operator $Q : L^1(\mathcal{T}_0) \to L^1(\mathcal{T}_0)$, called a *transport operator*, that satisfies

$$\int \mathbf{M}G(\xi(T))F\,d\mathbb{P} = \int G\,QF\,d\mathbb{P}$$

for any bounded F and G that are $\mathcal{U}_{-\infty}^0$ and $\mathcal{U}_0^\infty$-measurable respectively (see [8, Proposition 3.1, p. 640]). Here T is the temporal dependence range introduced in condition (FDT). To show the existence of an invariant density for this operator we used the classical Nash–Aronson inequalities for fundamental solutions of parabolic equations (see [1, Theorem 1, p. 891]). They state that for any $T > 0$ there exists a positive deterministic constant c_1, depending only on T, V_∞ and the dimension d, such that for all $-T < s < t < 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we have

$$\text{(3.1)} \qquad \frac{1}{c_1(t-s)^{d/2}} \exp\left\{-\frac{c_1|\mathbf{x}-\mathbf{y}|^2}{t-s}\right\} \ge p^\omega(s, \mathbf{x}; t, \mathbf{y}) \ge \frac{c_1}{(t-s)^{d/2}} \exp\left\{-\frac{|\mathbf{x}-\mathbf{y}|^2}{c_1(t-s)}\right\}.$$

Here $p^\omega(s, \mathbf{x}; \cdot, \cdot)$ is the transition probability density of the diffusion satisfying the stochastic differential equation (1.1) with the initial condition $\mathbf{x}(s) = \mathbf{x}$. With the help of estimates (3.1) we concluded ([8, (3.1), p. 640]) that there exists a constant $c_2 > 0$, depending only on T, V_∞, such that

$$\text{(3.2)} \qquad QF \ge c_2 \int F\,d\mathbb{P}, \qquad \forall F \ge 0, \mathcal{U}_{-\infty}^0\text{-measurable}.$$

As a consequence of the above estimate and [10, Theorem 5.6.2] we concluded (see [8, Lemma 3.2, p. 642]) the existence of an invariant density H_*

for Q. Thanks to (3.2) this density satisfies

$$H_* \geq c_2. \tag{3.3}$$

In addition it can be deduced from the formula for the operator Q (see [8, erratum, formula (3.1), p. 341]) and upper bounds on the transition probabilities of (3.1) that

$$H_* \leq c_3, \tag{3.4}$$

with c_3 depending on the same parameters as c_2.

Define

$$h_*(\omega) := \frac{1}{T} \int_{-T}^{0} \int_{\mathbb{R}^d} p^\omega(t, \mathbf{y}; 0, \mathbf{0}) H_*(T_{t,\mathbf{y}}\omega)\, d\mathbf{y}\, dt, \tag{3.5}$$

where $p^\omega(\cdot,\cdot;\cdot,\cdot)$ is the transition probability density corresponding to the diffusion given by (1.1). Note that since H_* is $\mathcal{U}^0_{-\infty}$-measurable, so is h_*. The following result has been shown in [8, Theorem 2.1, p. 639 and (3.5), p. 643].

THEOREM 3.1. *Suppose that the field* $\mathbf{u}(\cdot)$ *satisfies conditions* (S), (RH), (C), (FDT), (ACFDDT) *and* $\kappa > 0$. *Then the quasi-Lagrangian process* $\eta(t) := \mathbf{u}(t, \mathbf{x}(t))$, $t \geq 0$, *is stationary and ergodic over the probability space* $(\Omega \times \Sigma, \mathcal{B} \otimes \mathcal{A}, \mathbb{P}_* \otimes W)$, *where* $\mathbb{P}_*(d\omega) := h_*(\omega)\mathbb{P}(d\omega)$.

It is also straightforward to verify (see [8, pp. 643–644]) that h_* satisfies the conclusions of Theorem 2.7.

