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# THE WIGNER SEMI-CIRCLE LAW AND THE HEISENBERG GROUP

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**Abstract.** The Wigner Theorem states that the statistical distribution of the eigenvalues of a random Hermitian matrix converges to the semi-circular law as the dimension goes to infinity. It is possible to establish this result by using harmonic analysis on the Heisenberg group. In fact this convergence corresponds to the topology of the set of spherical functions associated to the action of the unitary group on the Heisenberg group.

On the vector space $V_n = Herm(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) one considers a Gaussian probability measure

$$\mathbb{P}_n(dx) = \frac{1}{C_n} \exp(-\gamma \operatorname{tr}(x^2)) m(dx),$$

where $m$ is the Lebesgue measure. For a Borel set $B \subset \mathbb{R}$, the random variable $\xi_B^{(n)}$ is defined by

$$\xi_B^{(n)}(x) = \frac{1}{n} \sharp\{\text{eigenvalues of } x \text{ in } B\}.$$

We write its expectation as

$$\mathbb{E}_n(\xi_B^{(n)}) = \mu_n(B).$$

This defines a probability measure $\mu_n$ on $\mathbb{R}$; this is the *statistical distribution of the*

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*eigenvalues.* If $\varphi$ is a bounded Borel function on $\mathbb{R}$,

$$\int_{\mathbb{R}} \varphi(t)\mu_n(dt) = \frac{1}{n}\int_{V_n} \mathrm{tr}(\varphi(x))\mathbb{P}_n(dx).$$

Here $\varphi(x)$ is the $n \times n$ Hermitian matrix defined by using the functional calculus: if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $x$, then the eigenvalues of $\varphi(x)$ are $\varphi(\lambda_1), \dots, \varphi(\lambda_n)$.

The *semi-circle law* of radius $a > 0$ is the probability measure $\sigma_a$ on $\mathbb{R}$ defined by

$$\int_{\mathbb{R}} \varphi(t)\sigma_a(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} \varphi(t)\sqrt{a^2 - t^2}dt.$$

The Wigner Theorem says that, after scaling, the statistical distribution of the eigenvalues $\mu_n$ converges weakly to the semi-circle law $\sigma_a$ with radius

$$a = \sqrt{\frac{\beta}{\gamma}}, \quad \beta = \dim_{\mathbb{R}} \mathbb{F} = 1, 2, 4.$$

More precisely:

Theorem 1 (Wigner). *For a bounded continuous function $\varphi$ on $\mathbb{R}$,*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi\Big(\frac{t}{\sqrt{n}}\Big)\mu_n(dt) = \frac{2}{\pi a^2}\int_{-a}^{a} \varphi(u)\sqrt{a^2 - u^2}du.$$

[Wigner, 1955, 1958].

This means that, for large $n$, the density of eigenvalues is approximately

$$\frac{2}{\pi a^2}\sqrt{na^2 - \lambda^2},$$

if $|\lambda| \leq a\sqrt{n}$, and 0 if $|\lambda| \geq a\sqrt{n}$.

To a random matrix $X$ with values in $Herm(n, \mathbb{F})$ one associates the random probability measure $\mu_X$:

$$\mu_X = \frac{1}{n}(\delta_{\lambda_1(X)} + \dots + \delta_{\lambda_n(X)}),$$

where $\lambda_1(X), \dots, \lambda_n(X)$ are the eigenvalues of $X$. It is called the *empirical eigenvalue distribution* of $X$.

