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BARGMANN REPRESENTATION OF q-COMMUTATION RELATIONS FOR $q > 1$ AND ASSOCIATED MEASURES

ILONA KRÓLAK
Mathematical Institute, University of Wrocław
Pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail: ikrol@math.uni.wroc.pl

Abstract. The classical Bargmann representation is given by operators acting on the space of holomorphic functions with the scalar product $\langle z^n|z^k\rangle_q = \delta_{n,k}[n]_q! = F(z^n\overline{z}^k)$. We consider the problem of representing the functional F as a measure for $q > 1$. We prove the existence of such a measure and investigate some of its properties like uniqueness and radiality. The above problem is closely related to the indeterminate Stieltjes moment problem.

1. Introduction. The q-commutation relations $a_j a_k^+ - q a_k^+ a_j = \delta_{jk} Id$ were introduced by Greenberg [Gre], Frisch and Bourret [FB]. They were intensively investigated by Bożejko and Speicher [BSp] as an interpolation between bosonic ($q = 1$) and fermionic ($q = -1$) cases. The one dimensional case, which is the main object of this paper, was studied by Bargmann [Ba]. We consider the Bargmann-Fock representation of the q-commutation relation which is given by operators acting on the space of homomorphic functions with the scalar product $\langle z^n|z^k\rangle_q = \delta_{n,k}[n]_q!$. For $q > 1$ we show a family of measures such that

$$\int_{\mathbf{C}} z^n \overline{z^k} d\mu(z) = \delta_{n,k}[n]_q!,$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [1]_q[2]_q \ldots [n]_q$. The problem turns out to be an indeterminate complex moment problem.

We shall make use of the language of the q-calculus. The q-binomial coefficients and the q-shifted factorial are defined as follows:

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

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$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j) \quad \text{with} \quad (a;q)_0 = 1,$$

We are looking for operators $a, a^+ = a^\star$ on some Hilbert space such that they satisfy the relation:

$$aa^+ - qa^+a = Id. \tag{$\star$}$$

The Bargmann representation of the above relation is given by

$$(a^+f)(z) = zf(z), \qquad (af)(z) = D_qf(z) = \begin{cases} \frac{f(qz)-f(z)}{z(q-1)} & \text{if } z \neq 0, \\ f'(0) & \text{if } z = 0. \end{cases}$$

The representation space $H^2(\mu_q)$ is the completion of the space of analytic functions on the disc $D_q = \{z \in \mathbf{C};\ |z|^2 < \frac{1}{1-q}\}$ with respect to the inner product:

$$\langle z^k, z^n \rangle = \delta_{n,k}[n]_q! = \int_{\mathbf{C}} z^n \overline{z}^k d\mu^q(z) \tag{$\star\star$}$$

It is known that for $q \in (0,1)$, $d\mu^q(z)$ is unique, radial and

$$d\mu^q(z) = (q;q)_\infty \sum_{k=0}^{\infty} \frac{q^k}{(q;q)_k}\, d\lambda_{r_k}(z), \qquad r_k = \frac{q^{\frac{k}{2}}}{\sqrt{1-q}},$$

where $d\lambda_{r_k}$ is the normalized Lebesgue measure on the circle with radius r_k.

However, we will prove, that for $q > 1$, there are many measures satisfying $(\star\star)$ and some of them are not radial.

This paper is organized as follows. In section 2 we construct a family of radial measures satisfying $(\star\star)$. In a subsequent section we estimate asymptotics of orthogonal polynomials with respect to a measure with moments $m_n = [n]_q!$, which enables us to describe all radial solutions of our problem. In section 4 we calculate extremal measures. Finally the last section is devoted to the existence and study of non-radial measures. We also give, in general, necessary and sufficient conditions for the existence of non-radial measures which orthogonalize monomials on the complex plane.

2. The existence of $\mu^q(dz)$.

For our further considerations we assume that the number q is greater than 1 and fixed. We are looking for a probability measure (or family of probability measures) which satisfies $(\star\star)$.

Using the polar coordinates we can write this measure as a product of its radial and angular part i.e.

$$d\mu(z) = d\mu_{(r)}(\varphi)d\nu(r),$$

where $\int_0^{2\pi} d\mu_{(r)}(\varphi) = 1$ and $\int_0^\infty r^{2n} d\nu(r) = [n]_q!$.

The above measure $d\mu(z)$ orthogonalizes monomials $\{z^n\}_{n\geq 0}$. That is why among solutions there are radial ones.

