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A PROJECTIVE CENTRAL LIMIT THEOREM AND INTERACTING FOCK SPACE REPRESENTATION FOR THE LIMIT PROCESS

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Abstract. Accardi et al. proved a central limit theorem, based on the notion of projective independence. In this note we use the symmetric projective independence to present a new version of that result, where the limiting process is perturbed by the insertion of suitable test functions. Moreover we give a representation of the limit process in 1-mode type interacting Fock space.

1. Introduction. The aim of this paper is to present a quantum central limit theorem based on the *symmetric projective independence* and give a GNS representation for the limit in an interacting Fock space (IFS).

Projective independence was introduced in [1] as a notion which abstracts, into an algebraic setting, the factorization rule to compute the mixed moments underlying the central limit theorem of [3]. Indeed, in the paper [3], the authors proved that each mean zero probability measure on the real line with finite moments of any order can be obtained as a central limit (in the sense of convergence of moments) of self-adjoint random variables, which are sum of creation, annihilation and preservation operators in 1-mode type interacting Fock spaces (see [2], [3], [6], [7] for more details on interacting Fock spaces). This result was also alternatively proved in [8] by a convolution of measures approach.

Moreover the projective independence was used by the authors to prove an algebraic central limit theorem and show that, under some conditions, a GNS representation for the limit can be realized in 1-mode type interacting Fock space.

In this note we investigate what happens, either for the central limit theorem or the GNS representation, after perturbing each term of the limiting process considered in [1] with a Riemann integrable function on the unit interval of the real line.

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The paper is organized as follows. Section 2 is devoted to recall some definitions, such as algebraic probability space, singleton condition and uniform boundedness of the mixed moments, which will be used in the successive sections. In Section 3 the notion of symmetric projective independence is given and its relation with the singleton condition is analyzed. This allows us to prove our main result, i.e. a central limit theorem. The main difference between the result presented in these notes and that obtained in [1] consists in the fact that here each term of the limiting process is perturbed by a Riemann integrable function on the unit interval. As a consequence the central limit theorem has to be modified: we need some stronger conditions on the limiting process and, moreover, the notion of independence used is less general with respect to that considered in [1]. In Section 4 we find a suitable representation for the limit obtained in our main theorem. In fact, by the reconstruction theorem of [4], the limit process of an algebraic central limit theorem is given by a family of elements of an algebraic probability space $(\mathcal{B}, \psi)$. We show that a GNS representation of such a space is realized by a 1-mode type interacting Fock space on $\mathbf{L}^2([0,1])$. More precisely the ψ-moments of a family of elements of $\mathcal{B}$ are the vacuum moments of creation and annihilation operators in 1-mode type interacting Fock space, whose test functions are exactly the perturbational terms of the limiting process.

2. Definitions and notations. This section can be seen as a collection of notations and definitions which are the preparatory tools to reach our main result.

An algebraic probability space is a pair $\{\mathcal{A}, \varphi\}$ where $\mathcal{A}$ is a unital $*$-algebra with unit 1 and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear, normalized ($\varphi(1) = 1$) and positive ($\varphi(a^*a) \geq 0$, for all $a \in \mathcal{A}$) functional.

If $(\mathcal{A}_i)_{i\in\mathcal{I}}$ is a family of (unital) $*$-subalgebras of $\mathcal{A}$, we will suppose that each $*$-subalgebra is generated by a set of generators $\{a_i^{\varepsilon};\ \varepsilon \in F\}$, where F is a finite set such that $F = F_s \cup F_a$ with $F_s \cap F_a = \emptyset$. F_s and F_a are called respectively the symmetric and non-symmetric part of F. The upper suffices in $(a_i^{\varepsilon})_{i\in\mathcal{I}}$ are needed by concrete examples of central limit theorems and are natural whenever $\{\mathcal{A}, \varphi\}$ can be constructed starting from the 1-mode type interacting Fock space (IFS), as shown in the following example (see[1] also).

Example 1. Let $\mathcal{H}$ be a separable Hilbert space and $\{\lambda_n\}_{n\in\mathbb{N}}$ a sequence of positive, real numbers such that $\lambda_0 = \lambda_1 := 1$ and for any $m > n$ $\lambda_m = 0$ if $\lambda_n = 0$. For each $n \geq 2$, on the algebraic tensor product space $\mathcal{H}^{\odot n}$ we define a pre-scalar product as follows:

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n \rangle := \lambda_n \prod_{j=1}^{n} \langle f_j, g_j \rangle$$

for any $f_1, \ldots, f_n, g_1, \ldots, g_n \in \mathcal{H}$. After taking quotient and completing one gets a Hilbert space $\mathcal{H}_n$ and, with the conventions $\mathcal{H}_0 := \mathbb{C}, \mathcal{H}_1 := \mathcal{H}$, the 1-mode type IFS over $\mathcal{H}$ with interacting functions $\{\lambda_n\}_{n\in\mathbb{N}}$ is given by

$$\Gamma(\mathcal{H},\{\lambda_n\}_{n\in\mathbb{N}}) := \mathbb{C}\Phi \oplus \mathcal{H} \oplus \bigoplus_{n=2}^{\infty} \mathcal{H}_n$$

where $\Phi := 1 \oplus 0 \oplus 0 \oplus \cdots$ denotes the vacuum vector.