3.2. *Approximation of a general flow by fields satisfying* (ACFDD). In this section we will construct a sequence of velocity fields satisfying the regularity condition (ACFDD) which approximate the velocity field $\mathbf{u}$ in an appropriate sense.

With no loss of generality we suppose that the probability space $\mathcal{T}_0$ is sufficiently rich to support a time-space stationary, Gaussian random field $\mathbf{g} : \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ that satisfies the following conditions:

- (G0) $\mathbf{g}$ is independent of $\mathbf{u}$.
- (G1) $\mathbf{g}$ is centered, i.e. $\mathbb{E}\mathbf{g}(0, \mathbf{0}) = 0$.
- (G2) Let $\mathcal{G}_a^b$ denote the σ-algebra generated by $\mathbf{g}(t, \mathbf{x})$, $a \leq t \leq b$, $\mathbf{x} \in \mathbb{R}^d$. Then $\mathbf{g}(s, \mathbf{x})$ and $\mathbf{g}(t, \mathbf{y})$ are uncorrelated when $|t - s| > T$, $\forall\, \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. This implies in particular (see [15, Theorem 10.2, p. 181]) that $\mathbf{g}(\cdot,\cdot)$ satisfies (FDT) with $\mathcal{G}^t_{-\infty}$ and $\mathcal{G}^\infty_{t+T}$ replacing $\mathcal{U}^t_{-\infty}$ and $\mathcal{U}^\infty_{t+T}$, respectively.
- (G3) $\mathbf{g}$ has realizations that are C^∞ $\mathbb{P}$-a.s.
- (G4) $\mathbf{g}$ satisfies condition (ACFDD).

To construct such a field let $W(dt, d\mathbf{x})$ be an $\mathbb{R}$-valued space-time white noise over $\mathcal{T}_0$ that is independent of $\mathbf{u}$, and $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a compactly supported C^∞ function with support in $(0,T)\times B_R(\mathbf{0})$ for some $R>0$, where $B_R(\mathbf{x})$ is the ball of radius $R>0$ centered at $\mathbf{x}$. Set $\Delta(\mathbf{x}) := (-\infty, x_1]\times \ldots \times (-\infty, x_d]$ and

$$g_1(t,\mathbf{x};\omega) := \int_{-\infty}^{t} \int_{\Delta(\mathbf{x})}\!\!\ldots\!\int g(t-s,\mathbf{x}-\mathbf{y})\, W(ds,d\mathbf{y})$$

and let $g_2,\ldots,g_d$ be independent copies of g_1. The field $\mathbf{g}=(g_1,\ldots,g_d)$ satisfies assumptions (G0)–(G4).

We define the velocity field $\mathbf{u}_n$ by

$$\mathbf{u}_n(t,\mathbf{x}) := \mathbf{u}(t,\mathbf{x}) + \frac{1}{n}\,\phi(\mathbf{g}(t,\mathbf{x})),$$

where $\phi:\mathbb{R}^d\to\mathbb{R}^d$ is given by $\phi(\mathbf{x}) := (1+|\mathbf{x}|^2)^{-1/2}\mathbf{x}$. Thanks to (G0)–(G4) it is easy to check that $\mathbf{u}_n$ satisfies all the conditions (S), (RH), (C), (FDT), (ACFDD). It is also easy to check that $\mathbf{u}_n$ and $\nabla_{\mathbf{x}}\mathbf{u}_n$ converge to $\mathbf{u}$ and $\nabla_{\mathbf{x}}\mathbf{u}$ respectively, as $n\to\infty$, uniformly on compact sets, $\mathbb{P}$-a.s.