PROPOSITION 2.1. *There is one to one correspondence between radial solutions of* $(\star\star)$ *and solutions of the problem:*

$$\int_0^\infty x^k d\overline{\nu}(x) = [k]_q!, \qquad k = 0,1,2,\ldots \tag{$\star\star\star$}$$

We will use methods of q-analysis. The following definitions and lemmas are analogs of suitable facts connected with the case $0 < q < 1$ (see [LM]).

PROPOSITION 2.2. *The operator $a = D_q$ has the following properties:*

1. $D_q(z^n) = [n]_q z^{n-1}$,
2. $D_q(f(z)h(z)) = (D_q f)(z)h(z) + f(qz)(D_q h)(z)$,
3. $D_q\left(\frac{f(z)}{h(z)}\right) = \frac{(D_q f)(z)h(z) - f(z)(D_q h)(z)}{h(z)h(qz)}$.

There is also an analogue of the integral.

PROPOSITION 2.3. *Assume that $f(x) = (D_q F)(x)$ and $F(\infty) = \lim_{x\to\infty} F(x)$ exists. Then*

$$\int_0^\infty f(x)d_q^a(x) \stackrel{def}{=} \sum_{k=-\infty}^{\infty} f(aq^k)q^k(q-1)a = F(\infty) - F(0),$$

if only the series is convergent for every $a > 0$.

For example all functions which decrease faster than $\frac{1}{x^2}$ as x tends to infinity and are bounded in the neighbourhood of 0, are integrable.

We can also define the q-exponential function:

PROPOSITION 2.4. *The expression*

$$f(z) = \exp_q(z) \stackrel{def}{=} \sum_{k=0}^{\infty} \frac{z^k}{[k]_q!}, \qquad q > 1,$$

defines an entire function with the property: $(D_q f)(z) = f(z)$. Additionally this function has the following infinite product representation:

$$\exp_q(x) = \prod_{k=0}^{\infty} (xq^{-k} - xq^{-k-1} + 1)^{-1}.$$

Sketch of the proof. Notice that the series above is absolutely convergent and apply D_q to each term. Since $(D_q f)(y) = f(y)$ for $y = q^{-1}x$ then $f(q^{-1}x) = \frac{f(x)}{x - q^{-1}x + 1}$. If we iterate this equality and use the continuity of f at 0 we get the infinite product representation.

PROPOSITION 2.5. *We have the following integration by parts procedure:*

$$\int_0^\infty D_q\left(\frac{x^n}{\exp_q(x)}\right)d_q^a(x) = \begin{cases} \left.\frac{x^n}{\exp_q(x)}\right|_\infty^0 = -\delta_{0,n}, \\ [n]_q \int_0^\infty \frac{x^{n-1}}{\exp_q(qx)} d_q^a(x) - \int_0^\infty \frac{x^n}{\exp_q(qx)} d_q^a(x). \end{cases}$$

Now we are ready to write explicitly the measure.

THEOREM 2.1. *If*

$$d\overline{\nu}^a(x) = a(q-1)\sum_{k=-\infty}^{\infty} \frac{q^k}{\exp_q(q^{k+1}a)}\delta_{q^k a}, \qquad a \in [1, q],$$

then for every $a > 0$, $\int_0^\infty x^n d\overline{\nu}^a(x) = [n]_q!$. Additionally for any $t \in \mathbf{R}_+$, there exists $a \in [1, q)$ such that $t \in \operatorname{supp} \overline{\nu}^a(x)$.

Proof. Proposition 2.5 implies:

$$\int_0^\infty \frac{x^n}{\exp_q(qx)} d_q^a(x) = [n]_q!.$$

By replacing the q-integral symbol by its definition we get:

$$[n]_q! = \sum_{k=-\infty}^{\infty} (aq^k)^n \frac{(q-1)aq^k}{\exp_q(q^{k+1}a)} = \int_0^\infty x^n d\overline{\nu}(x).$$

The second part of the assertion is obvious. ∎

3. Orthogonal polynomials and their asymptotics. Consider a probability measure $d\alpha(x)$ such that:

$$\int_0^\infty x^n d\alpha(x) = [n]_q!.$$

The problem of finding all probability measures which satisfy the condition given above is called the moment problem for the sequence $\{[n]_q!\}$. We will look for a solution using a sequence of orthogonal polynomials $R_n(x)$, i.e. $\int_{-\infty}^{\infty} R_n(x)R_m(x)d\alpha(x) = 0$ if $m \neq n$.

