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FINITE-DIMENSIONAL DIFFERENTIAL ALGEBRAIC GROUPS AND THE PICARD-VESSIOT THEORY

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Abstract. We make some observations relating the theory of finite-dimensional differential algebraic groups (the ∂_0-groups of [2]) to the Galois theory of linear differential equations. Given a differential field (K, ∂), we exhibit a surjective functor from (absolutely) split (in the sense of Buium) ∂_0-groups G over K to Picard-Vessiot extensions L of K, such that G is K-split iff $L = K$. In fact we give a generalization to "K-good" ∂_0-groups. We also point out that the "Katz group" (a certain linear algebraic group over K) associated to the linear differential equation $\partial Y = AY$ over K, when equipped with its natural connection $\partial - [A, -]$, is K-split just if it is commutative.

1. Introduction. Let (K, ∂) be a differential field of characteristic 0 with algebraically closed field k of constants. Let

$$(*) \qquad \partial Y = AY$$

be a linear differential equation over K. That is, Y is an n by 1 column vector of indeterminates and A is a n by n matrix over K. Let (L, ∂) be a Picard-Vessiot extension for $(*)$. The differential Galois group $Aut_\partial(L/K)$ is well-known to have the structure of an algebraic subgroup of $GL(n, k)$, so the group of k-points of some linear algebraic group G_k over k. On the other hand, another group G'_K, a linear algebraic subgroup now over K, was defined in [5] via the Tannakian point of view. The current paper was in part motivated by an informal question of Daniel Bertrand regarding the *differential algebraic* meaning of G'_K. In fact the Tannakian theory already equips G'_K with a "connection" $\nabla : \partial - [A, -]$, giving it the structure of a ∂_0-group over K (see section 2 for the definitions). The group of L-points (or even $\hat{K}$-points for $\hat{K}$ a differential closure of K) of this

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∂_0-group (G'_K, ∇) (see Definition 2.4) acts on the solution space of $(*)$ in L and should be viewed as the real "intrinsic" differential Galois group of $(*)$. (G'_K, ∇) is isomorphic *over* L to G_k equipped with the trivial connection, so is (absolutely) split in the sense of Buium [2]. We will point out that (G'_K, ∇) is K-split (that is, isomorphic over K to an algebraic group over k equipped with the trivial connection) just if G'_K is commutative. In particular $((G'_K)^0, \nabla)$ is K^{alg}-split iff $(G'_K)^0$ is commutative.

On the other hand we will point out that *any* ∂_0-group (G, ∇) which is defined over K and *absolutely* split, gives rise in a natural way to a Picard-Vessiot extension L of K: essentially L will be generated over K by a canonical parameter for an isomorphism of (G, ∇) with an algebraic group over k equipped with the trivial connection. Moreover any Picard-Vessiot extension of K arises in this way: given the equation $(*)$ above, a fundamental matrix U of solutions of $(*)$ will be a canonical parameter for an isomorphism between $(G_a^n, \partial - A)$ and G_a^n equipped with the trivial connection.

The observations in this paper are not too difficult. In fact the paper should be seen as an introduction to the Kolchin-Cassidy-Buium (and model-theoretic) theory of ∂_0-groups, for those familiar with the Picard-Vessiot theory (Galois theory of linear differential equations). Concerning the general theory of ∂_0-groups our only innovation is to bring into the picture rationality issues, the notion of being split *over* K, where K is an arbitrary (not necessarily differentially closed) differential field.

The rest of the paper is devoted to filling in the details of the above observations. At some point in section 3 some model-theoretic notation is used. The reader is referred to [11] for explanations.

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2. ∂_0-groups. The differential algebraic groups (of Kolchin [7]) are essentially just the group objects in Kolchin's category of differential algebraic varieties. From the model-theoretic point of view they are definable groups in a differentially closed field (see [11]). Such a group is said to be "finite-dimensional" if the differential function field of its connected component has finite transcendence degree (over the universal domain say). Finite-dimensional differential algebraic groups were exhaustively studied by Buium [2].

It will be convenient to introduce finite-dimensional differential algebraic groups (∂_0-groups) via a different formalism, that of Buium's "algebraic D-groups": a ∂_0-group will be an algebraic group equipped with what I will loosely refer to as a "connection". Let (K, ∂) be a differential field (of characteristic 0), with field $C_K = k$ of constants.

