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QUANTUM ITÔ ALGEBRA AND QUANTUM MARTINGALE

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Abstract. In this paper, we study a representation of the quantum Itô algebra in Fock space and then by using a noncommutative Radon-Nikodym type theorem we study the density operators of output states as quantum martingales, where the output states are absolutely continuous with respect to an input (vacuum) state. Then by applying quantum martingale representation we prove that the density operators of regular, absolutely continuous output states belong to the commutant of the $\star$-algebra parameterizing the quantum Itô algebra.

1. Introduction. Since the quantum stochastic calculus was introduced by Hudson and Parthasarathy in [10], stochastic integral representations of quantum martingales have been studied by many authors (see [11], [12], [17], [18] and the references cited therein) with applications to study of Markovian cocycles. In fact, the existence of a stochastic generator for a Markovian cocycle is verified by applying the integral representation of quantum martingale and then the cocycle is realized as the solution of a quantum stochastic differential equation (see [10], [16]).

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On the other hand, a quantum Itô algebra generalizing Itô algebra on the axiomatic level was defined in [2] by a family $\{d\Lambda(t,a) : a \in \mathfrak{a}\}$ of stochastic differentials $d\Lambda(t,a)$ with respect to $t \in \mathbf{R}_+$ with a indexing $\star$-semigroup $\mathfrak{a}$. Here $\mathfrak{a}$ is extended to a complex, in general infinite dimensional space, parameterizing $\star$-algebra with an involution, death $d \in \mathfrak{a}$ (a self-adjoint annihilator) and a positive linear $*$-functional l normalized as $l(d) = 1$, see Section 2. In fact, the quantum Itô algebra and the parameterizing $\star$-algebra can be identified. The classification of the general (noncommutative) quantum Itô algebra, and the representation theorem are established in [5] and [7]. The quantum Levy-Chinchin type decomposition theorem has been proved in [5] for finite-dimensional quantum Itô algebra, and in [6] for the infinite-dimensional case. Also, it was proved in [2, 3] that any (classical or quantum) stochastic noise described by a process $\mathbf{R}_+ \ni t \mapsto \Lambda(t,a)$ with independent increments, forming an Itô $\star$-algebra, can be represented in the (symmetric) Fock space $\mathcal{F}_0 = \Gamma(\mathcal{K})$ over the Hilbert space $\mathcal{K} = L^2_{\mathcal{E}}(\mathbf{R}_+)$ of $\mathcal{E}$-valued square-integrable functions on $\mathbf{R}_+$.

In this paper, we study a representation of the quantum Itô algebra in Fock space and families of output states defined on $*$-algebras of operators acting on a Fock scale of the representing Fock space of the quantum Itô algebra. By using a noncommutative Radon-Nikodym type theorem we study the density operators of output states as quantum martingales, where the output states are absolutely continuous with respect to a input (vacuum) state. Then by applying quantum martingale representation we prove that the density operators of regular, absolutely continuous output states belong to the commutant of the quantum Itô algebra.

The paper is organized as follows. In Section 2 we remind the definition of the general quantum Itô algebra given in [2], and their GNS-representation and Fock representation. In Section 3 we recall a rigging of Fock space which is necessary for our study. In Section 4 we recall quantum stochastic processes, specially quantum martingales, and Wick exponential of basic quantum stochastic processes. In Section 5 we study quantum input and output states, and their density operators as quantum martingales.

2. Quantum Itô algebra. Let $\mathfrak{a}$ be a noncommutative $\star$-algebra with a self-adjoint annihilator (death) $d \in \mathfrak{a}$, i.e., $d = d^\star$ and $\mathfrak{a}d = 0$. Let l be a normalized (as $l(d) = 1$) positive linear functional on $\mathfrak{a}$, i.e.,

$$l : \mathfrak{a} \to \mathbf{C}; \qquad l(a^\star a) \geq 0, \quad l(a^\star) = \overline{l(a)}, \quad a \in \mathfrak{a}.$$

Let $(\mathcal{E}, k)$ be a minimal Kolmogorov decomposition of l, i.e., $\mathcal{E}$ is a Hilbert space and k is a map from $\mathfrak{a}$ into $\mathcal{E}$ such that $l(a^\star b) = k(a)^* k(b)$ for any $a, b \in \mathfrak{a}$. We denote j the operator representation of $\mathfrak{a}$ on the Hilbert space $\mathcal{E}$ defined by

$$j : \mathfrak{a} \ni a \mapsto j(a) \in \mathcal{L}(\mathcal{E}), \qquad j(a)k(b) = k(ab), \quad a, b \in \mathfrak{a},$$

where $\mathcal{L}(\mathcal{E})$ is the algebra of all continuous linear operators on $\mathcal{E}$. It is directly seen that $j(a^\star b) = j(a^\star)j(b) = j(a)^* j(b)$ for any $a, b \in \mathfrak{a}$. Therefore, the functional l defines the GNS representation $a \mapsto \mathbf{a} = (a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ of $\mathfrak{a}$ in terms of the quadruples:

$$a^\bullet_\bullet = j(a), \quad a^\bullet_+ = k(a), \quad a^-_\bullet = k^*(a), \quad a^-_+ = l(a),$$

where $k^*(a) = k(a^\star)^*$.

REMARK. Since $(\mathcal{E}, k)$ is a minimal Kolmogorov decomposition of l, $\mathcal{E}$ can be considered as the completion of $\mathfrak{a}_0 = LS\{a \in \mathfrak{a} : l(a^\star a) \neq 0\}$ with respect to the norm $|\cdot|_{\mathcal{E}}$ induced by the inner product $\langle a, b\rangle_{\mathcal{E}} = l(a^\star b)$ for $a, b \in \mathfrak{a}_0$ and then k is given by

$$k(a) = \begin{cases} a, \; l(a^\star a) \neq 0; \\ 0, \; l(a^\star a) = 0, \qquad a \in \mathfrak{a}. \end{cases}$$

The operator representation j of $\mathfrak{a}$ is the left multiplication, i.e., $j(a)k(b) = j(a)b = ab$ for any $a, b \in \mathfrak{a}$.