Since the fields $\mathbf{u}$, $\mathbf{g}$ are jointly homogeneous, without any loss of generality we may assume that there exists a group $T_{t,\mathbf{x}}:\Omega\to\Omega$, $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^d$, satisfying assumptions (MP), (M), (SC) of Section 2.2 such that

$$\mathbf{u}(t,\mathbf{x};\omega) = \mathbf{u}(0,\mathbf{0};T_{t,\mathbf{x}}\omega),\quad \mathbf{g}(t,\mathbf{x};\omega) = \mathbf{g}(0,\mathbf{0};T_{t,\mathbf{x}}\omega), \tag{3.6}$$

for all $(t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^d$. Hence, in particular,

$$\mathbf{u}_n(t,\mathbf{x};\omega) = \mathbf{u}_n(0,\mathbf{0};T_{t,\mathbf{x}}\omega)\quad \text{for all } n\ge 1 \text{ and } (t,\mathbf{x})\in\mathbb{R}\times\mathbb{R}^d. \tag{3.7}$$

4. The proof of Theorem 2.1. Denote by $Q^{(n)}$ the transport operator corresponding to the field $\mathbf{u}_n$. Let $H_*^{(n)}$ be the corresponding invariant density and $\mathbb{P}_*^{(n)}(d\omega) := h_*^{(n)}(\omega)\mathbb{P}(d\omega)$, where

$$h_*^{(n)}(\omega) := \frac{1}{T}\int_{-T}^{0}\int_{\mathbb{R}^d} p_n^\omega(t,\mathbf{y};0,\mathbf{0}) H_*^{(n)}(T_{t,\mathbf{y}}\omega)\,d\mathbf{y}\,dt \tag{4.1}$$

and $p_n^\omega(\cdot,\cdot;\cdot,\cdot)$ denotes the transition probability density corresponding to the diffusion with drift $\mathbf{u}_n$. Thanks to (3.4) and the upper bound on the transition probability density in (3.1) one can find a deterministic constant $c_4>0$ independent of n such that

$$1/c_4 \le h_*^{(n)} \le c_4. \tag{4.2}$$

From (4.2) we conclude that $(h_*^{(n)})$ is L^2-weakly pre-compact. In the same fashion we deduce that there exists a constant c_5 independent of n such that

$$H_*^{(n)} \le c_5, \tag{4.3}$$

hence $(H_*^{(n)})$ is weakly pre-compact in L^2. Choose a subsequence (n_k), still denoted by (n), such that

$$\lim_{n\to\infty} h_*^{(n)} = h_*, \qquad \lim_{n\to\infty} H_*^{(n)} = H_*, \tag{4.4}$$

weakly in L^2. Note that since all $h_*^{(n)}$ are $\mathcal{U}_{-\infty}^0$-measurable, so also is h_*.

LEMMA 4.1. *We have*

$$h_*(\omega) = \frac{1}{T}\int_{-T}^{0}\int_{\mathbb{R}^d} p^\omega(t,\mathbf{y};0,\mathbf{0})H_*(T_{t,\mathbf{y}}\omega)\,d\mathbf{y}\,dt. \tag{4.5}$$

Proof. Denote the right hand side of (4.5) by $\widetilde{h}_*(\omega)$. From the Banach–Saks theorem (see [2, Theorem 1.8.4]) we conclude that there exists a subsequence $(H_*^{(n_k)})$ whose Cesàro means converge in the L^2-norm to H_*,

$$\lim_{m\to\infty}\left\|\frac{1}{m}\sum_{k=1}^{m} H_*^{(n_k)} - H_*\right\|_{L^2} = 0. \tag{4.6}$$

Hence, there exists a subsequence $(m') \subset \mathbb{N}$ such that

$$\lim_{m'\to\infty}\frac{1}{m'}\sum_{k=1}^{m'} H_*^{(n_k)} = H_* \qquad \mathbb{P}\text{-a.s.} \tag{4.7}$$

Repeating the same procedure we can choose a subsequence such that, in fact,

$$\lim_{m'\to\infty}\frac{1}{m'}\sum_{k=1}^{m'} h_*^{(n_k)} = h_* \qquad \mathbb{P}\text{-a.s.} \tag{4.8}$$