Definition 2.1. A ∂_0-*group* over K is a pair (G, ∇) where G is an algebraic group over K and ∇ is an extension of ∂ to a derivation of the structure sheaf $O_K(G)$ of G, commuting with co-multiplication. A homomorphism (over K) between (G_1, ∇_1) and (G_2, ∇_2) is the obvious thing (a K-homomorphism f of algebraic groups such that the corresponding map f^* between structure sheaves respects the respective derivations).

Rather quickly we will replace the derivation ∇ by a more accessible object. Given an algebraic group G over K, $T(G)$ will denote the tangent bundle of G, another algebraic group over K. $\tau(G)$ is a twisted version of $T(G)$ taking into account the derivation ∂ of K: working locally, if G is defined by polynomial equations $P_j(X_1,\ldots,X_n)=0$, then $\tau(G)$ is defined by the equations $\sum_{i=1}^n \partial P_j/\partial X_i(X_1,\ldots,X_n)Y_i + P_j^\partial(X_1,\ldots,X_n)=0$, where P^∂ is the result of applying ∂ to the coefficients of P. $\tau(G)$ has naturally the structure of an algebraic group over K with a surjective homomomorphism π to G (see [8]). If G is defined over k (the constants of K) then $\tau(G)$ identifies with $T(G)$ and π is as usual. A K-rational homomorphism f from G_1 to G_2 yields a K-rational homomorphism $\tau(f):\tau(G_1)\to\tau(G_2)$ commuting with π.

Remark 2.2. *If G is an algebraic group over K, then a ∂_0-group structure on G (that is, a derivation ∇ as in Definition 2.1) is equivalent to a K-rational homomorphic section $s:G\to\tau(G)$ of $\pi:\tau(G)\to G$.*

Proof. This is immediate: given a point a of G, ∇ determines a derivation of the local ring at a, yielding a point $s(a)$ in the fibre $\tau(G)_a$. ∇ commuting with co-multiplication is equivalent to s being a homomorphism.

Thus a ∂_0-group over K can be identified with a pair (G,s) where G is an algebraic group over K and the "connection" $s:G\to\tau(G)$ is a K-rational homomorphic section of π. A K-homomorphism f between such (G_1,s_1) and (G_2,s_2) is then a K-rational homomorphism f of algebraic groups such that $\tau(f).s_1 = s_2.f$. If G happens to be defined over the constants k of K then as mentioned above $\tau(G)=T(G)$, and we have at our disposal the "trivial connection" s_0, namely s_0 is the 0-section of $T(G)$.

Definition 2.3. Let (G,s) be a ∂_0-group over K. We say that (G,s) is K-*split* if it is isomorphic over K to some (G_0,s_0) where G_0 is defined over k and s_0 is the trivial connection.

Note that if X is a variety defined over K and $a\in X(\mathcal{U})$ is a $\mathcal{U}$-rational point of X then the expression $\partial(a)$ makes sense: working in an affine neighbourhood of a, defined over K, just apply ∂ to the coordinates of a. Moreover $\partial(a)$ is in $\tau(X)_a$.

Definition 2.4. Let (G,s) be a ∂_0-group over K. Let (L,∂) be a differential field extension of K. The group $(G,s)(L)$ of *L-rational points* of (G,s) is $\{g\in G(L):\partial(g)=s(g)\}$.

Let $(\mathcal{U},\partial)$ be a universal domain, that is, a differentially closed field containing K, of cardinality $\kappa >$ the cardinality of K, with the following properties: (i) any isomorphism between small (of cardinality $<\kappa$) differential subfields of $\mathcal{U}$ extends to a (differential) automorphism of $\mathcal{U}$, and (ii) if $K_1 < K_2$ are small differential fields then any embedding of K_1 in $\mathcal{U}$ extends to an embedding of K_2 in $\mathcal{U}$.

If (G,s) is a ∂_0-group over K, then $(G,s)(\mathcal{U})$, the set of points of (G,s) in $\mathcal{U}$, is a finite-dimensional differential algebraic group, defined over K, in the sense of Kolchin. Moreover any finite-dimensional differential algebraic group arises in this way (see [2]). We will often identify (G,s) with its group of $\mathcal{U}$ points, or sometimes with its group of $\hat{K}$-points where $\hat{K}$ is a differential closure of K. Also any ∂_0-group over K can naturally be also considered as a ∂_0-group over $\mathcal{U}$ (or over any differential field extending K).