The *quantum Itô algebra* defined in [2] is the linear span of differentials:

$$d\Lambda(t,a) = \Lambda(t+dt,a) - \Lambda(t,a), \qquad a \in \mathfrak{a}, \quad t \in \mathbf{R}_+ = [0,\infty)$$

for a family $\{\Lambda(a) : a \in \mathfrak{a}\}$ of operator-valued integrators $\Lambda(t,a)$ on a pre-Hilbert space satisfying the $\star$-semigroup conditions: $\Lambda(t,a^\star) = \Lambda(t,a)^\dagger$ and

$$d\Lambda(t,ab) = d\Lambda(t,a)d\Lambda(t,b), \qquad \sum \lambda_i d\Lambda(t,a_i) = d\Lambda\Big(t, \sum \lambda_i a_i\Big)$$

with mean values $\langle d\Lambda(t,a)\rangle = l(a)dt$ in a given vector state $\langle\cdot\rangle$, where for each $t \in \mathbf{R}_+$, $\Lambda(t,a)^\dagger$ is the Hermitian conjugate of the (unbounded) operator $\Lambda(t,a)$ which is defined on the pre-Hilbert space as the operator $\Lambda(t,a^\star)$, and

$$d\Lambda(t,a)d\Lambda(t,b) = d\left(\Lambda(t,a)\Lambda(t,b)\right) - d\Lambda(t,a)\Lambda(t,b) - \Lambda(t,a)d\Lambda(t,b).$$

Since l is normalized as $l(d) = 1$, dtI can be understood as $d\Lambda(t,d)$. By assuming that the parameterisation is exact such that $d\Lambda(t,a) = 0 \Rightarrow a = 0$, we can identify the Itô algebra $(d\Lambda(\mathfrak{a}), ldt)$ and the parameterising $\star$-algebra $(\mathfrak{a}, l)$.

As was proved in [2], a quantum stochastic stationary processes $\mathbf{R}_+ \ni t \mapsto \Lambda(t,a)$, $a \in \mathfrak{a}$ with $\Lambda(0,a) = 0$ and independent increment $d\Lambda(t,a) = \Lambda(t+dt,a) - \Lambda(t,a)$, forming a quantum Itô $\star$-algebra, can be represented in the Fock space $\mathcal{F}_0 \equiv \Gamma(L^2_{\mathcal{E}}(\mathbf{R}_+))$ over the $\mathcal{E}$-valued square integrable functions on $\mathbf{R}_+$ as $\Lambda^\nu_\mu(t, a^\mu_\nu) = a^\mu_\nu \Lambda^\nu_\mu(t)$, where

$$\Lambda(t,a) = \sum_{\mu=-}^{\bullet} \sum_{\nu=\bullet}^{+} a^\mu_\nu \Lambda^\nu_\mu(t) = a^\bullet_\bullet \Lambda^\bullet_\bullet(t) + a^\bullet_+ \Lambda^+_\bullet(t) + a^-_\bullet \Lambda^\bullet_-(t) + a^-_+ \Lambda^+_-(t),$$

is the canonical decomposition of Λ into the conservation $\Lambda^\bullet_\bullet$, creation $\Lambda^+_\bullet$, annihilation $\Lambda^\bullet_-$ and time $\Lambda^+_- = tI$ processes of quantum stochastic calculus having the mean values $< \Lambda^\nu_\mu(t) >= t\delta^\nu_+ \delta^-_\mu$ with respect to the vacuum state in $\mathcal{F}_0$. Thus the parameterising algebra $\mathfrak{a}$ can be always identified with $\star$-subalgebra of the Hudson-Parthasarathy (HP) algebra $\mathcal{Q}(\mathcal{E})$ of all quadruples $\mathbf{a} = (a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$, where $a^\mu_\nu : \mathcal{E}_\nu \to \mathcal{E}_\mu$ are bounded operators on $\mathcal{E}_\bullet = \mathcal{E}$, $\mathcal{E}_+ = \mathcal{E}_- = \mathbf{C}$ with the Hudson-Parthasarathy (HP) multiplication table [10]:

$$\mathbf{a} \bullet \mathbf{b} = (a^\mu_\bullet b^\bullet_\nu)^{\mu=-,\bullet}_{v=+,\bullet},$$

the unique death $\mathbf{d} = (\delta^\mu_- \delta^+_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$, and the involution $(a^\star)^\mu_{-\nu} = (a^\nu_\mu)^*$, where $-(-) = +$, $-\bullet = \bullet$ and $-(+) = -$.

3. Rigging of fock space. Let $\mathcal{H}$ be a (initial) complex Hilbert space with inner product $\langle\cdot|\cdot\rangle$ which is conjugate linear and linear in the first variable and second variable, respectively. Let $\mathcal{D}$ be a reflexive Fréchet subspace of $\mathcal{H}$ which can be considered as a projective limit with respect to an increasing sequence of Hilbertian norms $|\cdot|_p \geq |\cdot|_0 = |\cdot|_{\mathcal{H}}$

for $p \geq 0$, see [15]. In fact, $\mathcal{D} = \cap_{p\geq 0}\mathcal{D}_p$, where $\mathcal{D}_p$ is the completion of $\mathcal{D}$ with respect to the norm $|\cdot|_p$. Let $\mathcal{D}'$ denote the dual space of continuous antilinear (conjugate linear) functionals:

$$\zeta' : \mathcal{D} \ni \eta \mapsto \langle \eta | \, \zeta' \rangle \in \mathbf{C}$$

with respect to the canonical pairing $\langle \eta | \, \zeta' \rangle$ for $\eta, \zeta' \in \mathcal{H}$. The space $\mathcal{D}'$ will be equipped with the weak topology induced by its predual(=dual) $\mathcal{D}$. Let $\mathcal{B}(\mathcal{D})$ denote the linear space of all continuous sesquilinear forms $\langle \zeta | \, B\eta \rangle$ on $\mathcal{D}$ which is identified with the continuous linear operator (kernel) $B : \mathcal{D} \to \mathcal{D}'$. The space $\mathcal{B}(\mathcal{D})$ will be equipped with w^*-topology (induced by the predual $\mathcal{B}_*(\mathcal{D}) = \mathcal{D} \otimes \mathcal{D}$) which coincides with the weak topology on each bounded subset with respect to the norm $|\cdot|_p$. Let $\mathcal{L}(\mathcal{D})$ denote the algebra of all continuous linear operators $B : \mathcal{D} \to \mathcal{D}$ and then $\mathcal{L}(\mathcal{D}) \subset \mathcal{B}(\mathcal{D})$.

For each $B \in \mathcal{B}(\mathcal{D})$, $B^\dagger \in \mathcal{B}(\mathcal{D})$ is the Hermite conjugated form (kernel) defined by

$$\left\langle \zeta | \, B^\dagger \eta \right\rangle = \overline{\langle \eta | \, B\zeta \rangle}, \qquad \eta, \zeta \in \mathcal{D}.$$

Any operator $A \in \mathcal{L}(\mathcal{D})$ with $A^\dagger \in \mathcal{L}(\mathcal{D})$ can be uniquely extended to a weakly continuous operator onto $\mathcal{D}'$ as $(A^\dagger)^*$ which is denoted again by A, where A^* is the dual (adjoint) operator of A, i.e.,

$$A^* : \mathcal{D}' \to \mathcal{D}'; \qquad \langle \eta | \, A^*\zeta \rangle = \langle A\eta | \, \zeta \rangle \,, \qquad \eta \in \mathcal{D}, \ \zeta \in \mathcal{D}'.$$

In fact, for any $\eta, \zeta \in \mathcal{D}$ we have

$$\left\langle \eta | \, (A^\dagger)^*\zeta \right\rangle = \left\langle A^\dagger \eta | \, \zeta \right\rangle = \overline{\langle \zeta | \, A^\dagger \eta \rangle} = \langle \eta | \, A\zeta \rangle \,.$$

We say that the operator $A \in \mathcal{L}(\mathcal{D})$ with $A^\dagger \in \mathcal{L}(\mathcal{D})$ commutes with a sesquilinear form $B \in \mathcal{B}(\mathcal{D})$, i.e., $BA = AB$ if $\langle \zeta | \, BA\eta \rangle = \left\langle A^\dagger \zeta | \, B\eta \right\rangle$ for all $\eta, \zeta \in \mathcal{D}$. Note that for an operator $*$-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{D})$, the commutant

$$\mathcal{A}^{\mathrm{c}} = \{B \in \mathcal{B}(\mathcal{D}) : [A, B] = 0 \text{ for all } A \in \mathcal{A}\}$$

is weakly closed in $\mathcal{B}(\mathcal{D})$.