One shows also:

*For every $n$ let $X^{(n)}$ be a random matrix with values in $Herm(n, \mathbb{F})$, and law $\mathbb{P}_n$. Define*

$$Y(n) = \frac{1}{\sqrt{n}}X(n).$$

*Then the empirical eigenvalue distribution of $Y(n)$ almost surely converges weakly to the semi-circle law $\sigma_a$.*

In the original proof Wigner considers the moments of the measure $\mu_n$:

$$\mathfrak{M}_k(\mu_n) = \int_{\mathbb{R}} t^k \mu_n(dt) = \frac{1}{n}\int_{V_n} \mathrm{tr}(x^k)\mathbb{P}_n(dx),$$

and by combinatorial computations determines the asymptotics of $\mathfrak{M}_k(\mu_n)$ as $n$ goes to infinity: for $k$ fixed,

$$\mathfrak{M}_{2k}(\mu_n) \sim \left(\frac{\beta}{4\gamma}\right)^k \frac{(2k)!}{k!(k+1)!} n^k \qquad (n \to \infty).$$

Note that the moments of odd order vanish. On the other hand it is easy to compute the moments of the semi-circle law:

$$\mathfrak{M}_{2k}(\sigma_a) = \left(\frac{a^2}{4}\right)^k \frac{(2k)!}{k!(k+1)!}.$$

The proof by Pastur uses the Cauchy transform ([Pastur, 1972, 1996]). Recall that the Cauchy transform of a probability measure $\mu$ on $\mathbb{R}$ is the function $G_\mu$ defined on $\mathbb{C}\setminus\mathbb{R}$ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} \mu(dt).$$

For $\mu = \mu_n$, writing $G_{\mu_n} = G_n$,

$$G_n(z) = \frac{1}{n} \int_{V_n} \operatorname{tr}((zI - x)^{-1}) \mathbb{P}_n(dx).$$

After scaling one has to look at the functions

$$\tilde{G}_n(z) = \sqrt{n} G_n(\sqrt{n} z).$$

The proof amounts to showing that the functions $\tilde{G}_n$ converge,

$$\lim_{n\to\infty} \tilde{G}_n(z) = f(z),$$

and that $f$ is a holomorphic function satisfying

$$f(z)^2 - \frac{4}{a^2} z f(z) + \frac{4}{a^2} = 0.$$

Since $\operatorname{Im} G_n(z) < 0$ and hence $\operatorname{Im} f(z) < 0$ for $\operatorname{Im} z > 0$, necessarily

$$f(z) = \frac{2}{a^2}\big(z - \sqrt{z^2 - a^2}\big),$$

which is the Cauchy transform of the semi-circle law $\sigma_a$.

In this paper we will see that harmonic analysis on the Heisenberg group provides a proof of Wigner Theorem. It amounts to determining asymptotics of the Fourier transform of the measure $\mu_n$:

$$\widehat{\mu_n}(\tau) = \int_{\mathbb{R}} e^{i\tau t} \mu_n(dt) = \frac{1}{n} \int_{V_n} \operatorname{tr}(e^{i\tau x}) \mathbb{P}_n(dx).$$

In [Haagerup, Thorbjørnsen, 2003], the asymptotics of $\widehat{\mu_n}(\tau)$ are determined by using properties of Laguerre polynomials.

More general results are obtained by using logarithmic potential theory (see [Deift, 2000]). One defines the energy of a probability measure $\mu$ on $\mathbb{R}$ by

$$I(\mu) = \int_{\mathbb{R}^2} \log \frac{1}{|s-t|} \mu(ds)\mu(dt) + \int_{\mathbb{R}} Q(t)\mu(dt).$$

For $Q(t) = \gamma t^2$, the semi-circle law appears as equilibrium measure: measure which realizes the minimum of the energy.