DEFINITION 2.5. The ∂_0-group (G, s) over K is said to be (*absolutely*) *split* if it is $\mathcal{U}$-split, equivalently if it is L-split for some differential field $L > K$.

Note that (G, s) (over K) is absolutely split iff it is $\hat{K}$-split. In [2], Buium begins (and almost completes) the classification of (connected) ∂_0-groups over $\mathcal{U}$: the issue being to first determine which (connected) algebraic groups G over $\mathcal{U}$ have some "D-group" structure, that is, can be equipped with a suitable s, and secondly, to note that the space of D-group structures on G is, if nonempty, a principal homogeneous space for the set of rational homomorphic sections of the tangent bundle of G. It would be of interest to try to classify the ∂_0-groups over a given (say algebraically closed) differential field K, up to K-isomorphism, although possibly this is already implicit in Buium's work. In any case, one of the points of the current paper is that split but non-K-split ∂_0-groups over K are closely bound up with Picard-Vessiot extensions of K. This will be discussed in the next section. For the rest of this section I will give some examples and elementary facts about D-group structures on commutative algebraic groups over the constants, working over $\mathcal{U}$. C denotes the field of constants of $\mathcal{U}$. Note first that for such G any section $s : G \to T(G)$ can be identified with a homomorphism from G into its Lie algebra. (Canonically $T(G) = L(G) \times G$, so $s = (f, id)$ for a unique $f : G \to L(G)$.)

EXAMPLE 2.6 (D-group structures on commutative unipotent groups). Let $G = G_a^n$. A rational homomorphism from G to $L(G)$ is precisely a linear map from G to itself. Thus each D-group structure on G has the form (G, s_A) for some n by n matrix A over $\mathcal{U}$, where s_A is left matrix multiplication by A. Each such ∂_0-group is split (over $\mathcal{U}$): The set of $\mathcal{U}$-points of (G, s_A) is an n-dimensional vector space over C. Let $b_1, \ldots, b_n$ (thought of as column vectors) be a C-basis. Matrix multiplication by $(b_1, \ldots, b_n)$ yields an isomorphism between (G, s_0) and (G, s_A). This isomorphism need not be defined over the differential field generated by the coordinates of A.

EXAMPLE 2.7 (D-group structures on semiabelian varieties over the constants). Let A be a semiabelian variety over C. As above, D-group structures s on A are given by rational homomorphisms from A to the Lie algebra of A, of which there is only one, the 0 map. So the 0-section is the unique D-group structure on A.

EXAMPLE 2.8 (D-group structures on commutative algebraic groups over the constants). Let G be a connected commutative algebraic group defined over C. We will prove a special and easy case of a result from [2]: Let (G, s) be a D-group structure (over $\mathcal{U}$) on G. Then (G, s) is split if and only if the unipotent radical U of G (which note is also defined over C) is a D-subgroup of G (that is, $s(U)$ is contained in $L(U)$). Right to left is clear. Suppose now that s takes U into $L(U)$.

CLAIM. $s(G) = s(U)$.

Proof. Otherwise s induces a nonconstant homomorphism from the semiabelian variety G/U into $L(G)/L(U)$ which is impossible.

Let $H < G$ be the kernel of s. Using Example 2.6 we can write U as a direct sum of $H \cap U$ and a D-subgroup U_1 of U. By the claim G is the direct sum of H and U_1. As U_1 is split (by Example 2.6), G is split.

EXAMPLE 2.9 (A nonsplit D-group structure on $G_m \times G_a$.). Let $G = G_m \times G_a$. $T(G) = \tau(G)$ consists of the set of (x, y, u, v) where $x \neq 0$, and has group structure given by: $(x_1, y_1, u_1, v_1) \cdot (x_2, y_2, u_2, v_2) = (x_1x_2, y_1 + y_2, u_1x_2 + u_2x_1, v_1 + v_2)$. $\pi : T(G) \to G$ is projection on the first two coordinates. Let $s : G \to T(G)$ be: $s(x, y) = (x, y, xy, 0)$. Then s is a section of π as well as being a homomorphism. $(G, s)(\mathcal{U}) = \{(x, y) \in G : \partial x = xy, \partial y = 0\}$, which is isomorphic to the differential algebraic subgroup $\{x \in G_m : \partial(\partial x/x) = 0\}$ of G_m.