Put $\mathcal{K} = L^2_{\mathcal{E}}(\mathbf{R}_+)$ with norm $|\cdot|_{\mathcal{K}}$ and $\mathcal{F}_0 = \mathcal{F}_0(\mathcal{K}) = \Gamma(\mathcal{K})$ the Fock space over $\mathcal{K}$. Then the Fock space $\mathcal{F}_0$ can be represented as the Hilbert integral

$$\mathcal{F}_0 = \mathcal{F}_0(\mathcal{K}) = \Gamma(\mathcal{K}) := \int_{\Omega(\mathbf{R}_+)}^{\oplus} \mathcal{E}^{\otimes|\tau|} d\tau$$

of the functions $\varphi : \Omega(\mathbf{R}_+) \ni \tau \mapsto \varphi(\tau) \in \mathcal{E}^{\otimes|\tau|}$, where for $U \subset \mathbf{R}_+$, $\Omega(U)$ is the set of all finite subsets of U and $|\tau| = \#(\tau)$ is the number of elements in τ. For more study we refer to [1].

For each $p \in \mathbf{R}$ we denote $\mathcal{F}_p = \mathcal{F}_p(\mathcal{K})$ a (weighted) Fock space over $\mathcal{K}$ defined by

$$\mathcal{F}_p = \mathcal{F}_p(\mathcal{K}) := \int_{\Omega(\mathbf{R}_+)}^{\oplus} e^{p|\tau|} \mathcal{E}^{\otimes|\tau|} d\tau.$$

Then for any $p, q \in \mathbf{R}$ with $p \leq q$ we have $\mathcal{F}_q \subset \mathcal{F}_p$. The projective limit of $\{\mathcal{F}_p\}_{p\geq 0}$ is denoted by $\mathcal{F}$ and then we have a triplet:

$$\mathcal{F} = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{F}_p \subset \mathcal{F}_0 \subset \mathcal{F}^* \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{F}_{-p}.$$

Then $\mathcal{F}$ becomes a reflexive Fréchet space. In fact, for each $p \geq 0$ the strong dual space $\mathcal{F}_p^*$ of $\mathcal{F}_p$ with respect to $\mathcal{F}_0$ is identified with $\mathcal{F}_{-p}$ and the strong dual space $\mathcal{F}^*$ of $\mathcal{F}$ is identified with the inductive limit of $\{\mathcal{F}_{-p}\}_{p\geq 0}$. For each $p \in \mathbf{R}$, the norm on $\mathcal{F}_p$ is denoted by $\|\cdot\|_p$ and then

$$\|f^{\otimes}\|_p^2 = \int_{\Omega(\mathbf{R}_+)} e^{p|\tau|} \|f^{\otimes}(\tau)\|^2_{\mathcal{E}^{\otimes|\tau|}} d\tau := \sum_{n=0}^{\infty} \frac{e^{pn}}{n!} \left(\int_0^{\infty} |f(t)|^2_{\mathcal{E}} dt \right)^n = e^{e^p |f|^2_{\mathcal{K}}},$$

where for each $f \in \mathcal{K}$, $f^{\otimes} \in \mathcal{F}_p$ and $f^{\otimes}(\tau) = \otimes_{t\in\tau} f(t)$, $\tau \in \Omega(\mathbf{R}_+)$.

For each $p \geq 0$, put $\mathcal{G}_p = \mathcal{D}_p \otimes \mathcal{F}_p$ and let $\mathcal{G}_{-p}$ is the strong dual space of $\mathcal{G}_p$ with respect to $\mathcal{H} \otimes \mathcal{F}_0$. Then we have a triplet:

$$\mathcal{G} = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{G}_p \subset \mathcal{G}_0 \subset \mathcal{G}^* \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{G}_{-p}.$$

The norm on $\mathcal{G}_p$ is denoted by $\|\cdot\|_p$ again.

4. Quantum stochastic processes. For two locally convex spaces $\mathfrak{X}$, $\mathfrak{Y}$, we denote $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ the space of all continuous linear operators from $\mathfrak{X}$ into $\mathfrak{Y}$ which is equipped with the bounded convergence topology. Note that $\mathcal{G}^*$ and $\mathcal{G}'$ can be identified, where $\mathcal{G}'$ is the dual space of continuous antilinear functionals on $\mathcal{G}$. A family $\{\Xi_t\}_{t\geq 0}$ of operators in $\mathcal{L}(\mathcal{G}, \mathcal{G}')$ is called a *quantum stochastic process*.

From the following natural decomposition: for any $p \geq 0$

$$\mathcal{G}_p = \mathcal{G}_{p;t)} \otimes \mathcal{F}_{p;[t}, \qquad \mathcal{G}_{p;t)} = \mathcal{D}_p \otimes \mathcal{F}_{p;t)}, \tag{1}$$

where

$$\mathcal{F}_{p;t)} := \int^{\oplus}_{\Omega([0,t))} e^{p|\tau|} \mathcal{E}^{\otimes|\tau|} d\tau, \qquad \mathcal{F}_{p;[t} := \int^{\oplus}_{\Omega([t,\infty))} e^{p|\tau|} \mathcal{E}^{\otimes|\tau|} d\tau,$$

for each $\psi \in \mathcal{G}_p$ we have

$$\psi = \psi_{t)} \otimes \psi_{[t}, \qquad \psi_{t)} \in \mathcal{D}_p \otimes \mathcal{F}_{p;t)}, \quad \psi_{[t} \in \mathcal{F}_{p;[t}.$$

Therefore, we define the maps

$$E_{t)} : \mathcal{G}_p \ni \psi \mapsto \psi_{t)} \in \mathcal{D}_p \otimes \mathcal{F}_{p;t)}, \qquad E_{[t} : \mathcal{G}_p \ni \psi \mapsto \psi_{[t} \in \mathcal{F}_{p;[t},$$

and, put $\mathcal{G}_{t)} = E_{t)}(\mathcal{G})$. The space $\mathcal{G}_{t)}$ is embedded into $\mathcal{G}$ by the isometry:

$$E^{\dagger}_{t)} : \mathcal{G}_{t)} \ni \psi_{t)} \mapsto \psi \in \mathcal{G} \quad \text{as} \quad \psi(\tau) = \psi_{t)}(\tau_{t)}) \delta_{\emptyset}(\tau_{[t}),$$

where

$$\tau_{t)} = \tau \cap [0,t), \quad \tau_{[t} = \tau \cap [t,\infty), \qquad \tau \in \Omega(\mathbf{R}_+)$$

and

$$\delta_{\emptyset}(\tau) = \begin{cases} 1, \tau = \emptyset; \\ 0, \text{otherwise.} \end{cases}$$

The projectors $E_{t)}$ onto $\mathcal{G}_{t)}$ can be extended to $\mathcal{G}'$ as the adjoint of $E^{\dagger}_{t)}$.