The creation operator $A^+(f)$ with $f \in \mathcal{H}$ as the test function is defined in the following way:

$$A^+(f)\Phi := f,$$

$$(A^+(f)(f_1 \otimes \cdots \otimes f_n)) := f \otimes f_1 \otimes \cdots \otimes f_n, \quad \forall n \in \mathbb{N}, \quad \forall\, f_1, \ldots, f_n \in \mathcal{H}.$$

The annihilation operator $A(f)$ is defined as the adjoint of the creation operator:

$$A(f)\Phi := 0,$$

$$(A(f)(f_1 \otimes \cdots \otimes f_n)) := \frac{\lambda_n}{\lambda_{n-1}}\langle f, f_1\rangle f_2 \otimes \cdots \otimes f_n, \quad \forall n \in \mathbb{N}, \quad \forall\, f_1, \ldots, f_n \in \mathcal{H}$$

with the convention $\frac{0}{0} := 0$. Given $\alpha := (\alpha_m) \in l^\infty(\mathbb{R})$ and the identity operator $I \in \mathbf{B}(\mathcal{H})$, we define the preservation operator with intensity (α, I):

$$\Lambda_\alpha(I)(f_1 \otimes \cdots \otimes f_n) := \alpha_n(f_1 \otimes \cdots \otimes f_n) \;\; \forall n \in \mathbb{N}, \quad \forall\, f_1, \ldots, f_n \in \mathcal{H}.$$

Furthermore we consider

$$\mathcal{A} := *\text{-alg}\{I, A(f),\ \Lambda_\alpha(I) : f \in \mathcal{H}, \alpha = (\alpha_m) \in l^\infty(\mathbb{R})\},$$

$$\varphi := \langle \Phi, \cdot\Phi\rangle,$$

hence it follows $\{\mathcal{A}, \varphi\}$ is an algebraic probability space. If $\varepsilon \in \{-1, 0, 1\}$ and

$$A^\varepsilon(f, \alpha, I) := \begin{cases} A(f) & \text{if } \varepsilon = -1, \\ \Lambda_\alpha(I) & \text{if } \varepsilon = 0, \\ A^+(f) & \text{if } \varepsilon = 1, \end{cases}$$

then $F = \{-1, 0, 1\}$, $F_s = \{-1, 1\}$, $F_a = \{0\}$.

Let S be a nonempty ordered set. Recall that a partition of S is a family $\sigma = (V_i)_{i \in \mathcal{I}}$ of mutually disjoint nonempty subsets of S whose union is S and $\mathcal{I}$ is an index set. Any $V_i \in \sigma$ is called a block of the partition σ. Denote by $\mathcal{P}(S)$ the set of all partitions of S and for any $q \in \mathbb{N}^*$, let $\mathcal{P}(q) := \mathcal{P}(\{1, \ldots, q\})$. We identify two partitions $\sigma_1 = (V_i)_{i \in \mathcal{I}}$ and $\sigma_2 = (U_j)_{j \in \mathcal{I}}$ in $\mathcal{P}(S)$ if there exists π a permutation on $\mathcal{I}$ such that for any $i \in \mathcal{I}$ $V_i = U_{\pi(i)}$. A partition σ of S uniquely defines an equivalence relation $\sim_\sigma$ on S where, for each $i, j \in S$, $i \sim_\sigma j$ if and only if i, j belong to the same block of σ.

$\sigma = \{V_1, \ldots, V_l\} \in \mathcal{P}(S)$ is called a *pair partition* if $|V_i| = 2$ for any $i = 1, \ldots, l$, where $|\cdot|$ denotes the cardinality of a set.

Let $q \in \mathbb{N}^*$. Given a map $k : \{1, \ldots, q\} \to \mathcal{I}$, for any $l = 1, \ldots, q$ we indicate its image by k_l or $k(l)$ and denote

- Range(k) the range of k, i.e.

$$\text{Range}(k) := \{\overline{k}_1, \ldots, \overline{k}_m\} \subset \mathcal{I}, \quad m \in \mathbb{N}^*,\ m \leqslant q, \overline{k}_i \neq \overline{k}_j \text{ for } i \neq j$$

- for $j = 1, \ldots, m, V_{k,j} := k^{-1}(\overline{k}_j) = \{l \in \{1, \ldots, q\} : k(l) = \overline{k}_j\}$.

Clearly the $V_{k,j}$'s are the blocks of a partition $\sigma := \{V_{k,1}, \ldots, V_{k,m}\} \in \mathcal{P}(q)$. If $j = 1, \cdots, m$, for a fixed $V_{k,j} = k^{-1}(\overline{k}_j)$, denote $\varepsilon_{V_{k,j}}$ the restriction of ε to $V_{k,j}$, i.e. $\varepsilon_{V_{k,j}} := \{\varepsilon_l :\ l \in V_{k,j}\}$.

The symbol $\mathcal{I}^{\{1,\dots,q\}}$ denotes the set of all mappings from $\{1,\dots,q\}$ into $\mathcal{I}$. If k, $l\in\mathcal{I}^{\{1,\dots,q\}}$, we say that k is equivalent to l, and we write $k\approx l$, if they induce the same partition of $\{1,\dots,q\}$. Namely:

(i) $|\mathrm{Range}(k)|=|\mathrm{Range}(l)|=: m$;
(ii) $k^{-1}(\overline{k}_j)=l^{-1}(\overline{l}_j)$, for all $j=1,\dots,m$.

We denote $[k]:=\{l\in\mathcal{I}^{\{1,\dots,q\}}$ such that $k\approx l\}$ the $\approx$-equivalence class of k.

Conversely, any partition $\sigma\in\mathcal{P}(q)$ defines a unique equivalence class $[k]$ where k is any map taking constant value on every block of σ. Therefore we have a natural identification $\mathcal{I}^{\{1,\dots,q\}}{}_{/\approx}\equiv\mathcal{P}(q)$ and in the following we will often use the identification $[k]\equiv\sigma$.

In Section 3 we deal with projective independence and the related central limit theorem, where an important role is played by the singleton and the uniform boundedness of the mixed moments conditions.