We would like to estimate the asymptotics of $R_n(x)$ as $n \to \infty$. We will use the Leibniz expression for calculating D_q on the product of functions:

$$D_q^{(n)}(fg(x)) = \sum_{k=0}^{n} \binom{n}{k}_q [D_q^{(k)} g](x)[D_q^{(n-k)} f](q^k x). \tag{3.1}$$

As a consequence we get

Lemma 3.1.

$$D_q^{(k)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)\exp_q(qx) = \sum_{s=0}^{k} \binom{k}{s}_q \frac{[k]_q!}{[s]_q!} x^s (-1)^s q^{\frac{s(s+1-2k)}{2}}. \tag{3.2}$$

Proof. For $h(x) = f(q^p x)$ we have $[D_q h](x) = q^p[D_q f](q^p x)$. This implies $D_q(\exp_q(q^p x)) = q^p \exp_q(q^p x)$. Further

$$D_q\left(\frac{1}{\exp_q(q^p x)}\right) = q^p \frac{-1}{\exp_q(q^{p+1}x)}.$$

Finally we get

$$D_q^{(s)} \frac{1}{\exp_q(q^{-k+1}x)} = q^{-sk+1+2+\cdots+s} \frac{(-1)^s}{\exp_q(q^{-k+s+1}x)}.$$

Now we can apply Proposition 2.1, which completes the proof. ∎

Lemma 3.2. *Let* $\int_0^\infty x^n d\alpha(x) = [n]_q!$ *and*

$$Q_k(x) = D_q^{(k)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)\exp_q(qx), \qquad Q_0(x) = 1.$$

Then

$$\int_0^\infty Q_k(x)Q_m(x)d\alpha(x) = \delta_{m,k}[k]_q![k]_q!\, q^k.$$

Proof. It suffices to show the assertion in the case $d\alpha(x) = d\overline{\nu}^a(x)$ for an arbitrary measure $d\overline{\nu}^a(x)$ defined in Theorem 2.1, i.e. such that

$$\int_0^\infty f(x)d\overline{\nu}(x) = \int_0^\infty f(x)\frac{1}{\exp_q(qx)}d_q^a(x).$$

Let $s, k, m \in \mathbf{N}$ and $s \le k$, $m \le k$. Then the Leibniz rule (3.1) gives

$$D_q^{(s-1)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right) = \frac{xW_{k-1}(x)}{\exp_q(q^{s-k}x)},$$

where $W_{k-1}(x)$ has degree $(k-1)$.

Therefore

$$\begin{aligned} T &= \int_0^\infty D_q\left(D_q^{(s-1)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)x^m\right)d_q^1(x) \\ &= D_q^{(s-1)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)x^m\Bigg]_\infty^0 = \frac{W_{k-1}(x)x^{m+1}}{\exp_q(q^{s-k}x)}\Bigg]_\infty^0 = 0. \end{aligned}$$

By taking derivative under the integral we get:

$$\begin{aligned} T = q^m &\int_0^\infty D_q^{(s)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)x^m d_q^1(x) \\ &+ [m]_q \int_0^\infty D_q^{(s-1)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)x^{m-1} d_q^1(x), \end{aligned}$$

which implies the following recurrence formula:

$$\int_0^\infty D_q^{(s)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)x^m d_q^1(x) = -\frac{[m]_q}{q^m}\int_0^\infty D_q^{(s-1)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)x^{m-1} d_q^1(x).$$

Let $s = k$. Then iteration of the above formula m-times gives:

$$\int_0^\infty Q_k(x)x^m d\overline{\nu}(x) = (-1)^m [m]_q! q^{-\binom{m+1}{2}} \int_0^\infty D_q^{(k-m)}\left(\frac{x^k}{\exp_q(q^{-k+1}x)}\right)x^0 d_q^1(x).$$

For $k > m$ we have then

$$\int_0^\infty Q_k(x)x^m d\overline{\nu}(x) = 0.$$

However, for $k = m$ we get:

$$\begin{aligned} \int_0^\infty Q_k(x)x^k d\overline{\nu}(x) &= (-1)^k [k]_q! q^{-\binom{k+1}{2}} \int_0^\infty \frac{x^k}{\exp_q(q^{-k+1}x)} d_q^1(x) \\ &= (-1)^k [k]_q! q^{-\binom{k+1}{2}} \sum_{p=-\infty}^{\infty} \frac{(q-1)q^p q^{pk}}{\exp_q(q^{-k+1+p}x)} \\ &= (-1)^k [k]_q! q^{-\binom{k+1}{2}} q^{k^2+k} \sum_{s=-\infty}^{\infty} \frac{(q-1)q^s q^{sk}}{\exp_q(q^{s+1}x)} \\ &= (-1)^k [k]_q! q^{-\binom{k+1}{2}} q^{k^2+k} \int_0^\infty \frac{x^k}{\exp_q(qx)} d_q^1(x) \\ &= (-1)^k q^{\frac{(k+1)k}{2}} [k]_q! [k]_q!. \end{aligned}$$