Let $\mathbf{D}$ be the linear (dense) subspace of $\mathcal{G}$ such that for each $t \geq 0$, $E_{t)}(\mathbf{D}) \subset \mathbf{D}$. A quantum stochastic process $\{Z_t\}_{t\geq 0}$ of sesquilinear forms $\langle \psi | Z_t \psi \rangle$ given by linear operators $Z_t : \mathbf{D} \to \mathcal{G}'$ is called *adapted* if

$$Z_t(\psi) = Z_t(\psi_{t)}) \otimes \psi_{[t}, \qquad \psi \in \mathbf{D}.$$

For each linear operator $Z : \mathbf{D} \to \mathcal{G}'$, the *(vacuum) conditional expectation* $\mathbf{E}_{t)}(Z)$ of Z with respect to the past up to a time $t \in \mathbf{R}_+$ is defined by

$$\mathbf{E}_{t)}(Z)(\psi) = \mathbf{E}_{t)}(Z)(\psi_{t)} \otimes \psi_{[t)}) = \big(E_{t)}Z(\psi_{t)})\big) \otimes \psi_{[t}$$

for any $\psi \in \mathbf{D}$. A sesquilinear form $Z : \mathbf{D} \to \mathcal{G}'$ is said to be positive if $\langle \psi | \, Z\psi \rangle \geq 0$ for all $\psi \in \mathbf{D}$, and Hermitian if $Z^\dagger = Z$. Then for each $t \in \mathbf{R}_+$, $\mathbf{E}_{t)}$ is a positive projector, i.e., $\mathbf{E}_{t)}(Z) \geq 0$ if $Z \geq 0$ and

$$\mathbf{E}_{s)} \circ \mathbf{E}_{t)} = \mathbf{E}_{s)}, \qquad s < t.$$

Note that for each linear operator $Z : \mathbf{D} \to \mathcal{G}'$, $\{\mathbf{E}_{t)}(Z)\}_{t\geq 0}$ is an adapted sesquilinear form. From (1), we can easily prove that for each $t \in \mathbf{R}_+$, the conditional expectation $\mathbf{E}_{t)}$ can be extended to $\mathcal{B}(\mathcal{G}_p)$ for any $p \in \mathbf{R}$.

An adapted process $\{M_t\}_{t\geq 0}$ of sesquilinear forms $M_t : \mathbf{D} \to \mathcal{G}'$ is called a *quantum martingale* if $\mathbf{E}_{s)}(M_t) = M_s$ for any $0 \leq s \leq t$, i.e.,

$$\langle\!\langle \psi_{s)},\, M_t(\varphi_{s)})\rangle\!\rangle = \langle\!\langle \psi_{s)},\, M_s(\varphi_{s)})\rangle\!\rangle$$

for any $\psi, \varphi \in \mathbf{D}$, where $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the canonical bilinear form on $\mathcal{G} \times \mathcal{G}'$. For more study we refer to [11], [12], [17] and [18].

From now on, we assume that the Itô $\star$-algebra $\mathfrak{a}$ is realized as a $\star$-subalgebra of Hudson-Parthasarathy algebra $\mathcal{Q}(\mathcal{E})$ (see [7]) of all quadruples $\mathbf{a} = (a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$, $a \in \mathfrak{a}$.

Proposition 4.1. *For each $a \in \mathfrak{a}$, the Wick exponential operator*

$$W(t,a) =: \exp\{\Lambda(t,a)\} : \in \mathcal{L}(\mathcal{F})$$

is well-defined as the solution to the quantum stochastic differential equation:

$$dW(t,a) = W(t,a)d\Lambda(t,a), \qquad W(0,a) = I. \tag{2}$$

They give an analytic representation of the unital $\star$-semigroup $1 + \mathfrak{a}$ for the Itô $\star$-algebra with respect to the $\star$-product $a \star b = a^\star + a^\star b + b$, where 1 is a formal identity. Moreover, we have

$$W(t,a)^\dagger W(t,a) = W(t, a \star a), \qquad W(t,0) = I, \quad W(t,d) = e^t I.$$

For the proof we refer to [4] and [8]. More generally, for any function $g : \mathbf{R}_+ \to \mathfrak{a}$ such that the integral $\int_0^t \Lambda(dr, g(r))$ exists as an operator in $\mathcal{L}(\mathcal{F})$, the unique solution of the normal ordered differential equation:

$$dW(t,g) = W(t,g)\Lambda(dt, g(t)), \qquad W(0,g) = I$$

defines the Wick exponential:

$$W(t,g) =: \exp \int_0^t \Lambda(dr, g(r)) : . \tag{3}$$

5. States and quantum martingales. We start with the following *-representation theorem for a unital *-algebra $\mathfrak{A}$ with respect to a state φ on $\mathfrak{A}$.

Theorem 5.1 ([19]). *Let $\mathfrak{A}$ be a unital *-algebra and φ a state on $\mathfrak{A}$. Then there exist a Hilbert space $\mathcal{H}_\varphi$ and a *-representation π_φ of $\mathfrak{A}$, and a (strong) cyclic vector $h_\varphi \in$*

$\mathbf{D}(\pi_\varphi) \subset \mathcal{H}_\varphi$ *such that*

$$\varphi(a) = \langle h_\varphi | \, \pi_\varphi(a) h_\varphi \rangle \,, \qquad a \in \mathfrak{A},$$

where $\mathbf{D}(\pi_\varphi)$ *is a common dense domain of* $\pi_\varphi(a)$ *for all* $a \in \mathfrak{A}$.

Remark. In Theorem 5.1, $\{\pi_\varphi(a)h_\varphi \,|\, a \in \mathfrak{A}\}$ is dense in $\mathbf{D}(\pi_\varphi)$ in the induced topology on $\mathbf{D}(\pi_\varphi)$. For each finite subset S of $\mathfrak{A}$, the semi-norm $\| \cdot \|_S$ on $\mathbf{D}(\pi_\varphi)$ is defined by

$$\|f\|_S = \sum_{a \in S} |\pi_\varphi(a) f|_{\mathcal{H}_\varphi}.$$

The induced topology on $\mathbf{D}(\pi_\varphi)$ is defined as the topology generated by the neighborhoods:

$$N(f; S, \epsilon) = \{g \in \mathbf{D}(\pi_\varphi) \,|\, \|f - g\|_S < \epsilon\}.$$

Moreover, $\mathbf{D}(\pi_\varphi)$ is complete in the induced topology.