Definition 2.1. Let $\{\mathcal{A},\varphi\}$ be an algebraic probability space and$\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ a family of $*$-subalgebras of $\mathcal{A}$. A family $\{a_i^\varepsilon;\ i\in\mathcal{I},\varepsilon\in F\}$ in $\{\mathcal{A},\varphi\}$ such that for any $i\in\mathcal{I}$ and $\varepsilon\in F$, $a_i^\varepsilon\in\mathcal{A}_i$, is said to satisfy the *singleton condition* (with respect to φ) if for any $n\geqslant 1$, for any choice of $i_1,\dots,i_n\in\mathcal{I}$, $\varepsilon_1,\dots,\varepsilon_n\in F$

$$\varphi(a_{i_n}^{\varepsilon_n}\cdots a_{i_1}^{\varepsilon_1})=0 \tag{2.1}$$

whenever $\{i_1,\dots,i_n\}$ has a singleton i_s and $\varphi(a_{i_s}^{\varepsilon_s})=0$.

Definition 2.2. The family $\{a_i^\varepsilon;\ \varepsilon\in F,\, i\in\mathcal{I}\}$ in the algebraic probability space $\{\mathcal{A},\varphi\}$ is said to satisfy the condition of *uniform boundedness of mixed moments* if for each $m\in\mathbb{N}^*$, there exists a positive constant D_m such that

$$|\varphi(a_{i_m}^{\varepsilon_m}\cdots a_{i_1}^{\varepsilon_1})|\leqslant D_m \tag{2.2}$$

for any choice of $i_m,\dots,i_1\in\mathcal{I}$ and and $\varepsilon_m,\dots,\varepsilon_1\in F$.

3. Projective independence and central limit theorem. The notion of projective independence was introduced in [1]. In this paper we will use only its symmetric (i.e. $F_a=\emptyset$) part. The following notations are useful to describe it. Given $\{\mathcal{A},\varphi\}$ an algebraic probability space and a family $\{a_i^\varepsilon;\ \varepsilon\in F_s,\ i\in\mathcal{I}\}$ of elements of $\mathcal{A}$, for any $k:\{1,\dots,q\}\to\mathcal{I}$, and $a_{k_q}^{\varepsilon_q},\dots,a_{k_1}^{\varepsilon_1}\in\mathcal{A}$, we denote:

- $a^{\varepsilon V_{k,j}}:=\overleftarrow{\prod}_{s\in V_{k,j}}a_{\overline{k}_j}^{\varepsilon_s}$ where $\overleftarrow{\prod}$ denotes the product of the $a_{k_l}^{\varepsilon_l}$'s in the same order as they appear in $a_{k_q}^{\varepsilon_q}\cdots a_{k_1}^{\varepsilon_1}$ and, as usual, $V_{k,j}=k^{-1}(\overline{k}_j)$ for any j. We use the convention $\varphi(\overleftarrow{\prod}a^{\varnothing}):=1$.

Definition 3.1. Let $\{\mathcal{A},\varphi\}$ be an algebraic probability space. The family $\{a_i^\varepsilon;\ \varepsilon\in F_s,\ i\in\mathcal{I}\}$ of elements of $\mathcal{A}$ is called *φ-symmetric projectively independent* if for any $q\in\mathbb{N}^*$, any $\varepsilon=(\varepsilon_q,\dots,\varepsilon_1)\in F_s^q$, $k:\{1,\dots,q\}\to\mathcal{I}$, and $a_{k_q}^{\varepsilon_q},\dots,a_{k_1}^{\varepsilon_1}\in\mathcal{A}$, there exist $\omega(k,\varepsilon)\geq 0$ such that

$$\varphi(a_{k_q}^{\varepsilon_q}\cdots a_{k_1}^{\varepsilon_1})=\omega(k,\varepsilon)\prod_{j=1}^{|\mathrm{Range}(k)|}\varphi(a^{\varepsilon V_{k,j}}) \tag{3.1}$$

From now on we will write indifferently $\omega(k,\varepsilon)$ or $\omega(\tau,\varepsilon)$, where τ is the partition induced by the map k on $\{1,\dots,q\}$.

REMARK 3.1. This definition abstracts the situation described in the paper [3], relative to 1–mode type interacting Fock spaces (see Example 1). In that case the explicit form of F_s is $\{-1,+1\}$, whereas the coefficients $\omega(k,\varepsilon)$ are products of the symmetric Jacobi coefficients $\{\omega_n\}$ of the distribution uniquely associated with the interacting Fock space by the Accardi-Bożejko theorem (see [2] for more details). In order to be more explicit we present the following example.

EXAMPLE 2. Let $\mathcal{H}$ be a separable Hilbert space and consider the 1-mode type IFS $\Gamma(\mathcal{H},\{\lambda_n\}_{n\in\mathbb{N}})$ over $\mathcal{H}$ with interacting sequence $\{\lambda_n\}\subset\mathbb{R}_+$ and vacuum vector Φ, as introduced in Example 1. If $A^+(f)$ and $A(f)$ are respectively the creation and annihilation operators with test function f, $\varepsilon\in\{-1,1\}$ and

$$A^\varepsilon(f) := \begin{cases} A(f) & \text{if } \varepsilon=-1,\\ A^+(f) & \text{if } \varepsilon=1,\end{cases}$$

then $F_s=\{-1,1\}$. Let μ be the one dimensional symmetric distribution associated with the IFS with Jacobi parameters $(\omega_n := \frac{\lambda_n}{\lambda_{n-1}})_{n\in\mathbb{N}}$ (see [2], Theorem 5.2). Then, fixed $f_1,f_2\in\mathcal{H}$, one has

$$\langle\Phi, A(f_1)A(f_2)A^+(f_2)A^+(f_1)\Phi\rangle = \omega_1\cdot\omega_2\langle\Phi, A(f_1)A^+(f_1)\Phi\rangle\langle\Phi, A(f_2)A^+(f_2)\Phi\rangle.$$

Therefore $\omega(k,\varepsilon)=\omega_1\omega_2$.

