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REMARKS ON q-CCR RELATIONS FOR $|q| > 1$

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Abstract. In this paper we give a construction of operators satisfying q-CCR relations for $q > 1$:

$$A(f)A^*(g) - A^*(g)A(f) = q^N \langle f, g\rangle I$$

and also q-CAR relations for $q < -1$:

$$B(f)B^*(g) + B^*(g)B(f) = |q|^N \langle f, g\rangle I,$$

where N is the number operator on a suitable Fock space $\mathcal{F}_q(\mathcal{H})$ acting as

$$N x_1 \otimes \cdots \otimes x_n = n\, x_1 \otimes \cdots \otimes x_n.$$

Some applications to combinatorial problems are also given.

1. Introduction. Generalized Brownian motion (Gaussian random field) (GBM), $G(f)$ were introduced in our papers with R. Speicher [15, 16, 18], where the main examples came from the q-CCR relation for $q \in [-1, 1]$

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle I,$$

here f, g are in a real Hilbert space $\mathcal{H}$ and

$$G(f) = a(f) + a^*(f).$$

Other examples of (GBM) were constructed by M. Bożejko and M. Guţă [10], M. Bożejko and J. Wysoczański [19, 20], M. Guţă and H. Maassen [26, 27], M. Bożejko and H. Yoshida [21] and recently A. Buchholz [23] discovered a very interesting new class of (GBM).

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In this note we present a construction of q-CCR relations for $q > 1$ and q-CAR relations for $q < -1$.

We also give a simple application to a combinatorial problems on the set $\mathcal{P}_2(2n)$ of 2-partitions of the set $\{1, 2, \ldots, 2n\}$.

Namely for a 2-partition $\mathcal{V} \in \mathcal{P}_2(2n)$ we have

$$pbr(\mathcal{V}) + cr(\mathcal{V}) = \frac{1}{2} ip(\mathcal{V}).$$

Here $cr(\mathcal{V})$ is the number of crossings, which is given by the number of pairs of blocks of $\mathcal{V}$ which cross, that is:

$$cr(\mathcal{V}) = \sharp\{((a,b),(c,d)) : (a,b),(c,d) \in \mathcal{V}, \text{ and } a < c < b < d\}.$$

Also for $\mathcal{V} \in \mathcal{P}_2(2n)$ we define the number of pairs embracings introduced by de Medicis and Viennot [37] and also studied by A. Nica [39] as

$$pbr(\mathcal{V}) = \sharp\{((a,b),(c,d)) : (a,b),(c,d) \in \mathcal{V}, \text{ and } a < c < d < b\}.$$

In the same way we define the number of inner points

$$ip(\mathcal{V}) = \sum_{(i,j)\in\mathcal{V}} inpt(i,j),$$

where for a block $(i,j) \in \mathcal{V}$ we let $inpt(i,j)$ be the number of natural k with $i < k < j$.

2. Generalized Brownian motion. Let $\mathcal{H}$ be a real Hilbert space. A family of self-adjoint operators $G(f) = G(f)^*$, $f \in \mathcal{H}$, is called Generalized Gaussian random variables or Generalized Brownian Motion (GBM), if there exists a state ε on the von Neumann algebra generated by $G(f), f \in \mathcal{H}$ and a function

$$t : \bigcup_{n=1}^{\infty} \mathcal{P}_2(2n) \to \mathbb{C}$$

such that the following generalized Wick formula holds:

$$\varepsilon\left(G(f_1)\ldots G(f_k)\right) = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} t(\mathcal{V}) \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle & \text{if } k = 2n. \end{cases}$$

If the dimension of the Hilbert space is infinite, then the above definition is equivalent to the following one (see F. Lehner – II part, [35]): for each orthogonal linear map $O : \mathcal{H} \to \mathcal{H}$

$$\varepsilon\left(G(f_1)\ldots G(f_k)\right) = \varepsilon(G(O(f_1))\ldots G(O(f_k))).$$

A typical example of (GBM) was obtained by R. Speicher and myself [15, 16] in 1991 using q-CCR relations for $-1 \leq q \leq 1$, then putting $G(f) = a(f) + a^*(f)$ and knowing that $a(f)\Omega = 0$ we obtain the following Wick formula interpolating between $q = 1$ (Bose-Einstein statistics), $q = -1$ (Fermi-Dirac statistics) and $q = 0$ (Maxwell statistics):

$$\langle G(f_1)\ldots G(f_{2n})\Omega, \Omega\rangle = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{cr(\mathcal{V})} \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle.$$

Here $cr(\mathcal{V})$ is the number of crossings, which is given by the number of pairs of blocks of $\mathcal{V}$ which cross.