Let $\mathfrak{B}$ be the linear space of all sesquilinear forms in $\mathcal{B}(\mathcal{G})$ corresponding to operators $Z \in \mathcal{L}(\mathcal{G})$ with $Z^* \in \mathcal{L}(\mathcal{G})$ and then $\mathfrak{B}$ becomes a unital $*$-algebra with the conjugate linear involution $*$ which is considered as the Hermite conjugated form. The *quantum filtration* $\{\mathfrak{B}_t\}_{t>0}$ is defined as the increasing family of unital *-subalgebras $\mathfrak{B}_t$ of the adapted sesquilinear forms $Z_t \in \mathfrak{B}$. Let $p \in \mathbf{R}$ and $\omega_p : \mathfrak{B} \to \mathbf{C}$ an input state defined by

$$\omega_p(Z) = \langle\!\langle \psi_0 | \, Z \psi_0 \rangle\!\rangle_p, \qquad Y \in \mathfrak{B},$$

where $\langle\!\langle \cdot | \cdot \rangle\!\rangle_p$ is the inner product on $\mathcal{G}_p$ and $\psi_0 = \eta_0 \otimes \delta_\emptyset$ for some $\eta_0 \in \mathcal{D}$ with $|\eta_0| = 1$. We always assume that for each $t \in \mathbf{R}_+$, $\eta_0 \otimes \delta_\emptyset$ is a cyclic vector of $\mathfrak{B}_t$, i.e., $\mathfrak{B}_t(\eta_0 \otimes \delta_\emptyset)$ is dense in $\mathcal{G}_{t)} = \mathbf{E}_{t)}(\mathcal{G})$.

Definition 5.2. Let ρ and σ be states on a unital *-algebra $\mathfrak{A}$. Then we say that σ is ρ*-absolutely continuous* if for any family $\{\{a_{in}\}_{i=1}^{n_m}\}_{n=1}^{\infty}$ of finite sequences in $\mathfrak{A}$ with $\lim_{n\to\infty} \sum_{i,k=1}^{n_m} \rho(a_{in}^* a_{kn}) = 0$,

$$\lim_{n\to\infty} \sum_{i,k=1}^{n_m} \sigma(a_{in}^* a_{kn}) = 0.$$

For another definition of absolutely continuous of state with respect to another state, we refer to [9].

The following theorem shows that a quantum martingale plays an important role as the density operators of output states with respect to an input state which is a vector state given by the vacuum vector.

Theorem 5.3. *Let* $p \in \mathbf{R}$ *and* $\{\mathfrak{B}_t\}_{t \in \mathbf{R}_+}$ *a quantum filtration and* ς_t *an output state on* $\mathfrak{B}_t$ *for each* $t \in \mathbf{R}_+$ *such that*

$$\varsigma_t(Z) = \varsigma_s(Z), \qquad Z \in \mathfrak{B}_s, \qquad s \le t. \tag{4}$$

If for each $t \in \mathbf{R}_+$, ς_t *is* $\omega_p|_{\mathfrak{B}_t}$*-absolutely continuous, then there exists a (positive) kernel* $P_{\varsigma_t} \in \mathcal{L}(\mathcal{G}_p)$ *commuting with* $\mathfrak{B}_t$*, i.e.,* $P_{\varsigma_t} Z = Z P_{\varsigma_t}$, $Z \in \mathfrak{B}_t$*, in the right hand side,* Z *is extended onto* $\mathcal{G}^*$ *such that*

$$\varsigma_t(Z) = \langle\!\langle \psi_0 | \, P_{\varsigma_t} Z \, (\psi_0) \rangle\!\rangle_p = \omega_p(P_{\varsigma_t} Z), \qquad Z \in \mathfrak{B}_t. \tag{5}$$

Moreover, $\{P_{\varsigma_t}\}_{t\geq 0}$ is a quantum L^1-martingale, where L^1 means that

$$P_{\varsigma_t}(\psi_0,\psi_0) := \langle\!\langle \psi_0 |\, P_{\varsigma_t}(\psi_0)\rangle\!\rangle = 1.$$

In this case, P_{ς_t} is unique.

Proof. From Theorem 5.1, for each $t \in \mathbf{R}_+$, there exists a *-representation π_t of $\mathfrak{B}_t$ with respect to the state ς_t on a Hilbert space $\mathcal{H}_{\varsigma_t}$ and a cyclic vector $\upsilon_t \in \mathbf{D}(\pi_t) \subset \mathcal{H}_{\varsigma_t}$, i.e.,

$$\pi_t(YZ)f = \pi_t(Y)\pi_t(Z)f, \quad \langle f,\, \pi_t(Y)g\rangle = \langle \pi_t(Y^*)f,\, g\rangle\,, \qquad f,g \in \mathbf{D}(\pi_t)$$

and

$$\varsigma_t(Z) = \langle \upsilon_t,\, \pi_t(Z)\upsilon_t\rangle_{\mathcal{H}_{\varsigma_t}}\,, \qquad Z \in \mathfrak{B}_t.$$

Put $\mathcal{G}^\circ_{t)} = \mathfrak{B}_t(\eta_0 \otimes \delta_\emptyset)$ and then $\mathcal{G}^\circ_{t)}$ is a dense subspace of $\mathcal{G}_{t)}$ since $\eta_0 \otimes \delta_\emptyset$ is a cyclic vector of $\mathfrak{B}_t$. Define a linear operator R_t by

$$R_t : \mathcal{G}^\circ_{t)} \to \mathbf{D}(\pi_t); \qquad Z(\eta_0 \otimes \delta_\emptyset) \mapsto \pi_t(Z)\upsilon_t, \quad Z \in \mathfrak{B}_t.$$