LEMMA 3.1. *Let $\{a_i^\varepsilon;\varepsilon\in F_s,\, i\in\mathcal{I}\}$ be a family of φ-symmetric projectively independent elements of an algebraic probability space $\{\mathcal{A},\varphi\}$ with mean zero, i.e. $\varphi(a_i^\varepsilon)=0$ for any $\varepsilon\in F_s$, $i\in\mathcal{I}$. If $(\alpha_i)_{i\in\mathcal{I}}\subset\mathbb{C}$, then the family $\{\alpha_i a_i^\varepsilon;\varepsilon\in F_s,\, i\in\mathcal{I}\}$ satisfies the singleton condition. If in particular for any $i\in\mathcal{I}$ $\alpha_i:=1$, then $\{a_i^\varepsilon;\varepsilon\in F_s,\, i\in\mathcal{I}\}$ satisfies the singleton condition.*

Proof. Fix $q\in\mathbb{N}^*$ and $k:\{1,\dots,q\}\to\mathcal{I}$. Consider the product $(\alpha_{k_q}a_{k_q}^{\varepsilon_q})\cdots(\alpha_{k_1}a_{k_1}^{\varepsilon_1})$. If there exists $l\in\{1,\dots,q\}$ such that $|V_{k,l}|=1$, by the φ-symmetric projective independence, one has

$$\varphi((\alpha_{k_q}a_{k_q}^{\varepsilon_q})\cdots(\alpha_{k_1}a_{k_1}^{\varepsilon_1})) = \omega(k,\varepsilon)\Big(\prod_{j=1}^{q}\alpha_{k_j}\Big)\varphi(a_{k_l}^{\varepsilon_l})\prod_{\substack{j=1\\ j\neq l}}^{|\mathrm{Range}(k)|-1}\varphi(a^{\varepsilon_{V_{k,j}}})=0.$$

Hence the singleton condition is fulfilled. The last part of the statement clearly follows. ∎

LEMMA 3.2. *Let $\{a_n^\varepsilon;\varepsilon\in F_s,\, n\in\mathbb{N}\}$ be a family in $\{\mathcal{A},\varphi\}$ satisfying the uniform boundedness of the mixed moments and the singleton conditions. Let $\{f_n:\mathbb{C}\to\mathbb{C}\}_{n\in\mathbb{N}}$ be a family of bounded maps, i.e. there exists $\{M_n\}_{n\in\mathbb{N}}$, sequence of positive numbers such that for each $n\in\mathbb{N}$, $z\in\mathbb{C}$ $|f_n(z)|\le M_n$. Then, for any $m\in\mathbb{N}$, $k:\{1,\dots,m\}\to\mathbb{N}$ and $N\in\mathbb{N}$*

$$\lim_{N\to\infty}\frac{1}{N^{m/2}}\sum_{1\le k_1,\dots,k_m\le N}\varphi(a_{k_m}^{\varepsilon_m}f_m(z_m)\cdots a_{k_1}^{\varepsilon_1}f_1(z_1))$$

is equal to zero if m is odd. If $m = 2p$ it is equal to

$$\lim_{N\to\infty} \frac{1}{N^p} \sum_{\substack{\pi:\{1,\dots,2p\}\to\{1,\dots,p\}\\ 2-1 \text{ map}}} \sum_{\substack{\sigma:\{1,\dots,p\}\to\{1,\dots,N\}\\ \text{order preserving}}} \varphi(a^{\varepsilon_{2p}}_{\sigma\circ\pi(2p)} f_{2p}(z_{2p}) \cdots a^{\varepsilon_1}_{\sigma\circ\pi(1)} f_1(z_1)).$$

Proof. Indeed, arguing as in Lemmata 2.3 and 2.4 of [5]

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} \sum_{1\le k_1,\dots,k_m\le N} \varphi(a^{\varepsilon_m}_{k_m} f_m(z_m) \cdots a^{\varepsilon_1}_{k_1} f_1(z_1))$$

$$= \lim_{N\to\infty} \frac{1}{N^{m/2}} \sum_{1\le p\le m} \sum_{\substack{\pi:\{1,\dots,m\}\to\{1,\dots,p\}\\ \text{surjective}}} \sum_{\substack{\sigma:\{1,\dots,p\}\to\{1,\dots,N\}\\ \text{order preserving}}} \varphi(a^{\varepsilon_m}_{k_m} f_m(z_m) \cdots a^{\varepsilon_1}_{k_1} f_1(z_1))$$

where for all $k : \{1,\dots,m\} \to \{1,\dots,N\}$, p denotes the cardinality of the range of k. We show the limit above vanishes when $p < m/2$. In fact, let $C_{m,p}$ be the cardinality of the surjective maps from $\{1,\dots,m\}$ onto $\{1,\dots,p\}$ and $I_N(p)$ the cardinality of the order preserving maps from $\{1,\dots,p\}$ in $\{1,\dots,N\}$. Then, from (2.2),

$$\Big| N^{-m/2} \sum_{\substack{\pi:\{1,\dots,m\}\to\{1,\dots,p\}\\ \text{surjective}}} \sum_{\substack{\sigma:\{1,\dots,p\}\to\{1,\dots,N\}\\ \text{order preserving}}} \varphi(a^{\varepsilon_m}_{k_m} f_m(z_m) \cdots a^{\varepsilon_1}_{k_1} f_1(z_1)) \Big|$$

$$\le N^{-m/2} C_{m,p} D_m \Big| \prod_{j=1}^m f_j(z_j) \Big| I_N(p) \le N^{-m/2} C_{m,p} D_m \Big| \prod_{j=1}^m M_j \Big| \binom{N}{p}$$