To obtain the above Wick formula we need a deformed Fock space $\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})$ constructed by the completion of the free Fock space

$$\mathcal{F}(\mathcal{H}_{\mathbb{C}}) = \mathbb{C}\Omega \oplus \mathcal{H}_{\mathbb{C}} \oplus \mathcal{H}_{\mathbb{C}}^{\otimes 2} \oplus \dots$$

by introducing a new scalar product on $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ as follows. For $\xi, \eta \in \mathcal{H}_{\mathbb{C}}^{\otimes n}$ we define a q-deformed scalar product

$$\langle \xi, \eta \rangle_q = \langle P_q^{(n)} \xi, \eta \rangle,$$

where

$$P_q^{(n)} = \sum_{\pi \in \mathrm{S}(n)} q^{cr(\pi)} \pi,$$

and for a permutation $\pi \in \mathrm{S}(n)$,

$$cr(\pi) = \sharp \, \{(i,j) : 1 \leq i < j \leq n, \text{ and } \pi(i) > \pi(j)\}.$$

In the construction of $\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})$ we need the positivity of the operator $P_q^{(n)}$ for $-1 < q < 1$, which was done by[11] and [15, 16]. Then we form a creation operator

$$a^+(f)\xi = f \otimes \xi$$

and an annihilation operator

$$a_q(f) x_1 \otimes \ldots \otimes x_n = \sum_{k=1}^{n} q^{k-1} \langle f, x_k \rangle x_1 \otimes \ldots \otimes \check{x}_k \ldots \otimes x_n.$$

Hence for $f \in \mathcal{H}$ on the q-Fock space $\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})$ we have

$$[a^+(f)]^* = a_q(f).$$

Moreover for $-1 < q < 1$ and $\|f\| = 1$ we obtain

$$\|a(f)\| = \|a^+(f)\| = \begin{cases} (1-q)^{-1/2} & \text{if } 0 \leq q < 1, \\ 1 & \text{if } -1 < q \leq 0, \end{cases}$$

and the vacuum state

$$\varepsilon(\cdot) = \langle \cdot \, \Omega, \Omega \rangle$$

is a trace on the von Neumann algebra generated by $G(f), f \in \mathcal{H}$.

Also for $\|f\| = 1$, we have

$$\varepsilon(G(f)^k) = \int_{-\infty}^{\infty} x^k d\mu_q(x),$$

where

$$d\,\mu_q(x) = \frac{1}{2\pi} q^{-1/8} \Theta_1 \left(\frac{\vartheta}{\pi}, \frac{1}{2\pi i} \log q \right) dx,$$

here Θ_1 is the well-known Jacobi function (see Maassen, van Leuven [36]) and $2\cos\vartheta = x\sqrt{1-q}$.

The corresponding orthogonal polynomials with respect $d\mu_q(x)$ are q-Hermite polynomials satisfying the following recurrence relations, see Szegö [46]:

$$x H_n(x) = H_{n+1}(x) + (q^n - 1)/(q-1) H_{n-1}(x).$$

3. q**-CCR for** $q > 1$**.** In this section we prove the existence of q-Gaussian fields

$$G(f) = A(f) + A^+(f)$$

for $q > 1$ satisfying the following q-CCR relation:

$$A(f)A^*(g) - A^*(g)A(f) = q^N \langle f, g\rangle I,$$

where f, g are in a real Hilbert $\mathcal{H}$. The operator N is the number operator on a suitable Fock space $\mathcal{F}_q(\mathcal{H})$ acting as

$$N x_1 \otimes \cdots \otimes x_n = n\, x_1 \otimes \cdots \otimes x_n.$$

For this sake we consider the operator

$$\widetilde{P_q^{(n)}} = P_q^{(n)} w_0^{(n)} = w_0^{(n)}\, P_q^{(n)},$$

where $w_0^{(n)} = w_0$ is the permutation $(1, n)(2, n-1)(3, n-2)\ldots$.

Since for $\sigma \in S(n)$ we have

$$cr(\sigma w_0) = cr(w_0 \sigma) = \frac{1}{2} n(n-1) - cr(\sigma)$$

therefore we get

$$\widetilde{P_q^{(n)}} = P_q^{(n)} w_0^{(n)} = w_0^{(n)} P_q^{(n)} = q^{\frac{1}{2}n(n-1)} P_{1/q}^{(n)}.$$

So if $q > 1$, we have that the operator $\widetilde{P_q^{(n)}}$ is positive, since by the results of [11, 15, 16], the operator $P_{1/q}^{(n)}$ is positive.