**1. Mehta's formula.** Let $Q$ be a continuous real valued function on $\mathbb{R}$ such that, for all $m \geq 0$,

$$\int_{\mathbb{R}} |t|^m e^{-Q(t)} dt < \infty.$$

One considers on the space $V_n = Herm(n, \mathbb{F})$ the probability measure defined by

$$\mathbb{P}_n(dx) = \frac{1}{C_n} e^{-\operatorname{tr}(Q(x))} m(dx),$$

with

$$C_n = \int_{V_n} e^{-\operatorname{tr}(Q(x))} m(dx).$$

Let $f$ be a function on $V_n$, invariant under the unitary group:

$$f(uxu^*) = f(x) \quad (u \in U(n),\ x \in V_n).$$

Then $f(x)$ only depends on the eigenvalues $\lambda_1, \dots, \lambda_n$ of $x$,

$$f(x) = F(\lambda_1, \dots, \lambda_n),$$

where $F$ is a function on $\mathbb{R}^n$ which is symmetric, i.e. invariant under permutation. If $f$ is integrable, then, by a Weyl integration formula,

$$\int_{V_n} f(x) m(dx) = c_n \int_{\mathbb{R}^n} F(\lambda) |\Delta(\lambda)|^\beta d\lambda_1 \dots \lambda_n,$$

where $\Delta$ is the Vandermonde polynomial:

$$\Delta(\lambda) = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i).$$

If $f$ is integrable with respect to $\mathbb{P}_n$,

$$\int_{V_n} f(x) \mathbb{P}_n(dx) = \int_{\mathbb{R}^n} F(\lambda) q_n(\lambda) d\lambda_1 \dots d\lambda_n,$$

with

$$q_n(\lambda) = \frac{1}{Z_n} e^{-(Q(\lambda_1) + \cdots + Q(\lambda_n))} |\Delta(\lambda)|^\beta,$$

and

$$Z_n = \int_{\mathbb{R}^n} e^{-(Q(\lambda_1) + \cdots + Q(\lambda_n))} |\Delta(\lambda)|^\beta d\lambda_1 \dots \lambda_n.$$

In particular, if

$$f(x) = \frac{1}{n} \operatorname{tr}(\varphi(x)),$$

where $\varphi$ is a bounded measurable function on $\mathbb{R}$, then

$$F(\lambda) = \frac{1}{n} (\varphi(\lambda_1) + \cdots + \varphi(\lambda_n)),$$

and

$$\begin{aligned} \int_{\mathbb{R}} \varphi(t) \mu_n(dt) &= \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{R}^n} \varphi(\lambda_i) q_n(\lambda) d\lambda_1 \dots \lambda_n = \int_{\mathbb{R}^n} \varphi(\lambda_1) q_n(\lambda) d\lambda_1 \dots d\lambda_n \\ &= \int_{\mathbb{R}} \varphi(t) w_n(t) dt, \end{aligned}$$

with

$$w_n(t) = \int_{\mathbb{R}^{n-1}} q_n(t, \lambda_2, \ldots, \lambda_n) d\lambda_2 \ldots d\lambda_n.$$

In the case of $\mathbb{F} = \mathbb{C}$ ($\beta = 2$) the density of the statistical distribution $\mu_n$ of the eigenvalues can be expressed in terms of orthogonal polynomials. Let us consider the sequence of orthogonal polynomials $p_0, \ldots, p_m, \ldots$ with respect to the measure $e^{-Q(t)}dt$, normalized by the condition

$$p_m(t) = t^m + \cdots$$

Let $a_m$ denote the squared norm of $p_m$:

$$a_m = \int_{\mathbb{R}} |p_m(t)|^2 e^{-Q(t)} dt.$$

Define

$$h_m(t) = \frac{1}{\sqrt{a_m}} e^{-\frac{1}{2}Q(t)} p_m(t).$$

The functions $h_m$ are orthonormal in $L^2(\mathbb{R})$. Define also

$$K_n(s,t) = \sum_{k=0}^{n-1} h_k(s) h_k(t).$$

Up to an exponential factor, this is the Christoffel-Darboux kernel of the sequence of the orthogonal polynomials $p_m$. This is also the kernel of the orthogonal projection of $L^2(\mathbb{R})$ onto the subspace spanned by the functions $h_0, \ldots, h_{n-1}$. According to a notation introduced by Fredholm, let us write, for a kernel $K(s,t)$,

$$K\begin{pmatrix} s_1 & s_2 & \ldots & s_m \\ t_1 & t_2 & \ldots & t_m \end{pmatrix} = \det(K(s_i, t_j))_{1 \le i,j \le m}.$$