Since

$$\lim_{N\to\infty} N^{-m/2} \binom{N}{p} = \lim_{N\to\infty} \frac{1}{p!} N^{(p-m/2)}$$

the result is achieved. Up to slight modifications, the remaining part of the proof runs along the same arguments developed in [5], Lemma 2.4. ∎

Let $\{a^\varepsilon_n; \varepsilon \in F_s,\ n \in \mathbb{N}\}$ be a family of elements in an algebraic probability space $\{\mathcal{A}, \varphi\}$. Denote by $\mathcal{L}([0,1])$ the space of all complex valued Riemann integrable functions defined on $[0,1]$. For $1 \le n \le N$, $f \in \mathcal{L}([0,1])$, consider the centered sum

$$S_N(a^\varepsilon, f) := \sum_{n=1}^N a^\varepsilon_n f(\frac{n}{N})$$

where $\varepsilon \in F_s$.

THEOREM 3.1. *Let $\{a^\varepsilon_n; \varepsilon \in F_s,\ n \in \mathbb{N}\}$ be a family of elements of an algebraic probability space $\{\mathcal{A}, \varphi\}$ which is symmetric φ-projectively independent, with mean zero, i.e. $\varphi(a^\varepsilon_n) = 0$ for all $\varepsilon \in F_s$ and such that for any $\varepsilon_1, \varepsilon_2 \in F_s$, $\varphi(a^{\varepsilon_1}_n a^{\varepsilon_2}_n) = C(\varepsilon_1, \varepsilon_2)$ for any $n \in \mathbb{N}$. We also suppose such a family satisfies the uniform boundedness condition. If $m \ge 1$, $f_1, \dots, f_m \in \mathcal{L}([0,1])$, then*

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} \varphi(S_N(a^{\varepsilon_m}, f_m) \cdots S_N(a^{\varepsilon_1}, f_1))$$

$$= \lim_{N\to\infty} \frac{1}{N^{m/2}} \sum_{1\le k_1,\dots,k_m\le N} \varphi\left(a^{\varepsilon_m}_{k_m} f_m\left(\frac{k_m}{N}\right) \cdots a^{\varepsilon_1}_{k_1} f_1\left(\frac{k_1}{N}\right)\right)$$

is zero if m is odd and, if $m=2p$, is equal to

$$\lim_{N\to\infty}\frac{1}{N^p}\sum_{\substack{k:\{1,\dots,2p\}\to\{1,\dots,p\}\\ 2-1\ map}}\ \sum_{\substack{1\le k_1,\dots,k_{2p}\le N\\ j=1,\dots,p}}\varphi\left(a_{k_{2p}}^{\varepsilon_{2p}}f_{2p}\left(\frac{k_{2p}}{N}\right)\cdots a_{k_1}^{\varepsilon_1}f_1\left(\frac{k_1}{N}\right)\right)$$
$$=\sum_{\tau\in \mathrm{P.P.}(2p)}\omega(\tau,\varepsilon)\prod_{j=1}^{p}\left(C(\varepsilon_{l_j},\varepsilon_{r_j})\int_0^1 f_{l_j}(x)f_{r_j}(x)dx\right) \tag{3.2}$$

where P.P.$(2p)$ *denotes the set of all pair partitions of* $\{1,\dots,2p\}$ *and* $\{l_j,r_j\}_{j=1}^p$ *is the left-right index set relative to the pair partition* $\tau\in$ P.P.$(2p)$.

Proof. In fact

$$\frac{1}{N^{m/2}}\varphi(S_N(a^{\varepsilon_m},f_m)\cdots S_N(a^{\varepsilon_1},f_1))$$
$$=\frac{1}{N^{m/2}}\sum_{1\le k_1,\dots,k_m\le N}\varphi\left(a_{k_m}^{\varepsilon_m}f_m\left(\frac{k_m}{N}\right)\cdots a_{k_1}^{\varepsilon_1}f_1\left(\frac{k_1}{N}\right)\right). \tag{3.3}$$

From Lemma 3.1 it follows that the family $\{a_n^\varepsilon f(x),\varepsilon\in F_s,\ n\in\mathbb{N},f\in\mathcal{L}([0,1])\}$ satisfies the singleton condition and the same consequently occurs for $\{a_n^\varepsilon,\varepsilon\in F_s,\ n\in\mathbb{N}\}$. Moreover the family $\{a_n^\varepsilon;\varepsilon\in F_s,\ n\in\mathbb{N}\}$ verifies the uniform boundedness condition. Then, from Lemma 3.2, it follows that the limit of (3.3) can be different from zero only if $m=2p$ and k: $\{1,\dots,2p\}\to\{1,\dots,p\}$ is a $2-1$ map whose range $\{\overline{k}_1,\dots,\overline{k}_m\}$ takes values in $\{1,\dots,N\}$; it is well known that such a map induces a pair partition on $\{1,\dots,2p\}$. If $\{l_j,r_j\}:=k^{-1}(\overline{k}_j)$ with $l_j>r_j$, for all $j=1,\dots,p$, i.e. $k_{l_j}=k_{r_j}=\overline{k}_j$, the limit for $N\to\infty$ of the right hand side in (3.3) can be written as follows

$$\lim_{N\to\infty}\frac{1}{N^p}\sum_{\substack{k:\{1,\dots,2p\}\to\{1,\dots,p\}\\ 2-1\ \text{map}}}\ \sum_{\substack{1\le k_1,\dots,k_{2p}\le N\\ j=1,\dots,p}}\left(\prod_{j=1}^{p}(f_{l_j}f_{r_j})\left(\frac{k_h}{N}\right)\right)\varphi(\cdots a_{k_{l_j}}^{\varepsilon_{l_j}}\cdots a_{k_{r_j}}^{\varepsilon_{r_j}}\cdots). \tag{3.4}$$