Recalling that $p_n^\omega(\cdot,\cdot;\cdot,\cdot)$ denotes the transition probability density corresponding to the diffusion with drift $\mathbf{u}_n$, we can write

$$\begin{aligned}
&\int|\widetilde{h}_*(\omega) - h_*(\omega)|\,\mathbb{P}(d\omega) \qquad (4.9)\\
&= \int\bigg|\lim_{m'\to\infty}\frac{1}{T}\int_{-T}^{0}\int_{\mathbb{R}^d}\frac{1}{m'}\sum_{k=1}^{m'}[p^\omega(t,\mathbf{y};0,\mathbf{0})H_*(T_{t,\mathbf{y}}\omega)\\
&\qquad\qquad - p_{n_k}^\omega(t,\mathbf{y};0,\mathbf{0})H_*^{(n_k)}(T_{t,\mathbf{y}}\omega)]\,d\mathbf{y}\,dt\bigg|\,\mathbb{P}(d\omega)\\
&\le \int\limsup_{m'\to\infty}\frac{1}{T}\int_{0}^{T}\int_{\mathbb{R}^d}\frac{1}{m'}\sum_{k=1}^{m'}|p^\omega(0,\mathbf{0};t,\mathbf{y}) - p_{n_k}^\omega(0,\mathbf{0};t,\mathbf{y})|H_*(\omega)\,d\mathbf{y}\,dt\,\mathbb{P}(d\omega)\\
&\quad + \int\limsup_{m'\to\infty}\frac{1}{T}\int_{0}^{T}\int_{\mathbb{R}^d}\left|\frac{1}{m'}\sum_{k=1}^{m'}p_{n_k}^\omega(0,\mathbf{0};t,\mathbf{y})[H_*(\omega) - H_*^{(n_k)}(\omega)]\right|\,d\mathbf{y}\,dt\,\mathbb{P}(d\omega).
\end{aligned}$$

In the last expression we used the homogeneity of the random field. Denote by I_1, I_2 respectively the first and second expressions on the right hand side of (4.9).

Changing $H_*(\omega)$ and $H_*^{n_k}(\omega)$ on a set of measure 0 if necessary, we can assume that they are bounded for all $\omega \in \Omega$. The sequence $(p^{\omega}_{n_k}(0,\mathbf{0};t,\mathbf{y}))$ converges to $p^{\omega}(0,\mathbf{0};t,\mathbf{y})$ for all $\omega \in \Omega$ and $t \in (0,T]$, $\mathbf{y} \in \mathbb{R}^d$. Applying the Lebesgue dominated convergence theorem we see that $I_1 = 0$. Recalling (4.7) and applying the Lebesgue theorem again implies that I_2 also vanishes.

We define

$$\mathbb{P}_*(d\omega) := h_*(\omega)\,\mathbb{P}(d\omega). \tag{4.10}$$

Our next goal is to show that $\mathbb{P}_*$ satisfies the conclusions of Theorem 2.1.

Since $(\mathbf{u}_n(\cdot,\cdot))_{n\ge 0}$ converges uniformly on compact sets to $\mathbf{u}(\cdot,\cdot)$ together with appropriate spatial derivatives we conclude that $(\mathbf{x}_n(\cdot))_{n\ge 0}$ converges uniformly on compact intervals to $\mathbf{x}(\cdot)$ for $\mathbb{P}$-a.e. ω. Thus, also $(\mathbf{u}_n(\cdot,\mathbf{x}_n(\cdot)))_{n\ge 0}$ converges uniformly on compact intervals to $\mathbf{u}(\cdot,\mathbf{x}(\cdot))$.