THEOREM 2 (Mehta's formulas).

$$q_n(\lambda_1, \ldots, \lambda_n) = \frac{1}{n!} K_n \begin{pmatrix} \lambda_1 & \ldots & \lambda_n \\ \lambda_1 & \ldots & \lambda_n \end{pmatrix}, \qquad w_n(t) = \frac{1}{n} K_n(t,t) = \frac{1}{n} \sum_{k=0}^{n-1} h_k(t)^2.$$

[Mehta, 1991], pp. 91, 92.

In the following we will consider the case $Q(t) = t^2$ ($\gamma = 1$). Then the orthogonal polynomials $p_m$ are, up to normalization, the Hermite polynomials:

$$p_m(t) = 2^{-m} H_m(t),$$

and the $h_m$ are Hermite functions:

$$h_m(t) = (2^m m! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{1}{2}t^2} H_m(t).$$

**2. Harmonic analysis on the Heisenberg group.** We will recall classical facts about the representation theory of the Heisenberg group: Bargmann representation on the Fock space, and Schrödinger representation on $L^2(\mathbb{R}^d)$ (see [Faraut, 1987] for instance).

The Heisenberg group $H_d$ is the set $H_d = \mathbb{C}^d \times \mathbb{R}$ equipped with the product

$$(z,t) \cdot (z', t') = (z + z', t + t' + \mathrm{Im}\,(z'|z)).$$

The Fock space $\mathcal{H}_\lambda$ ($\lambda \in \mathbb{R}^*$) is the space of holomorphic functions $\varphi$ on $\mathbb{C}^d$ with

$$\|\varphi\|_\lambda^2 = \left(\frac{|\lambda|}{\pi}\right)^d \int_{\mathbb{C}^d} e^{-|\lambda|\|\zeta\|^2} |\varphi(\zeta)|^2 m(d\zeta) < \infty.$$

The Bargmann representation $T_\lambda$ is the representation of $H_d$ on $\mathcal{H}_\lambda$ given by:

$$(T_\lambda(z,t)\varphi)(\zeta) = e^{\lambda(it - \frac{1}{2}\|z\|^2 - (\zeta|z))}\varphi(\zeta + z).$$

The unitary group $U(d)$ acts on the Heisenberg group $H_d$ by automorphisms:

$$(z,t) \mapsto (uz,t) \quad (u \in U(d)),$$

and also on the Fock space $\mathcal{H}_\lambda$:

$$(\tau_u\varphi)(\zeta) = \varphi(u^{-1}\zeta),$$

so that

$$T_\lambda(uz,t) = \tau_u T_\lambda(z,t)\tau_{u^{-1}}.$$

Let $\mathcal{P}_m$ denote the space of polynomials in $d$ variables, homogeneous of degree $m$. The space $\mathcal{P}_m$, which is a subspace of the Fock space $\mathcal{H}_\lambda$, is invariant under $U(d)$, and irreducible for this action. Furthermore

$$\mathcal{H}_\lambda = \widehat{\bigoplus}_{m=0}^{\infty} \mathcal{P}_m.$$

Let $L^1(H_d)^\flat$ be the space of $U(d)$-invariant integrable functions (radial in $z$). It is a commmutative convolution algebra [Korányi, 1980]. The characters of the commutative Banach algebra $L^1(H_d)^\flat$ are of the form:

$$\chi(f) = \int_{H_d} f(z,t)\omega(z,t)m(dz)dt,$$

here $\omega$ is a bounded spherical function, i.e. a $U(d)$-invariant bounded continuous function on $H_d$ such that

$$\int_{U(d)} \omega((z,t)(uz',t'))\nu(du) = \omega(z,t)\omega(z',t'),$$

where $\nu$ is the normalized Haar measure of $U(d)$, or

$$\int_{U(d)} \omega(z + uz', t + t' + \mathrm{Im}(z^* uz'))\nu(du) = \omega(z,t)\omega(z',t').$$

The spectrum of the commutative Banach algebra can be identified with the set of bounded spherical functions, and its topology is induced by the topology of uniform convergence on compact sets.