Rather deeper results concern D-group structures on algebraic groups which cannot be defined over the constants. For example, an abelian variety A over $\mathcal{U}$ which is not isomorphic to an abelian variety over C has *no* D-group structure. A will nevertheless have finite-dimensional differential algebraic subgroups (defined by differential equations of order > 1) which correspond to D-group structures on extensions of A by unipotent groups. (See [2] and [8].) Such examples will not concern us too much in this paper. Moreover, the further away from the constants an algebraic group G is, the more rigid will be the space of D-group structures on G.

3. Relations with the Picard-Vessiot theory. We will take as our basic reference Bertrand's review [1]. Recall the basic set-up: K is a differential field (considered as a small subfield of $\mathcal{U}$) with algebraically closed field k of constants.

$$(*) \qquad \partial Y = AY$$

is a linear ODE over K. V^∂ denotes the solution space of $(*)$ in $\mathcal{U}$, an n-dimensional vector space over C (the constants of $\mathcal{U}$). $V^\partial(\hat{K})$ denotes the vectors in V^∂ whose coordinates are in $\hat{K}$. $V^\partial(\hat{K})$ is an n-dimensional vector space over k. The Picard-Vessiot extension L/K for $(*)$ is the (differential) field generated over K by the coordinates of elements of $V^\partial(\hat{K})$. (As $\hat{K}$ has the same constants as K, L has the same constants as K.) Let us fix a fundamental solution matrix U for $(*)$, namely the columns of U form a basis for $V^\partial(\hat{K})$ over k (and so also for V^∂ over C). Via U we obtain an isomorphism ρ_U between $Aut_\partial(L/K)$ and an (algebraic) subgroup of $GL(n, k)$: $\sigma(U) = U\rho_U(\sigma)$. Write this subgroup as $G_k(k)$, the group of k-rational points of a linear algebraic group G_k over k. Note that $G_k(k)$ is precisely the set of $\hat{K}$-points of the ∂_0-group (G_k, s_0) where s_0 is the trivial connection.

In [5], a somewhat different definition of the Galois group of $(*)$ was given, but now as an algebraic group G'_K over K. This was defined via the Tannakian theory. The usual notion of a connection on a vector space over the differential field (K, ∂) is an additive endomorphism $D : V \to V$ such that for any $\lambda \in K$, and $v \in V$, $D(\lambda v) = \partial(\lambda)v + \lambda D(v)$. Let $V = K^n$. From the equation $(*)$ we obtain a connection $D_V : \partial - A$ on V. D_V induces, on each K-vector space E constructed from V by iterating direct sums, tensor products, and duals, a connection D_E. $GL(n, K)$ acts on each of these vector spaces, and G'_K is defined to be $\{g \in GL(n, K) : g(W) = W$, for every K-subpace W of any construction E over V for which $D_E(W) \subseteq W\}$.

Note that $GL(n, K)$ acts on itself and thus on its own coordinate ring R over K. As remarked in [1], G'_K is precisely the stabilizer of the ideal $I \subset R$ consisting of polynomials which vanish on the fundamental matrix U of solutions of $(*)$. (And this does not depend on the choice of U.) In any case we obtain an algebraic group over K, and one can ask in what sense G'_K is the Galois group of $(*)$. Note that $G'_K \subseteq End(V) = V \otimes V^*$ and the latter K-vector space, itself a construction over V, is equipped with the connection $D_{End(V)} : \partial - [A, -]$ (that is, for $X \in End(V), D_{End(V)}(X) = \partial X - [A, X]$). (This connection is also implicit in [3]). In any case this connection equips G'_K with the structure of a ∂_0-group (G'_K, s) where $s(g) = [A, g]$. It is this ∂_0-group (or rather its group of $\hat{K}$-points), which should be considered as the canonical (or intrinsic) Galois group of $(*)$. At this point we make use of model-theoretic/differential algebraic language. Working in $\mathcal{U}$, $tp(-/K)$ means type over K in the sense of differential fields, and $tp_f(-/K)$ means type over K in the sense of fields. Let $G_1 = (G'_K, s)(\mathcal{U})$.

REMARK 3.1. *$G_1 = \{g \in GL(n,\mathcal{U}) : \partial g = [A, g]$ and for any $U_1 \in GL(n,\mathcal{U})$ realising $tp_f(U/K)$ and independent (in the sense of fields) from g over K, $tp_f(gU_1/K) = tp_f(U_1/K)\}$.*

Proof. Clear.