Then for any $Y, Z \in \mathfrak{B}_t$ we have

$$R_t\left(Z(Y(\eta_0 \otimes \delta_\emptyset))\right) = R_t\left(ZY(\eta_0 \otimes \delta_\emptyset)\right) = \pi_t(Z)\pi_t(Y)\upsilon_t = \pi_t(Z)R_t(Y(\eta_0 \otimes \delta_\emptyset))$$

which implies that for any $Z \in \mathfrak{B}_t$

$$R_t Z = \pi_t(Z)R_t \quad \text{on} \quad \mathcal{G}^\circ_{t)} \tag{6}$$

and R_t is continuous with respect to the Hilbert space norm in $\mathcal{H}_{\varsigma_t}$. In fact, since ς_t is $\omega_p|_{\mathfrak{B}_t}$-absolutely continuous, if $\lim_{n\to\infty}\sum_{i=1}^{m_n} Z_{in}(\eta_0 \otimes \delta_\emptyset) = 0$ in $\mathcal{G}_{p;t)}$, then

$$\lim_{n\to\infty}\sum_{i,k=1}^{m_n} \omega_p|_{\mathfrak{B}_t}(Z^*_{in}Z_{kn}) = \lim_{n\to\infty}\left\|\sum_{i=1}^{m_n} Z_{in}(\eta_0 \otimes \delta_\emptyset)\right\|_p^2 = 0$$

and

$$\begin{aligned}\lim_{n\to\infty}\left\|R_t\Big(\sum_{i=1}^{m_n} Z_{in}(\eta_0 \otimes \delta_\emptyset)\Big)\right\|^2_{\mathcal{H}_{\varsigma_t}} &= \lim_{n\to\infty}\left\|\sum_{i=1}^{m_n} \pi_t(Z_{in})\upsilon_t\right\|^2_{\mathcal{H}_{\varsigma_t}} \\ &= \lim_{n\to\infty}\sum_{i,k=1}^{m_n} \varsigma_t(Z^*_{in}Z_{kn}) = 0.\end{aligned}$$

Since $\mathcal{G}^\circ_{t)}$ is dense in $\mathcal{G}_{t)}$ and so in $\mathcal{G}_{p;t)}$, R_t can be extended to $\mathcal{G}_{p;t)}$ as a continuous operator into $\mathcal{H}_{\varsigma_t}$. The extension is denoted by the same symbol. On the other hand, the adjoint $R^*_t : \mathcal{H}_{\varsigma_t} \to \mathcal{G}_{p;t)}$ of R_t satisfying that $R^*_t\pi_t(Z) = ZR^*_t$ (on $\mathbf{D}(\pi_t)$) by (6). Now, for each $t \in \mathbf{R}_+$ we put $P_{\varsigma_t} = R^*_t R_t : \mathcal{G}_{p;t)} \to \mathcal{G}_{p;t)}$. Then, by ampliation, P_{ς_t} can be extended to $\mathcal{G}_p$ as a continuous linear operator which is denoted by the same symbol. Furthermore, P_{ς_t} commutes with all $Z \in \mathfrak{B}_t$ and

$$\langle\!\langle \eta_0 \otimes \delta_\emptyset |\, P_{\varsigma_t}Z(\eta_0 \otimes \delta_\emptyset)\rangle\!\rangle_p = \langle R_t(\eta_0 \otimes \delta_\emptyset),\, \pi_t(Z)R_t(\eta_0 \otimes \delta_\emptyset)\rangle = \langle \upsilon_t,\, \pi_t(Z)\upsilon_t\rangle = \varsigma_t(Z)$$

which proves (5). Moreover, $\{P_{\varsigma_t}\}_{t\geq 0}$ becomes a quantum martingale due to the property that $\varsigma_s = \varsigma_t|_{\mathfrak{B}_s}$ for all $s \leq t$. In fact, for any $s \leq t$ and $Y, Z \in \mathfrak{B}_s$, we have

$$\begin{aligned}\langle\langle Y(\eta_0\otimes\delta_\emptyset)|\, P_{\varsigma_t}Z(\eta_0\otimes\delta_\emptyset)\rangle\rangle_p &= \langle\langle \eta_0\otimes\delta_\emptyset|\, P_{\varsigma_t}Y^*Z(\eta_0\otimes\delta_\emptyset)\rangle\rangle_p\\ &= \varsigma_t(Y^*Z)=\varsigma_s(Y^*Z)\\ &= \langle\langle \eta_0\otimes\delta_\emptyset|\, P_{\varsigma_s}Y^*Z(\eta_0\otimes\delta_\emptyset)\rangle\rangle_p\\ &= \langle\langle Y(\eta_0\otimes\delta_\emptyset)|\, P_{\varsigma_s}Z(\eta_0\otimes\delta_\emptyset)\rangle\rangle_p.\end{aligned}$$

Moreover, we have

$$P_{\varsigma_t}(\psi_0,\psi_0)=\langle\langle \eta_0\otimes\delta_\emptyset|\, P_{\varsigma_t}(\eta_0\otimes\delta_\emptyset)\rangle\rangle_p=\varsigma_t(1)=1.$$

The uniqueness of P_{ς_t} is immediate by the density of $\mathcal{G}^\circ_{t)}$. ∎

Let $\{\mathfrak{g}_t\}_{t\geq 0}$ be the increasing family of $\star$-semigroups $\mathfrak{g}_t$ of all step functions $g:\mathbf{R}_+\to\mathfrak{a}$, $g(s)=0$ for all $s\geq t$ under the $\star$-product

$$(g\star h)(s)=g(s)^\star+g(s)^\star h(s)+h(s),\qquad g,h\in\mathfrak{g}_t,\quad s\geq 0.$$

Remark. If for each $t\in\mathbf{R}_+$, $\mathfrak{B}_t$ contains all $|\zeta><\eta_0|\otimes W(t,g)$ with ζ in a dense subset of $\mathcal{D}$ and $g\in\mathfrak{g}_t$, then $\eta_0\otimes\delta_\emptyset$ is a cyclic vector of $\mathfrak{B}_t$ from the minimality of the Kolmogorov decomposition k of l.

Let $\mathfrak{G}$ be the unital $*$-algebra consisting of all operators $W(t,g)$ for $t\in\mathbf{R}_+$ and $g\in\mathfrak{g}_t$. For each $t\in\mathbf{R}_+$, let $\mathfrak{G}_t$ be the unital $*$-algebra consisting of all operators $W(s,g)\in\mathcal{L}(\mathcal{F})$ for $s\leq t$ and $g\in\mathfrak{g}_s$. Then $\{\mathfrak{G}_t\}_{t\geq 0}$ is an increasing family of unital $*$-subalgebras of $\mathfrak{G}$. Let $p\in\mathbf{R}$ and $\varrho_p:\mathfrak{G}\to\mathbf{C}$ a state defined by

$$\varrho_p(W(t,g))=\langle\langle \delta_\emptyset|\, W(t,g)\delta_\emptyset\rangle\rangle_p,\qquad W(t,g)\in\mathfrak{G}.$$

Then as a simple application of Theorem 5.3 we have the following corollary.