Take $0 < t_1 < \ldots < t_k$, $\mathbf{x}_1,\ldots,\mathbf{x}_k \in \mathbb{R}^d$ and $f_1,\ldots,f_k : \mathbb{R}^d \to \mathbb{R}$ bounded continuous functions. Since $\mathbf{u}_n(\cdot,\cdot)$ satisfies condition (ACFDD), we can use (3.6) and (3.7) of [8] to deduce that for any $h \ge 0$,

$$\int_\Omega \mathbf{M}\prod_{i=1}^k f_i(\mathbf{u}_n(t_i+h,\mathbf{x}_i+\mathbf{x}_n(t_i+h)))\,\mathbb{P}_*^{(n)}(d\omega) = \int_\Omega \mathbf{M}\prod_{i=1}^k f_i(\mathbf{u}_n(t_i,\mathbf{x}_i+\mathbf{x}_n(t_i)))\,\mathbb{P}_*^{(n)}(d\omega). \tag{4.11}$$

Recall that $\mathbf{M}$ is the expectation w.r.t. the probability measure W corresponding to Brownian paths $\mathbf{w}(\cdot)$. Therefore, letting $n \to \infty$ gives

$$\int_\Omega \mathbf{M}\prod_{i=1}^k f_i(\mathbf{u}(t_i+h,\mathbf{x}_i+\mathbf{x}(t_i+h)))\,\mathbb{P}_*(d\omega) = \int_\Omega \mathbf{M}\prod_{i=1}^k f_i(\mathbf{u}(t_i,\mathbf{x}_i+\mathbf{x}(t_i)))\,\mathbb{P}_*(d\omega). \tag{4.12}$$

Hence, $\mathbb{P}_*$ is stationary. The proof of ergodicity is postponed until Section 5.3 after we present the proof of ergodicity of the abstract environment process.

5. The proof of Theorem 2.7

5.1. *Stationarity.* Let $\mathcal{U}$ denote the sub-σ-algebra of $\mathcal{B}$ generated by $\mathbf{u}(t,\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d$. Let h_* of Theorem 2.7 be given by (4.4). Thanks to (4.2) we conclude that $h_* \ge 1/c_4$, so (2.6) holds. Moreover, (4.12) implies that (2.7) holds for all F that are bounded and $\mathcal{U}_0^\infty$-measurable. We also have the following.

LEMMA 5.1. *Suppose that $F \in B(\Omega)$ and $t \ge 0$. Let $F_{\mathcal{U}} \in B(\Omega)$ be such that $F_{\mathcal{U}} = \mathbb{E}[F \,|\, \mathcal{U}]$ $\mathbb{P}$-a.s. Then*

$$\mathbb{E}[P^t F \,|\, \mathcal{U}] = P^t F_{\mathcal{U}} \quad \mathbb{P}\text{-}a.s.$$

Proof. Suppose that G is bounded and $\mathcal{U}$-measurable. Then

$$\begin{aligned}(5.1)\quad \int \mathbb{E}[P^t F \,|\, \mathcal{U}] G \, d\mathbb{P} &= \int P^t F G \, d\mathbb{P} \\ &\overset{(2.5)}{=} \int \int_{\mathbb{R}^d} p^\omega(0, \mathbf{0}; t, \mathbf{x}) F(T_{t,\mathbf{x}}\omega) G(\omega) \, d\mathbf{x} \, \mathbb{P}(d\omega) \\ &= \int \int_{\mathbb{R}^d} p^{T_{-t,-\mathbf{x}}\omega}(0, \mathbf{0}; t, \mathbf{x}) F(\omega) G(T_{-t,-\mathbf{x}}\omega) \, d\mathbf{x} \, \mathbb{P}(d\omega)\end{aligned}$$

the last expression being the consequence of the homogeneity of $\mathbb{P}$. Since $p^\omega(0, \mathbf{0}; t, \mathbf{x})$ is $\mathcal{U}$-measurable, being a fundamental solution of the Kolmogorov forward equation with $\mathcal{U}$-measurable coefficients, the left hand side of (5.1) equals

$$\int \int_{\mathbb{R}^d} p^{T_{-t,-\mathbf{x}}\omega}(0, \mathbf{0}; t, \mathbf{x}) F_{\mathcal{U}}(\omega) G(T_{-t,-\mathbf{x}}\omega) \, d\mathbf{x} \, \mathbb{P}(d\omega) = \int P^t F_{\mathcal{U}} G \, d\mathbb{P}. \ \blacksquare$$