By the definition of symmetric φ-projective independence, (3.4) is equal to:

$$\lim_{N\to\infty}\sum_{\tau:=\{l_j,r_j\}_{j=1}^p\in\mathrm{P.P.}(2p)}\omega(\tau,\varepsilon)\frac{1}{N^p}$$
$$\times\sum_{\substack{1\le k_{l_j}=k_{r_j}\le N\\ j=1,\dots,p}}\left(\prod_{j=1}^{p}\left[\left(f_{l_j}f_{r_j}\right)\left(\frac{k_{l_j}}{N}\right)\right]\varphi(a_{k'_{l_j}}^{\varepsilon_{l_j}}a_{k'_{r_j}}^{\varepsilon_{r_j}})\right) \tag{3.5}$$

where in the last equality we used the natural identification $[k]\equiv\tau$ between the set of $2-1$ maps $\{1,\dots,2p\}\to\{1,\dots,p\}$ and P.P.$(2p)$. Since

$$\prod_{j=1}^{p}\left(\frac{1}{N}\sum_{k_{l_j}=k_{r_j}=1}^{N}(f_{l_j}f_{r_j})\left(\frac{k_{l_j}}{N}\right)\varphi(a_{k_{l_j}}^{\varepsilon_{l_j}}a_{k_{r_j}}^{\varepsilon_{r_j}})\right)$$
$$=\prod_{j=1}^{p}C(\varepsilon_{l_j},\varepsilon_{r_j})\left(\frac{1}{N}\sum_{k_{l_j}=k_{r_j}=1}^{N}(f_{l_j}f_{r_j})\left(\frac{k_{l_j}}{N}\right)\right)$$

on the right hand side above we recognize Riemann sums. Therefore the limit (3.5) is equal to

$$\sum_{\tau\in\mathrm{P.P.}(2p)}\omega(\tau,\varepsilon)\prod_{j=1}^{p}\Big(C(\varepsilon_{l_j},\varepsilon_{r_j})\int_0^1 f_{l_j}(x)f_{r_j}(x)dx\Big).\ \blacksquare$$

REMARK 3.2. The symmetric central limit theorem in [1] is achieved without the Riemann integrable functions: in this sense the result above could seem more general. On the other hand in [1] the authors performed the proof under a weaker condition on the mean covariance. Namely the condition above such that for any $\varepsilon_1,\varepsilon_2\in F_s$, $\varphi(a_n^{\varepsilon_1}a_n^{\varepsilon_2})=C(\varepsilon_1,\varepsilon_2)$ for any $n\in\mathbb{N}$, is there replaced by

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\varphi(a_k^{\varepsilon_1}a_k^{\varepsilon_2})=C(\varepsilon_1,\varepsilon_2).$$

4. Representation of the limit process. Throughout this section we will take $F_s:=\{-1,+1\}$ and for any $a\in\mathcal{A}$, $a^{-1}=(a^1)^*$. Our goal consists in finding Fock representations for the limit process arising from Theorem 3.1. In fact, as a consequence of the reconstruction theorem by Accardi, Frigerio and Lewis (see [4]), one knows that there exist an algebraic probability space $(\mathcal{B},\psi)$ and random variables a_ψ^{-1},a_ψ^1 in this space such that

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N^{m/2}}\varphi(S_N(a^{\varepsilon_m},f_m)\cdots S_N(a^{\varepsilon_1},f_1))\\&=\sum_{\tau\in\mathrm{P.P.}(2p)}\omega(\tau,\varepsilon)\prod_{j=1}^{p}\Big(C(\varepsilon_{l_j},\varepsilon_{r_j})\int_0^1 f_{l_j}(x)f_{r_j}(x)dx\Big)=\psi(a_\psi^{\varepsilon_m}\cdots a_\psi^{\varepsilon_1}).\end{aligned}\tag{4.1}$$

If $(\mathcal{H}_\psi,\Phi_\psi)$ is a GNS space of $(\mathcal{B},\psi)$, then

$$\psi(a_\psi^{\varepsilon_m}\cdots a_\psi^{\varepsilon_1})=\langle\Phi_\psi,A^{\varepsilon_m}\cdots A^{\varepsilon_1}\Phi_\psi\rangle$$

where the A^{ε_j}'s are operators in $\mathcal{H}_\psi$. We would like to write them concretely as operators of creation and annihilation in a suitable Fock space. To this purpose it seems necessary to make some constraints on the family $\{a_i^\varepsilon,\ \varepsilon\in F_s,i\in\mathcal{I}\}$ in $\{\mathcal{A},\varphi\}$ as defined in Section 3, i.e. we need something more than the projective independence. Therefore we suppose that for any $\varepsilon_1,\varepsilon_2\in F_s$

$$C(\varepsilon_1,\varepsilon_2)=\begin{cases}c>0 & \text{if }\varepsilon_1=-1,\ \varepsilon_2=1,\\ 0 & \text{otherwise,}\end{cases}\tag{4.2}$$

and without loss of generality we take $c=1$. Under this assumption, there are some terms which do not give any contribution to the sum $\sum_{\tau\in\mathrm{P.P.}(2p)}$ above. In order to identify them, we write $\varepsilon\in\{-1,1\}_+^{2p}$ if $\varepsilon\in\{-1,1\}^{2p}$and

- $\sum_{j=1}^{2p}\varepsilon(j)=0$;
- for any $k=1,\dots,2p$ $\sum_{j=1}^{k}\varepsilon(j)\ge 0$.