LEMMA 3.2. *G_1 acts faithfully (by left matrix multiplication) on V^∂. Moreover this action is precisely the group of permutations of V^∂ induced by automorphisms of the differential field $\mathcal{U}$ which fix $K \cup C$ pointwise.*

Proof. We start with

CLAIM. Let $g \in GL(n,\mathcal{U})$. Then $g \in G_1$ if and only if $tp(gU/K \cup C) = tp(U/K \cup C)$.

Proof. Note first that $tp(U/K) = r(x)$ say is determined by (i) $tp_f(U/K) = r_f(x)$, and (ii) $\partial x = Ax$. Note also that $r(x)$ has a unique extension to a complete type $r'(x)$ say, over $K \cup C$ (otherwise there would be new constants in $L = K(U)$, which there are not).

Suppose first that $tp(gU/K) = r$. Let U_1 realise r independently from g over K in the sense of differential fields. As U and U_1 are bases for V^∂ over C, $U_1 = UB$ for some $B \in GL(n, C)$. Now U, gU and U_1 each realise r' (over $K \cup C$). It follows that $gU_1 = g(UB) = (gU)(B)$ also realizes r', in particular r. As U_1 is independent from g over K in the sense of fields, and (as $\partial(U_1) = AU_1$ and $\partial(gU_1) = A(gU_1)$) $\partial(g) = [A, g]$, we see from Remark 3.1 that $g \in G_1$.

The other direction of the Claim follows by reversing the argument.

The claim gives us a bijection between the set of permutations σ of V^∂ induced by $Aut_\partial(\mathcal{U}/K \cup C)$ and G_1: σ goes to g where $\sigma(U) = gU$. In fact the action of σ on V^∂ is identical to the action of g by left matrix multiplication: if $v \in V^\partial$ (a column vector), then $v = Uc$ for some column vector of constants, and $\sigma(v) = \sigma(Uc) = \sigma(U)c = (gU)c = g(Uc) = gv$). The map ($\sigma$ to g) is clearly a group isomorphism.

COROLLARY 3.3. *$G_1(\hat{K})$ $(= (G'_K, s)(\hat{K}))$ acts on $V^\partial(\hat{K})$ (by left matrix multiplication) inducing an isomorphism with $Aut_\partial(L/K)$.*

Proof. The first part is immediate from the lemma, using the fact that $\hat{K}$ is homogeneous over K in the model-theoretic sense. As L is generated over K by the points of $V^\partial(\hat{K})$ the second part also follows.

REMARK 3.4. *G_1 as above is also the intrinsic definable automorphism group of V^∂ over C in the model-theoretic sense.*

Explanation. If P and Q are $\emptyset$-definable sets in a saturated model M of a stable theory and P is Q-internal, then the group (G, P) of permutations of P induced by automorphisms of M which fix Q pointwise, is isomorphic to some definable (in M) group action on P. This is due in full generality to Hrushovski [4], and an exposition appears in chapter 7 of [10]. The Picard-Vessiot theory is a special case, as (working over K), V^∂ is C-internal. (In fact Poizat [12] was the first to give a model-theoretic explication of the Picard-Vessiot theory and Kolchin's more general strongly normal theory.) However, even in the general model-theoretic context, there are various incarnations of the definable automorphism group and its action on P: the intrinsic case is where G and its action are $\emptyset$-definable and G lives in P^{eq}. The other case depends on the choice of a "fundamental set of solutions" u from P: G lives in Q^{eq}, is defined over the canonical base of $tp(u/Q)$ and in general requires the parameter u to define its action on P. In any case, transplanted to the Picard-Vessiot situation, it is $(G_{K'}, s)$ which is the intrinsic group, and (G_k, s_0) (s_0 being the trivial connection) which is the non-canonical group.

Note that the ∂_0-group (G'_K, s) (where $s(-) = [A, -]$) is L-split. It is isomorphic to (G_k, s_0) by the map ρ_U: $gU = U\rho_U(g)$. Note also that we obtain easily a simple definition of G_k as an algebraic group over k: Let $U_1, \dots, U_s$ realize the distinct nonforking extensions of $tp(U/K)$ over $K(U)$. Let $U_i = UB_i$ for $B_i \in GL(n, C)$. Let $p_i = tp_f(B_i/k)$.Then G_k is precisely the stabilizer of $\{p_1, \dots, p_s\}$ in $GL(n)$.