We repeat the construction as before; we define a new scalar product on $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ as follows. For $\xi, \eta \in \mathcal{H}_{\mathbb{C}}^{\otimes n}$ we define a q-deformed scalar product

$$\langle \xi, \eta\rangle_q = \langle \widetilde{P_q^{(n)}} \xi, \eta\rangle.$$

Let $\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})$ be the completion of the free Fock space under the new scalar product. The creation operator is equal to

$$A^+(f)\xi = f \otimes \xi, \quad \text{for } \xi \in \mathcal{F}_q(\mathcal{H}_{\mathbb{C}}) \text{ and } f \in \mathcal{H}.$$

One can calculate that the annihilation operator

$$A(f) = [A^+(f)]^*$$

is of the form:

$$A(f)(x_1 \otimes \ldots \otimes x_n) = \sum_{k=1}^{n} q^{n-k} \langle f, x_k\rangle\, x_1 \otimes \ldots \otimes \check{x}_k \ldots \otimes x_n = q^{n-1} a_{1/q}(f) x_1 \otimes \ldots \otimes x_n.$$

Remark 3.1. If $\mathcal{H}$ is one-dimensional and $q > 1$, then the left and right q-annihilation operators are the same and

$$A(f) = a_q(f) \; for \;\; f \in \mathcal{H}.$$

This fact will be useful for us later.

The proof follows from the fact that if $\mathcal{H} = \mathbb{C} f$, then

$$A(f) f \otimes \cdots \otimes f = (1 + q + \cdots + q^{n-1})\langle f, f\rangle = (q^{n-1} + \cdots + q + 1)\langle f, f\rangle = a_q(f)\; f \otimes \cdots \otimes f.$$

Now we can state

PROPOSITION 3.1. *(i) For $f, g \in \mathcal{H}$, and $q > 1$, we have q-CCR relations*

$$A(f)A^*(g) - A^*(g)A(f) = q^N \langle f, g\rangle I,$$

where

$$N(x_1 \otimes \ldots \otimes x_n) = n(x_1 \otimes \ldots \otimes x_n).$$

(ii) If

$$f_j \in \mathcal{H}, \ \ G(f_j) = A(f_j) + A^+(f_j),$$

then the following Wick formula holds:

$$\langle G(f_1)\ldots G(f_{2n})\Omega, \Omega\rangle = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{pbr(V)} \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle.$$

(iii) For $f \in \mathcal{H}$ the operator $G(f) = A(f) + A^+(f)$ is symmetric and of the deficiency index $(1,1)$ and the corresponding orthogonal polynomials are q-Hermite for $q > 1$

$$xH_n(x) = H_{n+1}(x) + \frac{q^n - 1}{q - 1} H_{n-1}(x).$$

Since $A(f) = A^+(f)^*$, $A^*(f)$ is the closure of $A^+(f)$.

Proof. (i) Since

$$A(f) = q^{n-1} a_{1/q}(f) \text{ on } \mathcal{H}^{\otimes n}$$

and also

$$a_{1/q}(f)a^+(g) - q^{-1}a^+(g)a_{1/q}(f) = \langle f, g\rangle I,$$

multiplying the last equation by q^n we get

$$(q^n a_{1/q}(f))a^+(g) - a^+(g)(q^{n-1}a_{1/q}(f)) = q^n\langle f, g\rangle I,$$

and this gives

$$A(f)A^*(g) - A^*(g)A(f) = q^N \langle f, g\rangle I.$$

The proof of (ii) is classical and coming by simple induction on n, so we omit it.

To get (iii) we need to consider the one-dimensional case and it is well known (see Królak [32]) that for $q > 1$ we get undeterminate moment problem, so the operator has index $(1,1)$, (see Achieser [2], Ismail, Manson [31]) and from the construction of the creation and annihilation operators we get q-Hermite polynomials. ■

4. Combinatorial applications.

Next we obtain de Medicis-Viennot (see [37]) interesting combinatorial formula:

COROLLARY 4.1. *For $q \in \mathbb{C}$ we have*

$$\sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{cr(\mathcal{V})} = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{pbr(\mathcal{V})}.$$

Proof. Since for $q > 1$ and $\mathcal{H} = \mathbb{C}e$ is one dimensional and $\|e\| = 1$, we have

$$\varepsilon((A(e) + A^+(e))^{2n}) = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{pbr(\mathcal{V})}.$$

On the other hand

$$A^+(e) = a^+(e) \ \text{ and } \ A(e) = a_q(e).$$

Therefore by [15, 16] we get

$$\varepsilon((A(e)+A^{+}(e))^{2n}) = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{cr(\mathcal{V})}.$$

Hence for $q>1$ we obtain

$$\sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{cr(\mathcal{V})} = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{pbr(\mathcal{V})},$$

and then by analytic continuation that equality holds for all $q\in\mathbb{C}$. ■