Corollary 5.4. *If $\{\sigma_t\}_{t\in\mathbf{R}_+}$ is a family of states $\sigma_t:\mathfrak{G}_t\to\mathbf{C}$ such that*

$$\sigma_t(Z)=\sigma_s(Z),\qquad Z\in\mathfrak{G}_s,\quad s\leq t$$

and, for each $t\geq 0$, σ_t is $\varrho_p|_{\mathfrak{G}^t}$-absolutely continuous, then there exists a unique L^1-martingale $\{P_{\sigma_t}\}_{t\geq 0}$ of operators $P_{\sigma_t}\in\mathcal{L}(\mathcal{F}_p)$ commuting with $\mathfrak{G}_t$ such that

$$\sigma_t(Z)=\langle\langle \delta_\emptyset|\, P_{\sigma_t}Z\delta_\emptyset\rangle\rangle_p=\varrho_p(P_{\sigma_t}Z),\qquad Z\in\mathfrak{G}_t.$$

Let $p\in\mathbf{R}$ and $\lambda_p:\mathfrak{g}=\cup_{t\geq 0}\mathfrak{g}_t\to\mathbf{C}$ be a state defined by

$$\lambda_p(g)=\langle\langle \delta_\emptyset|\, W(t,g)\delta_\emptyset\rangle\rangle_p,\qquad g\in\mathfrak{g}_t\subset\mathfrak{g}.$$

Theorem 5.5. *If $\{\rho_t\}_{t\in\mathbf{R}_+}$ is a family of states $\rho_t:\mathfrak{g}_t\to\mathbf{C}$ such that*

$$\rho_t(g)=\rho_s(g),\qquad g\in\mathfrak{g}_s,\quad s\leq t \tag{7}$$

and, for each $t\geq 0$, ρ_t is $\lambda_p|_{\mathfrak{g}_t}$-absolutely continuous, then there exists a unique L^1-martingale $\{K_{\rho_t}\}_{t\geq 0}$ of operators $K_{\rho_t}\in\mathcal{L}(\mathcal{F}_p)$ commuting with $\{W(t,g):g\in\mathfrak{g}_t\}$, i.e.,

$$K_{\rho_t}W(t,g)=W(t,g)K_{\rho_t},\qquad g\in\mathfrak{g}_t$$

such that

$$\rho_t(g)=\langle\langle \delta_\emptyset|\, K_{\rho_t}W(t,g)\delta_\emptyset\rangle\rangle_p=\lambda_p(K_{\rho_t}W(t,g)),\qquad g\in\mathfrak{g}_t.$$

Proof. Since for $s\leq t$ and $g\in\mathfrak{g}_s\subset\mathfrak{g}_t$, by applying (3) we have

$$W(t,g)=:\exp\int_0^t\Lambda(dr,g(r)):=:\exp\int_0^s\Lambda(dr,g(r)):=W(s,g),$$

where $g(r) = 0$ for all $r \geq s$. Therefore, for each $t \geq 0$, $\mathfrak{G}_t = \{W(t,g) : g \in \mathfrak{g}_t\}$ and so consider the family of states $\{\sigma_t\}_{t\in\mathbf{R}_+}$ defined by

$$\sigma_t(Z) = \rho_t(g), \qquad Z = W(t,g) \in \mathfrak{G}_t, \quad g \in \mathfrak{g}_t.$$

Then from (7), we have

$$\sigma_t(Z) = \sigma_s(Z), \qquad Z \in \mathfrak{G}_s, \quad s \leq t$$

and, for each $t \geq 0$, σ_t is $\varrho_p|_{\mathfrak{G}^t}$-absolutely continuous since ρ_t is $\lambda_p|_{\mathfrak{g}_t}$-absolutely continuous. Therefore, by Corollary 5.4, the proof is immediate. ∎

Let $p \geq q$. A quantum martingale $\{M_t\}_{t\geq 0}$ in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ is said to be *regular with respect to a Radon measure* $\mathfrak{m}$ on $\mathbf{R}_+$, or simply *regular* if for all $0 \leq v < u$ and $\phi \in \mathcal{G}_{p;v]}$, $\psi \in \mathcal{G}_{-q;v]}$

$$\|(M_u - M_v)\phi\|_q^2 \leq \|\phi\|_p^2\, \mathfrak{m}([v,u]), \qquad \|(M_u^* - M_v^*)\psi\|_{-p}^2 \leq \|\psi\|_{-q}^2\, \mathfrak{m}([v,u]).$$

If $\{M_t\}_{t\geq 0}$ in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ is a regular quantum martingale, then M_t admits the following integral representation:

$$M_t = \alpha I + \sum_{\mu=-}^{\bullet}\sum_{\nu=\bullet}^{+} \Lambda_\mu^\nu(t, M_\nu^\mu) \quad \Big(\text{equivalently, } dM_t = \sum_{\mu=-}^{\bullet}\sum_{\nu=\bullet}^{+} M_\nu^\mu d\Lambda_\mu^\nu(t)\Big) \tag{8}$$

for some complex number α, where $M_+^- = 0$. For the proof, we refer to [12] and [18].

Proposition 5.6. *Let $p \geq q$ and let $\{M_t\}_{t>0}$ a regular martingale in $\mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ such that for each $t \in \mathbf{R}_+$, M_t admits the integral representation* (8) *and commuting with $\mathfrak{G}_t$, i.e.,*

$$[W(t,g), M_t] = W(t,g)M_t - M_tW(t,g) = 0, \qquad g_t \in \mathfrak{g}_t.$$

Then the coefficient $\mathbf{M} = (M_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ commutes with $\mathbf{a} = (a_\nu^\mu)_{\nu=+,\bullet}^{\mu=-,\bullet}$ for all $a \in \mathfrak{a}$ in the sense of HP-product, i.e.,

$$[\mathbf{a}, \mathbf{M}] = (a_\bullet^\mu M_\nu^\bullet - M_\bullet^\mu a_\nu^\bullet)_{\nu=+,\bullet}^{\mu=-,\bullet} = 0.$$

Proof. Since M_t commutes with $\mathfrak{G}_t$, M_t commutes with $W(s,a)$ for any $0 \leq s \leq t$ and $a \in \mathfrak{a}$. Therefore, $W(t,a)$ commutes with dM_t for any $a \in \mathfrak{a}$. On the other hand, for any $a \in \mathfrak{a}$ by (2) we have

$$M_t dW(t,a) = W(t,a)M_t d\Lambda(t,a) = W(t,a)d\Lambda(t,a)M_t$$

for the last identity we use the adaptedness of M_t. Hence M_t commutes with $dW(t,a)$ for any $a \in \mathfrak{a}$. Therefore, by applying the quantum Itô formula (see [1], [10]), we have

$$d[M_t, W(t,a)] = [dM_t, W(t,a)] + [M_t, dW(t,a)] + [dM_t, dW(t,a)] = 0$$

which implies that

$$d[M_t, W(t,a)] = \sum_{\mu=-}^{\bullet}\sum_{\nu=\bullet}^{+} (M_\bullet^\mu W(t,a)a_\nu^\bullet - W(t,a)a_\bullet^\mu M_\nu^\bullet)\, d\Lambda_\mu^\nu(t)$$

$$= W(t,a)\sum_{\mu=-}^{\bullet}\sum_{\nu=\bullet}^{+} (M_\bullet^\mu a_\nu^\bullet - a_\bullet^\mu M_\nu^\bullet)\, d\Lambda_\mu^\nu(t) = 0.$$

Therefore, by the independence of the integrators $d\Lambda_\mu^\nu$ (see [13], [14]), we have $\mathbf{M} \bullet \mathbf{a} = \mathbf{a} \bullet \mathbf{M}$. ∎

Let $p \in \mathbf{R}$ and let $\mathcal{A}_p$ the quantum Itô algebra consisting of all $(M^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ for a regular martingale $M = \{M_t\}_{t\geq 0} \subset \mathcal{L}(\mathcal{F}_p)$ and $(a^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet}$ for all $a \in \mathfrak{a}$. If $\{M_t\}_{t>0}$ is a regular martingale in $\mathcal{L}(\mathcal{F}_p)$ such that M_t commutes with $\mathfrak{G}_t$ for each $t \in \mathbf{R}_+$, then by Proposition 5.6, $(M_{t}{}^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet} \in \mathfrak{a}^{\mathrm{c}}$, where $\mathfrak{a}^{\mathrm{c}}$ is the commutant of $\mathfrak{a}$.