Note that the set X of realizations of $tp(U/K)$ is a left principal homogeneous space for G_1 $(= (G'_K, s)(\mathcal{U}))$, and a right principal homogeneous space for $G_2 = G_k(C)$ $(= (G_k, s_0)(\mathcal{U}))$ (and likewise working with $\hat{K}$-rational points), where the actions commute ($g(xh) = (gx)h$ for $g \in G_1, x \in X, h \in G_2$). That is, X is a (differential algebraic) bi-torsor for (G_1, G_2) defined over K. It follows that G_1 is isomorphic *over K* to G_2 just if G_1 (so also G_2) is commutative. This kind of thing (in the general model-theoretic framework of definable automorphism groups) was already observed in passing in [4]. In any case we will give some details.

Let us start with a general lemma:

LEMMA 3.5. *Let (H_1, X, H_2) be an abstract bi-torsor. That is, X is an (abstract) left principal homogeneous space for the (abstract) group H_1, an (abstract) right principal homogeneous space for the (abstract) group H_2 and the left and right actions commute. For $x \in X$, let ρ_x be the isomorphism between H_1 and H_2 defined by $hx = x\rho_x(h)$. Let $h \in H_1$. The following are equivalent:*

(i) h is in the centre of H_1.
(ii) for all $x \in X$, $\rho_x = \rho_{hx}$.
(iii) for some $x \in X$, $\rho_x = \rho_{hx}$.

Proof. (i) implies (ii). Assume $h \in Z(H_1)$. Let $g \in H_1$ and $x \in X$. Then $ghx = x\rho_x(g)\rho_x(h)$ and $hgx = x\rho_x(h)\rho(g)$. But also $ghx = (hx)\rho_{hx}(g) = x\rho_x(h)\rho_{hx}(g)$. So as $gh = hg$ we see that $\rho_{hx}(g) = \rho_x(g)$. As $g \in H_1$ was arbitrary, we see that $\rho_{hx} = \rho_x$.

(ii) implies (iii) is immediate.

(iii) implies (i) follows by reversing the proof of (i) implies (ii).

Let us now return to the differential situation: L is the Picard-Vessiot extension of K for the equation $\partial Y = AY$ over K, G'_K is the Katz group and $s(-)$ is $[A, -]$. U is a fundamental matrix of solutions of $(*)$ (and $L = K(U)$).

Corollary 3.6. *(G'_K, s) is K-split if and only if G'_K is commutative.*

Proof. Recall the notation: $G_1 = (G'_K, s)(\mathcal{U})$, $G_2 = G_k(C)$, and let X be the space of realisations of $tp(U)$. ρ_U is the isomorphism between G_1 and G_2: $gU = U\rho_U(g)$.

Firstly, let us suppose that G'_K is commutative. Then so is G_1 and by the previous lemma, $\rho_U = \rho_{gU}$ for all $g \in G_1$. But X is precisely the set of such gU ($g \in G_1$), so ρ_U is fixed by K-automorphisms of the differential field $\mathcal{U}$ so is defined over K: (G'_K, s) is K-split.

Conversely, suppose (G'_K, s) is K-split. So there is a K-definable isomorphism f between G_1 and some ∂_0-group G_3 of the form (H, s_0) where H is an algebraic group over C and s_0 is the trivial connection. Then H must be defined over k. $\rho_U.f^{-1}$ is then an isomorphism between G_3 and G_2 defined over $\hat{K}$. As both G_3 and G_2 are the groups of C-points of algebraic groups defined over the algebraically closed field k, and k is the constants of $\hat{K}$, it follows that $\rho_U.f^{-1}$ is defined over k (and is actually an isomorphism of algebraic groups). Thus (the differential algebraic isomorphism) ρ_U is defined over K. So for each $g \in G_1$, $\rho_U = \rho_{gU}$. By Lemma 3.5, G_1 is commutative. But easily G_1 is Zariski-dense in G'_K, whereby G'_K is commutative.

With the same notation:

Corollary 3.7. *$((G'_K)^0, s)$ is K^{alg} split iff $(G'_K)^0$ is commutative.*

Proof. Let L_1 be the compositum $K^{alg}L$. Then L_1 is the Picard-Vessiot extension of K^{alg} for the equation $(*)$, with Katz group $(G'_K)^0$. Now apply the previous corollary.