For $f \in L^1(H_d)$, define

$$T_\lambda(f) = \int_{H_d} T_\lambda(z,t) f(z,t)m(dz)dt.$$

If $f$ is $U(d)$ invariant: $f \in L^1(H_d)^\flat$:

$$f(uz,t) = f(z,t),$$

then the operator $T_\lambda(f)$ commutes with the $U(d)$-action:

$$\tau_u T(z,t) = T(z,t)\tau_u \quad (u \in U(d)).$$

Therefore, by Schur's Lemma, $\mathcal{P}_m$ is an eigenspace for $T_\lambda(f)$: there is a complex number $\hat{f}(\lambda, m)$ such that, for $\varphi \in \mathcal{P}_m$,

$$T_\lambda(f)\varphi = \widehat{f}(\lambda, m)\varphi.$$

The map $f \mapsto \hat{f}(\lambda, m)$ is a character of $L^1(H_d)^\flat$, hence can be written

$$\widehat{f}(\lambda, m) = \int_{H_d} f(z,t)\omega_{\lambda,m}(z,t)m(dz)dt,$$

where $\omega_{\lambda,m}$ is a *spherical function.* (The functions $\omega_{\lambda,m}$ can be written in terms of Laguerre polynomials.)

The monomials of degree $m$ form an orthogonal basis of $\mathcal{P}_m$. Let $u_\alpha$ be the normalized monomials:

$$u_\alpha(\zeta) = \sqrt{\frac{|\lambda|^{|\alpha|}}{\alpha!}}\zeta^\alpha,$$

with

$$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_d = m, \qquad \alpha! = \alpha_1! \ldots \alpha_d!.$$

PROPOSITION 3.

$$\omega_{\lambda,m}(z,t) = \frac{1}{d_m}\sum_{|\alpha|=m}(T_\lambda(z,t)u_\alpha|u_\alpha), \quad \textit{where} \quad d_m = \dim \mathcal{H}_m = \frac{(m+d-1)!}{m!(d-1)!}.$$

*Proof.* If $f \in L^1(H_d)^\flat$, $\varphi \in \mathcal{P}_m$ with $\|\varphi\|_\lambda = 1$, then

$$(T_\lambda(f)\varphi|\varphi)_\lambda = \hat{f}(\lambda, m),$$

and this can be written

$$\hat{f}(\lambda, m) = \int_{H_d} f(z,t)(T_\lambda(z,t)\varphi|\varphi)m(dz)dt.$$

and also

$$\begin{aligned}\hat{f}(\lambda, m) &= \frac{1}{d_m}\sum_{|\alpha|=m}\int_{H_d} f(z,t)(T_\lambda(z,t)u_\alpha|u_\alpha)m(dz)dt \\ &= \int_{H_d} f(z,t)\Big(\frac{1}{d_m}\sum_{|\alpha|=m}(T_\lambda(z,t)u_\alpha|u_\alpha)\Big)m(dz)dt.\end{aligned}$$

Therefore

$$\omega_{\lambda,m}(z,t) = \frac{1}{d_m}\sum_{|\alpha|=m}(T_\lambda(z,t)u_\alpha|u_\alpha). \ \blacksquare$$

Let $S(\mathbb{C}^d)$ denote the unit sphere in $\mathbb{C}^d$, and $\sigma$ the normalized uniform measure on $S(\mathbb{C}^d)$. For $\mu \geq 0$, define

$$\omega_\mu(z,t) = \int_{S(\mathbb{C}^d)} e^{i\mu \mathrm{Im}\,(z|\zeta)}\sigma(d\zeta).$$

The function $\omega_\mu$ is spherical.