Let $\tau\in\mathrm{P.P.}(2p)$ and $\{l_j,r_j\}_{j=1}^p$ be the left-right index set relative to τ. It is easy to check (4.2) implies that, if $\varepsilon=(\varepsilon(l_1),\dots,\varepsilon(l_p),\varepsilon(r_1),\dots,\varepsilon(r_p))$ does not belong to $\{-1,1\}_+^{2p}$, the corresponding term in the summation is zero. Furthermore from (4.2) one has that the nonzero contributions are determined exactly by those $\varepsilon \in \{-1,1\}_+^{2p}$ such that for any $j = 1, \dots, p$, $\varepsilon(l_j) = -1$ and $\varepsilon(r_j) = 1$. To avoid the introduction of new symbols, whenever we shall write $\varepsilon \in \{-1,1\}_+^{2p}$ we will require the conditions $\varepsilon(l_j) = -1$ and $\varepsilon(r_j) = 1$ are satisfied. Moreover we assume:

1. For any $q \in \mathbb{N}^*$, any $\varepsilon \in \{-1,1\}^m$, $k : \{1, \dots, m\} \to \mathcal{I}$
$$\varphi(a_{k_m}^{\varepsilon_m} \cdots a_{k_1}^{\varepsilon_1}) = 0 \tag{4.3}$$
when there is a crossing in k, i.e. there exist $h < i < j < l$ such that $k_h = k_j$, $k_i = k_l$. As a consequence, only the noncrossing pair partitions appear in the sum on the left hand side of (4.1). Since it is known (see [7], Lemma 22.6 for details) that any noncrossing pair partition τ in $\{1, \dots, 2p\}$ is uniquely determined by $\varepsilon \in \{-1,1\}_+^{2p}$, from now on we will write $\omega(\tau, \varepsilon)$ as $\omega(\varepsilon)$.
2. (Factorization principle) For any $\varepsilon \in \{-1,1\}_+^{2p}$
$$\varphi(a_{k_{2p}}^{\varepsilon_{2p}} \cdots a_{k_1}^{\varepsilon_1}) = \varphi\Big(\prod_{h=l_{d_1}}^{r_{d_1}} a_{k_h}^{\varepsilon_h}\Big) \cdots \varphi\Big(\prod_{h=l_{d_{m+1}}}^{r_{d_{m+1}}} a_{k_h}^{\varepsilon_h}\Big) \tag{4.4}$$
where m and $\{d_j\}_{j=1}^{m+1}$ are determined by $\varepsilon = \{l_h, r_h\}_{h=1}^{p}$ and $1 \le m < 2p$, $1 = d_1 < \cdots < d_{m+1} \le 2p$, $r_{d_h} = l_{d_{h-1}} + 1$, $h = 2, \dots, m+1, r_{d_1} = 1$, $l_{d_{m+1}} = 2p$. Each block $\{\varepsilon_{l_{d_j}}, \dots, \varepsilon_{r_{d_j}}\}$, $j = 1, \dots, m+1$ is called a connected component of the partition ε.
3. (Rule to compute the mixed moments) Let us introduce the following notation:
$$\omega_1 := \omega(\varepsilon = \{-1, 1\})$$
and generally, for any $n \ge 2$
$$\omega_n := \omega(\varepsilon = \{\underbrace{-1, \dots, -1}_{n\text{-times}}, \underbrace{1, \dots, 1}_{n\text{-times}}\}).$$
Let us take
$$\varphi(a_{k_1}^{-1} a_{k_1}^{1}) = \omega_1 \tag{4.5}$$
and, if $\varepsilon \in \{-1,1\}_+^{2p}$ and
$$\varphi(a_{k_{2p}}^{\varepsilon_{2p}} \cdots a_{k_1}^{\varepsilon_1}) = \prod_{j=1}^{r} \omega_j^{l_j} \tag{4.6}$$
where $r \le p$, $l_j \in \mathbb{N}$, $j = 1, \dots, r$, then
$$\varphi(a_{k_{2p+1}}^{-1} a_{k_{2p}}^{\varepsilon_{2p}} \cdots a_{k_1}^{\varepsilon_1} a_{k_{2p+1}}^{1}) = \omega_1 \prod_{j=1}^{r} \omega_{j+1}^{l_j}. \tag{4.7}$$
For example, if
$$\varphi(a_3^{-1} a_3^1 a_2^{-1} a_1^{-1} a_1^1 a_2^1) = \omega_1^2 \omega_2$$
then
$$\varphi(a_4^{-1} a_3^{-1} a_3^1 a_2^{-1} a_1^{-1} a_1^1 a_2^1 a_4^1) = \omega_1 \omega_2^2 \omega_3.$$

By means of (4.4),…,(4.7), one can inductively compute all the mixed moments.

Fix $\varepsilon \in \{-1,1\}^{2p}_{+}$, and, as in [3], we introduce the depth function of the string ε, i.e. the map $d_\varepsilon : \{1,\dots,2p\} \to \{0,\pm1,\dots,\pm 2p\}$ such that for any $j \in \{1,\dots,q\}$

$$d_\varepsilon(j) := \sum_{k=1}^{j} \varepsilon(k).$$

Let $\mathcal{H} := \mathbf{L}^2([0,1])$. If

$$\lambda_0 := 1, \ \lambda_1 := \omega_1 \tag{4.8}$$

and for any $n \geq 2$

$$\lambda_n := \lambda_{n-1}\omega_n \tag{4.9}$$

we consider the 1-mode type IFS over $\mathcal{H}$ as in Example 1. Since by the definition each of the ω_n's is nonnegative, from (4.8) and (4.9), it follows that the λ_n's are nonnegative for any $n \in \mathbb{N}$. As a consequence, on the algebraic n-th tensor product $\mathcal{H}^{\odot n}$, we define a pre-scalar product in the following way: for any $f_1,\dots,f_n,g_1,\dots,g_n \in \mathcal{H}$

$$\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n \rangle := \lambda_n \prod_{k=1}^{n} \langle f_k, g_k \rangle.$$