To finish the argument for the stationarity note that since h_* is $\mathcal{U}_{-\infty}^0$-measurable we can write

$$\begin{aligned}\int P^t F h_* \, d\mathbb{P} = \int \mathbb{E}[P^t F \,|\, \mathcal{U}] h_* \, d\mathbb{P} &\overset{\text{Lemma 5.1}}{=} \int P^t F_{\mathcal{U}} h_* \, d\mathbb{P} \\ &= \int F_{\mathcal{U}} h_* \, d\mathbb{P} = \int F h_* \, d\mathbb{P}\end{aligned}$$

and (2.7) is satisfied.

5.2. *Ergodicity.* Here we assume that condition (ET) holds. Note that for each $t > 0$ the operator P^t is conservative in the sense of Hopf decomposition (see [6, II, Chapter]). According to [6, Theorem A, p. 21] it suffices to show that any $A \in \mathcal{B}$ satisfying

$$(5.2)\qquad P^t \mathbf{1}_A = \mathbf{1}_A \quad \text{for some } t > 0$$

must be trivial, i.e. $\mathbb{P}(A) = 0$ or 1. Suppose that A satisfies (5.2) and $\mathbb{P}(A) > 0$. With no loss of generality we may assume that $A \in \mathcal{U}$. Indeed, otherwise consider $F := \mathbb{E}[\mathbf{1}_A \,|\, \mathcal{U}]$ that is non-constant, $\mathcal{U}$-measurable and such that $P^t F = F$. This, again by [6], implies the existence of a non-trivial set $B \in \mathcal{U}$ such that $P^t \mathbf{1}_B = \mathbf{1}_B$.

We have

$$\int_\Omega \int_{\mathbb{R}^d} \mathbf{1}_A(T_{t,\mathbf{y}}\omega) p^\omega(0, \mathbf{0}, t, \mathbf{y}) \mathbf{1}_{A^c}(\omega) \, d\mathbf{y} \, \mathbb{P}(d\omega) = 0,$$

which in turn implies

$$\int \mathbf{1}_A(T_{t,\mathbf{y}}\omega)\mathbf{1}_{A^c}(\omega)\,\mathbb{P}(d\omega) = 0 \tag{5.3}$$

for m-a.e. $\mathbf{y} \in \mathbb{R}^d$, where m denotes the Lebesgue measure on $\mathbb{R}^d$. Stochastic continuity of the group $T_{\cdot,\cdot}$ implies that (5.3) holds in fact for all $\mathbf{y} \in \mathbb{R}^d$ and in particular also for $\mathbf{y} = \mathbf{0}$. Therefore, $\mathbf{1}_A(T_{t,\mathbf{0}}\omega) = \mathbf{1}_A(\omega)$ $\mathbb{P}$-a.s., which implies $\mathbb{P}(A) = 1$ by (ET).

5.3. *Ergodicity of the quasi-Lagrangian process.* Suppose that A is a Borel subset of $C([0,\infty);\mathbb{R}^d)$ that satisfies (2.3) and $\mathbb{P}_* \otimes W[\eta(\cdot) \in A] \in (0,1)$. Assume that Ω and the group $T_{\cdot,\cdot}$ are as in Remark 2.5. Let $\xi(\cdot)$ be the corresponding environment process with (P^t) the corresponding semigroup of transition probabilities. By (FDT) the group $T_{\cdot,\cdot}$ is temporally ergodic. Then $F := \mathbf{M}\mathbf{1}_A(\eta(\cdot))$ is non-constant and, by (2.3), satisfies $P^hF = F$ for any $h > 0$. This contradicts the conclusion of the previous section, and shows that there are no non-trivial shift invariant sets.

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