THEOREM 4. *The spectrum of $L^1(H_d)^\flat$, i.e. the set of bounded spherical functions, can be described as follows:*

$$\mathrm{Spect}(L^1(H_d)^\flat) = \{\omega_{\lambda,m} \mid \lambda \in \mathbb{R}^*,\ m \in \mathbb{N}\} \cup \{\omega_\mu \mid \mu \geq 0\}.$$

[Korányi, 1980].

The topology of the spectrum $\mathrm{Spect}(L^1(H_d)^\flat)$ is given by the following embedding into $\mathbb{R}^2$:

$$(\lambda, m) \mapsto (\lambda, 4(m + d/2)|\lambda|), \qquad \mu \mapsto (0, \mu^2).$$

([Bougerol, 1981]). In particular:

Corollary 5.

$$\lim_{\lambda \to 0, 4m|\lambda| \to \mu^2} \omega_{\lambda,m}(z,t) = \omega_\mu(z,t),$$

*uniformly on compact sets.*

Theorem 4 can be proven by using the fact that the spherical functions are eigenfunctions of the differential operators on $H_d$ which are left $H_d$-invariant, and invariant under the $U(d)$-action. The algebra of these operators is generated by the sublaplacian $\Delta_0$ and the first order differential operator $T$:

$$\Delta_0 = \sum_{j=1}^{d} (X_j^2 + Y_j^2), \qquad T = \frac{\partial}{\partial t},$$

where, writting $z = x + iy$,

$$X_j = \frac{\partial}{\partial x_j} - y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} + x_j \frac{\partial}{\partial t},$$

and

$$\begin{aligned} \Delta_0\, \omega_{\lambda,m} &= -4(m + d/2)|\lambda|\, \omega_{\lambda,m}, \quad T\omega_{\lambda,m} = i\lambda\, \omega_{\lambda,m}, \\ \Delta_0\, \omega_\mu &= -\mu^2\, \omega_\mu, \quad T\omega_\mu = 0. \end{aligned}$$

**3. Schrödinger representation.** The Schrödinger representation $U_\lambda$ of the Heisenberg group $H_d$ on $L^2(\mathbb{R}^d)$ is given by

$$(U_\lambda(z,t)f)(\xi) = e^{i\lambda(t + (x|y) + 2(y|\xi))} f(\xi + x),$$

where $z = x + iy$.

The representations $T_\lambda$ and $U_\lambda$ are equivalent. The Hermite-Weber transform:

$$A_\lambda : \mathcal{H}_\lambda \to L^2(\mathbb{R}^d),$$

given by

$$(A_\lambda \varphi)(\xi) = \left(\frac{2|\lambda|}{\pi}\right)^{\frac{1}{4}} \varphi\left(-\frac{1}{2|\lambda|}\frac{\partial}{\partial \xi}\right) e^{-|\lambda| \|\xi\|^2},$$

intertwines both representations:

$$A_\lambda T_\lambda = U_\lambda A_\lambda,$$

and maps monomials to Hermite functions, modified by a suitable scaling. Recall the Hermite functions:

$$h_m(x) = (2^m m! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_m(x).$$

For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, let

$$h_\alpha(\xi) = h_{\alpha_1}(\xi_1) \dots h_{\alpha_d}(\xi_d),$$

and, for $\lambda \in \mathbb{R}^*$,

$$h_\alpha^\lambda(\xi) = (2|\lambda|)^{-\frac{d}{4}} h_\alpha(\sqrt{2|\lambda|}\xi).$$