By taking quotient and completing, $\mathcal{H}^{\odot n}$ becomes a Hilbert space, which will be denoted by $\mathcal{H}_n$. The 1-mode type IFS over $\mathcal{H}$ with interacting sequence $\{\lambda_n\}_n$ is

$$\Gamma(\mathcal{H}, \{\lambda_n\}_n) := \mathbb{C}\Phi \oplus \bigoplus_{n\geq 1} \mathcal{H}_n$$

where Φ is the vacuum vector. Since

$$\lambda_n = 0 \Longrightarrow \lambda_m = 0 \ \ \forall m \geq n, \ n \in \mathbb{N}$$

one defines, as in Example 1 the creation operator with test function $f \in \mathcal{H}$ and the annihilation operator as its adjoint.

LEMMA 4.1. *The family of creation and annihilation operators is symmetrically projectively independent with respect to the vacuum state $\langle \Phi, \cdot \Phi \rangle$ in 1-mode type IFS over $\mathcal{H}$.*

Proof. In fact for any $p \in \mathbb{N}$, any $\varepsilon = (\varepsilon(l_1),\dots,\varepsilon(l_p),\varepsilon(r_1),\dots,\varepsilon(r_p)) \in \{-1,1\}^{2p}_{+}$, $f_1,\dots,f_{2p} \in \mathcal{H}$, by Lemma 4.2 of [3], one has

$$\langle \Phi, A_{2p}^{\varepsilon(2p)}(f_{2p}) \cdots A_1^{\varepsilon(1)}(f_1)\Phi \rangle = \prod_{j=1}^{p} \omega_{d_\varepsilon(r_j)} \langle f_{l_j}, f_{r_j} \rangle$$

where $\{l_j, r_j\}_{j=1}^{p}$ is the unique noncrossing pair partition induced by ε. The statement follows after noticing that the terms of the product on the right hand side above depend only on ε. ■

Let f be an element of $\mathcal{L}([0,1])$ and $Q(f) := A(f) + A^+(f)$ the field operator in 1-mode type IFS over $\mathcal{H}$. As in example 1, we use the following notation: for any $\varepsilon \in \{-1,1\}$

$$A^\varepsilon(f) := \begin{cases} A(f) & \text{if } \varepsilon = -1, \\ A^+(f) & \text{if } \varepsilon = 1. \end{cases}$$

Moreover we introduce the convention such that for any $f \in \mathcal{L}([0,1])$, $\varepsilon \in \{-1,1\}$

$$f^{\varepsilon} := \begin{cases} \overline{f} & \text{if } \varepsilon = -1, \\ f & \text{if } \varepsilon = 1. \end{cases}$$

The next result gives us the Fock representation of the limit process.

THEOREM 4.1. *The limit process* $\{a_{\psi}^{-1}, a_{\psi}^{1}\}$ *is represented in* $\Gamma(\mathcal{H}, \{\lambda_n\})$, *that is*

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} \varphi(S_N(a^{\varepsilon_m}, f_m^{\varepsilon_m}) \cdots S_N(a^{\varepsilon_1}, f_1^{\varepsilon_1})) = \langle \Phi, Q(f_m) \cdots Q(f_1)\Phi \rangle$$

where $f_j \in \mathcal{L}([0,1])$ *for any* $j = 1, \ldots, m$.

Proof. Indeed

$$\langle \Phi, Q(f_m) \cdots Q(f_1)\Phi \rangle = \sum_{\varepsilon \in \{-1,1\}^m} \langle \Phi, A^{\varepsilon_m}(f_m) \cdots A^{\varepsilon_1}(f_1)\Phi \rangle. \tag{4.10}$$

One can check that the right hand side of (4.10) is equal to (see Lemma 4.2 of [3] for details)

$$\sum_{\varepsilon \in \{-1,1\}_+^{2p}} \Big[\prod_{j=1}^{p} \omega_{d_\varepsilon(r_j)} \langle f_{l_j}, f_{r_j} \rangle \Big] \tag{4.11}$$

where $m = 2p$ and $\{l_j, r_j\}_{j=1}^p$ is the noncrossing pair partition determined by ε. By the one to one correspondence between $\{-1,1\}_+^{2p}$ and the set $NC\text{P.P.}(2p)$ of noncrossing pair partitions on $\{1, 2, \ldots, 2p\}$, we write (4.11) as

$$\sum_{NC\text{P.P.}(2p)} \Big[\prod_{j=1}^{p} \omega_{d_\varepsilon(r_j)} \langle f_{l_j}, f_{r_j} \rangle \Big].$$

On the other hand, from (4.1) and the assumption 1., one has:

$$\lim_{N\to\infty} \frac{1}{N^{m/2}} \varphi(S_N(a^{\varepsilon_m}, f_m^{\varepsilon_m}) \cdots S_N(a^{\varepsilon_1}, f_1^{\varepsilon_1}))$$
$$= \sum_{NC\text{P.P.}(2p)} \omega(\varepsilon) \prod_{j=1}^{p} \Big(\int_0^1 \overline{f}_{l_j}(x) f_{r_j}(x) dx \Big).$$

From (4.4)–(4.7), it follows that

$$\omega(\varepsilon) = \prod_{j=1}^{p} \omega_{d_\varepsilon(r_j)}. \quad \blacksquare$$

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