These results give a cheap way of producing split but non-K-split D-group structures on noncommutative connected algebraic groups over C (for suitable G and algebraically closed K).

Corollary 3.8. *Let G be a connected noncommutative algebraic subgroup of $GL(n,\mathcal{U})$, defined over some field k of constants. Let A be a generic (in the sense of differential fields) point over k of the Lie algebra $L(G) < M_n(\mathcal{U})$ of G. Let K be the algebraic closure of the differential field generated over k by the coordinates of A, and let $s(-) = [A, -]$. Then (G, s) is defined over K, and is absolutely split but not K-split.*

Proof. We may assume that $A = (\partial g)g^{-1}$ for g a generic point over k of G (in the sense of differential fields). Then $L = K(g)$ is a Picard-Vessiot extension of K for the equation $\partial Y = AY$ with Katz group G.

So we have established one way of obtaining split but non-K-split ∂_0-groups from Picard-Vessiot extensions of K.

Finally we will give another relationship between these two classes of objects. Our notation (K, k, $\partial Y = AY$, U, $L = K(U)$) is as before. Let us first note that the solution space V^∂ of $\partial Y = AY$ is (the set of $\mathcal{U}$-points of) (G_a^n, s_A) (see Example 2.6). Moreover the fundamental matrix of solutions U is a "canonical parameter" for a (differential algebraic) isomorphism of (G_a^n, s_A) with (G_a^n, s_0) (multiplication by U). (To say that U is a canonical parameter means that moving U moves the isomorphism). It easily follows as in above arguments that (G_a^n, s_A) is K-split iff U has its coordinates in K. More generally we have:

PROPOSITION 3.9. *Let K be a differential subfield of $\mathcal{U}$ with algebraically closed field k of constants. Let (G, s) be an absolutely split (not necessarily linear) ∂_0-group defined over K. Let $u \in \hat{K}$ be a canonical parameter (over K) for some differential algebraic isomorphism between G_1 and some (H, s_0) where H is an algebraic group over k and s_0 is the trivial connection. Then*

(i) $L = K < u >$ (the differential field generated over K by u) is a Picard-Vessiot extension of K whose Katz group is an algebraic subgroup of $Aut(G)$.
(ii) (G, s) is K-split iff $L = K$.
(iii) The "map" taking (G, s) to L establishes a functor from the class of absolutely split ∂_0-groups over K (up to K-isomorphism) to the class of Picard-Vessiot extensions of K (inside $\hat{K}$ or equivalently up to isomorphism over K).
(iv) The functor in (iii) is surjective.

Proof. (i) Let $G_1 = (G, s)(\mathcal{U})$. As G_1 is absolutely split there is an isomorphism f defined over $\hat{K}$ between $G_2 = (H, s_0)(\mathcal{U})$ (for some algebraic group H defined over k) and G_1. By elimination of imaginaries in differentially closed fields, there is some tuple u from $\hat{K}$ such that $f = f_u$ is defined over $K < u >$ and such that for any u' realising $tp(u/K)$, $f_{u'} = f_u$ iff $u = u'$. (This is what we mean by u being a canonical parameter over K for f.) In any case we may identify u with f_u, and similarly for any realisation u_1 of $r(x) = tp(u/K)$. For each such u_1, $u^{-1}.u_1$ is a (definable) automorphism g of G_2. $u_1 = u.g$, so clearly $u_1 \in K < u, C >$. As $L = K < u >$ has the same constants as K, it follows that L is a strongly normal extension of K in the sense of Kolchin [6]. To see that L is a Picard-Vessiot extension of K it is enough to show that the (extrinsic) Galois group of L over K is linear. Working in $\mathcal{U}$, this extrinsic Galois group G_3 say, is the set of C-points of an algebraic group defined over k. Clearly G_3 acts definably (over k) and faithfully on G_2 as (group) automorphisms. As all this is going on inside the constants C, the action is rational. Thus the connected component of G_3 embeds (rationally) over k into (the group of C-points of) $GL(L)$ where L is the Lie algebra of G_2. Thus the connected component of G_3,and so G_3 itself, is definably the group of C-points of a linear algebraic group over k. So L is a Picard-Vessiot extension of K. The element u gives an isomorphism between G_3 and a K-definable subgroup G_4 of $Aut(G_1)$. G_4 (or rather its group of $\hat{K}$-points) is the intrinsic Galois group of L over K. It is not hard to see that the Katz group is a K-algebraic subgroup G' of $Aut(G)$. (That is, G_4 is the ∂_0-group (G'_1, s) for a suitable connection s.)