By using the Rodrigues formula

$$H_m(x) = e^{x^2}\left(-\frac{d}{dx}\right)^m e^{-x^2},$$

one sees that

$$A_\lambda u_\alpha^\lambda = h_\alpha^\lambda.$$

The Hermite-Weber transform is a unitary isomorphism whose inverse

$$B_\lambda : L^2(\mathbb{R}^d) \to \mathcal{H}_\lambda$$

is the Segal-Bargmann transform:

$$(B_\lambda)(\zeta) = \left(\frac{2|\lambda|}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}|\lambda|\zeta^t\zeta} \int_{\mathbb{R}^d} e^{-|\lambda|(\zeta-\xi)^t(\zeta-\xi)} f(\xi)m(d\xi).$$

Proposition 6.

$$\omega_{\lambda,m}(z,t) = \frac{1}{d_m}\sum_{|\alpha|=m}(U_\lambda(z,t)h_\alpha^\lambda | h_\alpha^\lambda).$$

*Proof.* In fact

$$(U_\lambda(z,t)h_\alpha^\lambda|h_\alpha^\lambda) = (U_\lambda(z,t)A_\lambda u_\alpha^\lambda|A_\lambda u_\alpha^\lambda) = (A_\lambda T_\lambda(z,t)u_\alpha^\lambda|A_\lambda u_\alpha^\lambda)$$

because $A_\lambda$ intertwines $U_\lambda$ and $T_\lambda$, and, since $A_\lambda$ is a unitary isomorphism, the above equals $(T_\lambda(z,t)u_\alpha^\lambda|u_\alpha^\lambda)_\lambda$. The statement then follows from Proposition 3. ∎

Now take $z = iy$ $(x = 0)$, $y \in \mathbb{R}^d$, $t = 0$:

$$\omega_{\lambda,m}(iy,0) = \int_{\mathbb{R}^d} e^{2i\lambda(y|\xi)}\left(\frac{1}{d_m}\sum_{|\alpha|=m}|h_\alpha^\lambda(\xi)|^2\right)m(d\xi).$$

Let us consider the probability measure $\eta_m$ on $\mathbb{R}^d$ given by

$$\eta_m(d\xi) = \left(\frac{1}{d_m}\sum_{|\alpha|=m}|h_\alpha^\lambda(\xi)|^2\right)m(d\xi),$$

and let $\hat{\eta}_m$ denote its Fourier transform

$$\hat{\eta}_m(x) = \int_{\mathbb{R}^d} e^{i(x|\xi)}\eta_m(d\xi).$$

The above result can be written:

$$\hat{\eta}_m(\sqrt{2|\lambda|}y) = \omega_{\lambda,m}(iy,0).$$

On the other hand

$$\omega_\mu(iy,0) = \int_{S(\mathbb{C}^d)} e^{i\mu \mathrm{Re}\,(y|\zeta)}\sigma(d\zeta).$$

Recall that $\sigma$ is the normalized uniform measure on the unit sphere $S(\mathbb{C}^d)$. By projecting $\sigma$ onto $\mathbb{R}^d$:

$$\omega_\mu(iy,0) = \frac{1}{A_d}\int_{B(\mathbb{R}^d)} e^{i\mu(y|\xi)}(1-\|\xi\|^2)^{\frac{d}{2}-1}m_n(d\xi),$$

where $B(\mathbb{R}^d)$ is the unit ball in $\mathbb{R}^d$, and

$$A_d = \frac{\pi^{\frac{d}{2}}\Gamma(d/2)}{(d-1)!}.$$

Let us also consider the probability measure $\eta$ on $\mathbb{R}^d$ given by, for a bounded Borel function $f$,

$$\int_{\mathbb{R}^d} f(\xi)\eta(d\xi) = \frac{1}{A_d}\int_{B(\mathbb{R}^d)} f(\xi)(1-\|\xi\|^2)^{\frac{d}{2}-1}m(d\xi).$$

Then, if $\hat{\eta}$ denotes its Fourier transform,

$$\hat{\eta}(\mu y) = \omega_\mu(iy, 0).$$

Recall the classical Lévy-Cramér Theorem:

THEOREM 7 (Lévy-Cramér). *Let $\mu_m$ be a sequence of probability measures on $\mathbb{R}^d$. One assumes that the sequence of the Fourier transforms*

$$\psi_m(x) = \int_{\mathbb{R}^d} e^{i(x|\xi)}\mu_m(d\xi)$$

*converges at every point:*

$$\lim_{m\to\infty} \psi_m(x) = \psi(x),$$

*ant that the limit function $\psi$ is continuous at 0. Then the measures $\mu_m$ converge weakly to a probability measure $\mu$ whose $\psi$ is the Fourier transform.*

As an application we will obtain:

THEOREM 8. *For a bounded continuous function $f$ on $\mathbb{R}^d$,*

$$\lim_{m\to\infty}\int_{\mathbb{R}^d} f\left(\frac{\xi}{\sqrt{2m}}\right)\eta_m(d\xi) = \frac{1}{A_d}\int_{B(\mathbb{R}^d)} f(x)(1-\|x\|^2)^{\frac{d}{2}-1}m(dx).$$

*Proof.* Let us scale the measure $\eta_m$, and consider the measure $\tilde{\eta}_m$:

$$\int_{\mathbb{R}^d} f(\xi)\tilde{\eta}_m(d\xi) = \int_{\mathbb{R}^d} f\left(\frac{\xi}{\sqrt{2m}}\right)\eta_m(d\xi).$$

Its Fourier transform is

$$\widehat{\tilde{\eta}}_m(x) = \hat{\eta}_m\left(\frac{x}{\sqrt{2m}}\right).$$

Recall Corollary 4:

$$\lim_{\lambda\to 0, 4m|\lambda|\to\mu^2} \omega_{\lambda,m}(z,t) = \omega_\mu(z,t).$$

For $\lambda = \frac{1}{4m}$,

$$\widehat{\tilde{\eta}}_m(y) = \hat{\eta}_m\left(\frac{y}{\sqrt{2m}}\right) = \omega_{\frac{1}{4m},m}(iy).$$

Therefore, since $\hat{\eta}(y) = \omega_1(iy)$,

$$\lim_{m\to\infty}\widehat{\tilde{\eta}}_m(y) = \hat{\eta}(y).$$

Hence Theorem 8 follows from the Lévy-Cramér Theorem (Theorem 7). ∎

For $d = 2$, we obtain, for a bounded continuous function $f$ on $\mathbb{R}^2$,

$$\lim_{m\to\infty} \int_{\mathbb{R}^2} f\left(\frac{\xi_1}{\sqrt{2m}}, \frac{\xi_2}{\sqrt{2m}}\right)\left(\frac{1}{m+1} \sum_{\alpha_1+\alpha_2=m} h_{\alpha_1}(\xi_1)^2 h_{\alpha_2}(\xi_2)^2\right) d\xi_1 d\xi_2$$
$$= \frac{1}{\pi} \int_{x_1^2+x_2^2\leq 1} f(x_1, x_2)dx_1dx_2.$$

and by projecting onto $\mathbb{R}$:

Corollary 9. *For a bounded continuous function $\varphi$ on $\mathbb{R}$,*

$$\lim_{m\to\infty} \int_{\mathbb{R}} \varphi\left(\frac{\xi}{\sqrt{2m}}\right)\left(\frac{1}{m+1} \sum_{k=0}^{m} h_k(\xi)^2\right) d\xi = \frac{2}{\pi} \int_{-1}^{1} \varphi(x)\sqrt{1-x^2}dx.$$

By using the second Mehta's formula (Theorem 2), this proves Wigner Theorem.

Remark. In his paper [Biane, 1997], Biane establishes a link between the semi-circle law and harmonic analysis on the Heisenberg group $H_d$, but this link is of a different nature than the one we consider in the present paper. In particular Biane considers asymptotics as the dimension of the Heisenberg group goes to infinity, whereas we keep this dimension fixed.

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