Let $\{\mathfrak{B}_t\}_{t\in\mathbf{R}_+}$ be a quantum filtration satisfying that for any $0 \leq s \leq t$, $\mathbf{E}_{s)}Z \in \mathfrak{B}_s$ for any $Z \in \mathfrak{B}_t$, where $[\mathbf{E}_{s)}Z]Y(\eta_0 \otimes \delta_\emptyset) = \mathbf{E}_{s)}(ZY(\eta_0 \otimes \delta_\emptyset)))$ for any $Y \in \mathfrak{B}$.

A family $\{\varsigma_t\}_{t\geq 0}$ of output state ς_t on $\mathfrak{B}_t$ for each $t \in \mathbf{R}_+$ is said to be *regular with respect to a Radon measure* $\mathfrak{m}$ on $\mathbf{R}_+$, or simply *regular* if for any $0 \leq s < t$ and $Z \in \mathfrak{B}_s$, $Y \in \mathfrak{B}_t$

$$|\varsigma_t(Z^*Y) - \varsigma_s(Z^*\mathbf{E}_{s)}Y)|^2 \leq \mathfrak{m}([s,t])\omega_p(Z^*Z)\omega_p(Y^*Y). \tag{9}$$

THEOREM 5.7. *Let $\{\mathfrak{B}_t\}_{t\in\mathbf{R}_+}$ be a quantum filtration and ς_t an output state on $\mathfrak{B}_t$ for each $t \in \mathbf{R}_+$ such that (4) holds. If for each $t \in \mathbf{R}_+$, ς_t is $\omega_p|_{\mathfrak{B}_t}$-absolutely continuous and regular, then the martingale $\{P_{\zeta_t}\}_{t\geq 0}$ given as in Theorem 5.3 is regular.*

Proof. Let $0 \leq s \leq t$. Then by (9) for any $Z \in \mathfrak{B}_s$ and $Z \in \mathfrak{B}_t$

$$\begin{aligned}|\langle\!\langle (P_{\varsigma_t} - P_{\varsigma_s})\, Z(\eta\otimes\delta_\emptyset) |\, Y(\eta\otimes\delta_\emptyset)\rangle\!\rangle_p|^2 &= |\varsigma_t(Z^*Y) - \varsigma_s(Z^*\mathbf{E}_{s)}Y)|^2 \\ &\leq \mathfrak{m}([s,t])\omega_p(Z^*Z)\omega_p(Y^*Y) \\ &= \mathfrak{m}([s,t])\, \|\, Z(\eta_0\otimes\delta_\emptyset)\,\|_p^2\, \|\, Y(\eta_0\otimes\delta_\emptyset)\,\|_p^2\end{aligned}$$

which implies that for any $Z \in \mathfrak{B}_s$

$$\|\,(P_{\varsigma_t} - P_{\varsigma_s})\, Z(\eta\otimes\delta_\emptyset)\,\|_p^2 \leq \mathfrak{m}([s,t])\,\|\, Z(\eta_0\otimes\delta_\emptyset)\,\|_p^2 .$$

Since $\eta_0 \otimes \delta_\emptyset$ is a cyclic vector of $\mathfrak{B}_t$, $\{P_{\zeta_t}\}_{t\geq 0}$ is regular. ∎

THEOREM 5.8. *If $\{\sigma_t\}_{t\in\mathbf{R}_+}$ is a family of states $\sigma_t : \mathfrak{G}_t \to \mathbf{C}$ such that*

$$\sigma_t(Z) = \sigma_s(Z), \qquad Z \in \mathfrak{G}_s, \quad s \leq t$$

and, for each $t \geq 0$, σ_t is $\varrho_p|_{\mathfrak{G}^t}$-absolutely continuous and regular, then there exists a unique L^1-martingale $\{P_{\sigma_t}\}_{t\geq 0}$ of operators $P_{\sigma_t} \in \mathcal{L}(\mathcal{F}_p)$ with $(P_{\sigma_t}{}^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet} \in \mathfrak{a}^{\mathrm{c}}$ such that

$$\sigma_t(Z) = \langle\!\langle \delta_\emptyset |\, P_{\sigma_t} Z\delta_\emptyset\rangle\!\rangle_p, \qquad Z \in \mathfrak{G}_t.$$

Proof. Since σ_t is $\varrho_p|_{\mathfrak{G}^t}$-absolutely continuous, by Corollary 5.4 $P_{\sigma_t} \in \mathcal{L}(\mathcal{F}_p)$. On the other hand, P_{σ_t} commutes with $\mathfrak{G}_t$ and hence $(P_{\sigma_t}{}^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet} \in \mathfrak{a}^{\mathrm{c}}$. ∎

THEOREM 5.9. *Let $\{\rho_t\}_{t\in\mathbf{R}_+}$ be a family of states $\rho_t : \mathfrak{g}_t \to \mathbf{C}$ such that*

$$\rho_t(g) = \rho_s(g), \qquad g \in \mathfrak{g}_s, \quad s \leq t$$

and, for each $t \geq 0$, ρ_t is $\lambda_p|_{\mathfrak{g}_t}$-absolutely continuous. If $\{\sigma_t\}_{t\in\mathbf{R}_+}$ is regular, where σ_t is defined by

$$\sigma_t(Z) = \rho_t(g), \qquad Z = W(t,g) \in \mathfrak{G}_t, \quad g \in \mathfrak{g}_t,$$

then there exists a unique L^1-martingale $\{K_{\rho_t}\}_{t\geq 0} \subset K_{\rho_t} \in \mathcal{L}(\mathcal{F}_p)$ with $(K_{\rho_t}{}^\mu_\nu)^{\mu=-,\bullet}_{\nu=+,\bullet} \in \mathfrak{a}^{\mathrm{c}}$ such that

$$\rho_t(g) = \langle\!\langle \delta_\emptyset |\, K_{\rho_t} W(t,g)\delta_\emptyset\rangle\!\rangle_p, \qquad g \in \mathfrak{g}_t.$$

Proof. The proof is similar to the proof of Theorem 5.8 as an application of Theorem 5.5. ∎

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