(ii) Note that L depends only on G_1 (not on G_2 or the isomorphism f_u): if g_w were an isomorphism of G_1 with another ∂_0 group G'_2 of the form (H', s_0) (H' defined over k), where $w \in \hat{K}$ is a canonical parameter for g_w, then the induced isomorphism between G_2 and G'_2 "lives in" k, whence $K < w > = K < u > = L$. So G_1 is K-split iff $u \in K$ iff $L = K$.

(iii) If G_1' is another absolutely split ∂_0-group over K which is isomorphic over K to G_1, and $w \in \hat{K}$ is a canonical parameter for an isomorphism witnessing the splitting, then as in (ii) but using also the isomorphism between G_1 and G_1', we see that $K(w) = K(u)$. So we get a map F from K-isomorphism types of absolutely split ∂_0-groups over K to Picard-Vessiot extensions of K. To say that this is a functor means that if f is an embedding of G_1 into G_2 defined over K then the P-V extension of K corresponding to G_1 is a subfield of that corresponding to G_1'. This is clear from the construction of L and above remarks.

(iv) Finally by the remarks preceding the proposition, any Picard-Vessiot extension of K is in the image of F.

Corollary 3.10. *Let (K, ∂) be a differential field with algebraically closed field of constants. Then the following are equivalent:*

(i) K has no proper Picard-Vessiot extensions.
(ii) Any ∂_0-group defined over K which is absolutely split is already K-split.

I will end with some remarks about general ∂_0-groups (not necessarily absolutely split). Let K be an algebraically closed differential field. Let G be a ∂_0-group defined over K. Call G *K-good* if $G(K) = G(\hat{K})$. A *K-form* of G is a ∂_0-group over K which is isomorphic (but not necessarily over K) to G. We then have the following strengthening of Corollary 3.10:

Proposition 3.11. *The following are equivalent:*

(i) K has no proper Picard-Vessiot extensions.
(ii) Any K-good ∂_0-group has a unique K-form up to K-isomorphism.

Explanation. We work with definability in the differentially closed field $\mathcal{U}$. Let G be a K-good connected ∂_0-group (so also K-definable). Let G_1 be a K-form of G. So G_1 is a ∂_0-group over K which is definably isomorphic to G. Let $u \in \hat{K}$ be a canonical parameter for a definable isomorphism (which we also call u) between G and G_1. All we have to do is show that $L = K < u >$ is a Picard-Vessiot extension of K. First note that as K is algebraically closed and $K < L < \hat{K}$,

(a) $K < u >$ has the same constants as K.

Next we want to show that L is a differential Galois extension of K in the sense of [9]. By that paper it suffices to see that the set X of realizations of $tp(u/K)$ is a principal homogeneous space for a ∂_0-group H defined over K such that $H(\hat{K}) = H(K)$. Well, for any w realizing $tp(u/K)$, clearly $w = u \circ f$ for a unique definable automorphism f of G. Let H be the (definable) group of automorphisms of G obtained this way. H can be considered as a ∂_0-group defined over K, and as $G(\hat{K}) = G(K)$, also $H(\hat{K}) = H(K)$. So we see

(b) L is a differential Galois extension of K (in the sense of [9]) and moreover $Aut(L/K)$ is isomorphic to $H(\hat{K})$.

Finally (by [9]) we need to see that the ∂_0-group H is definably isomorphic to the group of $\mathcal{C}$-points of a linear algebraic group defined over $\mathcal{C}$. We use the differential Lie algebra $L^{\partial}(G)$ as introduced by Kolchin in Chapter VIII of [7]. As G is a ∂_0-group, $L^{\partial}(G)$ is a finite-dimensional vector space over $\mathcal{C}$. The ∂_0-group H (a group of definable automorphisms of G) embeds in $GL(L^{\partial}(G))$ giving us what we want. (We have just observed here that if G is any connected ∂_0-group and H is a ∂_0-group which acts definably on G as a group of automorphisms, then H definably embeds in $GL_n(\mathcal{C})$ for some n.) Thus L is a Picard-Vessiot extension of K, and is a proper extension of K just if G_1 is not definably isomorphic over K to G.

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