Now we present another application of the q-Gaussian field to get a new combinatorial identity between the three functions on 2-partitions of $2n$-element set:

$$cr(\mathcal{V}),\ pbr(\mathcal{V}) \text{ and } ip(\mathcal{V}).$$

From Proposition 3.1 and Theorem 6 in Bożejko and Yoshida's paper [21], putting there $s=q^{1/2}$ and q^{-1} instead q we obtain the following proposition:

PROPOSITION 4.1. *If $q\geq 1$, $f_j\in\mathcal{H}$ and $G(f_j)=A(f_j)+A^{+}(f_j)$ are q-Gaussian random variables, then*

$$\begin{aligned}\langle G(f_1)\dots G(f_{2n})\Omega,\Omega\rangle &= \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{1/2ip(\mathcal{V})-cr(\mathcal{V})} \prod_{(i,j)\in\mathcal{V}} \langle f_i,f_j\rangle\\ &= \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} q^{pbr(\mathcal{V})} \prod_{(i,j)\in\mathcal{V}} \langle f_i,f_j\rangle.\end{aligned}$$

As a corollary we get the following combinatorial result:

COROLLARY 4.2. *For 2-partitions $\mathcal{V}\in\mathcal{P}_2(2n)$ we have:*

$$pbr(\mathcal{V})+cr(\mathcal{V}) = \frac{1}{2}\, ip(\mathcal{V}).$$

The proof of the corollary follows at once from the following lemma:

LEMMA 4.1. *If for $f_j\in\mathcal{H}$ and $t_i:\mathcal{P}_2(2n)\to\mathbb{C}$ we have*

$$\sum_{\mathcal{V}\in\mathcal{P}_2(2n)} t_1(\mathcal{V}) \prod_{(i,j)\in\mathcal{V}} \langle f_i,f_j\rangle = \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} t_2(\mathcal{V}) \prod_{(i,j)\in\mathcal{V}} \langle f_i,f_j\rangle,$$

then

$$t_1(\mathcal{V}) = t_2(\mathcal{V}) \text{ for all } \mathcal{V}\in\mathcal{P}_2(2n).$$

Proof. It is not difficult to show that for a given 2-partition $\tilde{\mathcal{V}}\in\mathcal{P}_2(2n)$ we can find a suitable family $f_j\in\mathcal{H}$ such that

$$\sum_{\mathcal{V}\in\mathcal{P}_2(2n)} t_1(\mathcal{V}) \prod_{(i,j)\in\mathcal{V}} \langle f_i,f_j\rangle = t_1(\tilde{\mathcal{V}})$$

and this finishes the proof of Lemma 4.1 and Corollary 4.2.

5. Remarks on q-CAR relations for $q<-1$. For $q<-1$, a possible definition of q-CAR relations is the following:

$$B(f)B^{*}(g)+B^{*}(g)B(f) = |q|^{N}\langle f,g\rangle I.$$

For the construction of such relations we introduce again a new scalar product on the free Fock space

$$\mathcal{F}(\mathcal{H}_{\mathbb{C}}) = \mathbb{C}\Omega \oplus \mathcal{H}_{\mathbb{C}} \oplus \dots$$

using as before the new positive operator

$$\widehat{P}_q^{(n)} = |q|^{\binom{n}{2}} P_{1/q}^{(n)}.$$

Then if we take, as always, the new creation as the free left creation operator $B^+(f) = a^+(f)$, and the annihilation operator $B(f) = |q|^{N-1} a_{1/q}(f)$ then one can verify that the q-CAR relation holds:

$$B(f)B^*(g) + B^*(g)B(f) = |q|^N \langle f, g\rangle I.$$

Similarly one can show that the following proposition holds:

Proposition 5.1. *If $q \geq 1, f_j \in \mathcal{H}$ and $G(f_j) = B(f_j) + B^+(f_j)$ are q-Gaussian random variables, then*

$$\begin{aligned}\langle G(f_1)\dots G(f_{2n})\Omega, \Omega\rangle &= \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} |q|^{1/2ip(\mathcal{V})} q^{-cr(\mathcal{V})} \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle \\ &= \sum_{\mathcal{V}\in\mathcal{P}_2(2n)} |q|^{pbr(\mathcal{V})} (-1)^{cr(\mathcal{V})} \prod_{(i,j)\in\mathcal{V}} \langle f_i, f_j\rangle.\end{aligned}$$

Also $B^*(f)$ is the closure of $B^+(f)$.

Remark 5.1. As in previous cases also for $q < -1$, we obtain a similar recurrence relation for the q-Hermite polynomials from the above construction of the q-creation and q-annihilation operators:

$$xH_n(x) = H_{n+1}(x) + (-1)^{n-1}\frac{q^n - 1}{q - 1} H_{n-1}(x).$$

Problem 5.1. *Try to get the q-Mehler formula for $q < -1$ similar to that in the case $q > 1$, which was done in the paper of Ismail and Masson [31].*

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