Summing up:

$$\int_0^\infty Q_k(x)Q_m(x)d\overline{\nu}(x) = 0 \quad \text{for } k \neq m$$

and

$$\int_0^\infty Q_k(x)Q_k(x)d\overline{\nu}(x) = \int_0^\infty Q_k(x)(-1)^k q^{-\frac{k(k-1)}{2}} x^k d\overline{\nu}(x) = q^k [k]_q![k]_q!. \ \blacksquare$$

Now we will estimate polynomials $Q_n(x)$ as n tends to infinity. First, we calculate the generating function for

$$L_n(x) = \frac{Q_n(x)}{[n]_q!}, \quad \text{i.e.} \quad G(r,x) = \sum_{s=0}^\infty r^n L_n(x).$$

Secondly we will look at the Taylor coefficients of the resulting function at zero. Finally, we apply the Darboux method (see Lemma 3.4).

To calculate $G(r,x)$ we will use the binomial theorem (see [GR] for the proof).

THEOREM 3.1. *Let $0 < q < 1$ and*

$$h_a(z) = \sum_{n=0}^\infty \frac{(a;q)_n}{(q;q)_n} z^n.$$

Then $h_a(z)$ is holomorphic in $\{z; |z| < 1\}$ and

$$h_a(z) = \lim_{p\to\infty} \frac{(az;q)_p}{(z;q)_p} = \frac{(az;q)_\infty}{(z;q)_\infty}. \tag{3.3}$$

In the next lemma we calculate the generating function for $L_n(x)$.

LEMMA 3.3. *If $L_n(x) = \frac{Q_n(x)}{[n]_q!}$ then*

$$G(r,x) = \sum_{n=0}^\infty r^n L_n(x) = \sum_{s=0}^\infty \frac{(-rx)^s}{[s]_q!(r;q^{-1})_{s+1}} q^{-\binom{s}{2}}. \tag{3.4}$$

and defines a holomorphic function in $\{z; |z| < 1\}$.

Proof. We know that

$$L_n(x) = \sum_{s=0}^n \frac{[n]_q!}{[n-s]_q!} \frac{1}{[s]_q![s]_q!} (-1)^s q^{\binom{s+1}{2}} q^{-sn} x^s.$$

Let $\sum_{n=0}^\infty r^n L_n(x) = \sum_{s=0}^\infty A_s(r)x^s$. We want to calculate $A_s(r)$. The following equality holds:

$$\frac{[n+s]_q!}{[n]_q!} = [s]_q! q^{sn} \frac{(q^{-(s+1)};q^{-1})_n}{(q^{-1};q^{-1})_n}. \tag{3.5}$$

For simplicity we will write $[m]_q = [m]$. Remember that $q > 1$. Then

$$\begin{aligned} A_s(r) &= \frac{(-1)^s}{[s]![s]!} q^{\binom{s+1}{2}} \sum_{n=s}^{\infty} \frac{[n]!}{[n-s]!} r^n q^{-sn} = \frac{(-1)^s}{[s]![s]!} q^{\binom{s+1}{2}} q^{-s^2} r^s \sum_{n=0}^{\infty} r^n q^{-sn} \frac{[n+s]!}{[n]!} \\ &= \frac{(-1)^s}{[s]![s]!} q^{\binom{s+1}{2}} q^{-s^2} r^s [s]! \sum_{n=0}^{\infty} r^n \frac{(q^{-(s+1)}; q^{-1})_n}{(q^{-1}; q^{-1})_n} \\ &\quad \text{[Now we use the binomial theorem]} \\ &= \frac{(-1)^s}{[s]![s]!} q^{\binom{s+1}{2}} q^{-s^2} r^s [s]! \frac{(q^{-(s+1)} r; q^{-1})_\infty}{(r; q^{-1})_\infty} \\ &= \frac{(-1)^s}{[s]![s]!} q^{\binom{s+1}{2}} q^{-s^2} r^s [s]! \frac{1}{(r; q^{-1})_{s+1}} = \frac{(-r)^s}{[s]!(r; q^{-1})_{s+1}} q^{-\binom{s}{2}}. \end{aligned} \tag{3.6}$$

This proves (3.4). Further for $|r| < 1 - \epsilon$ and arbitrary k we have $|(r; q^{-1})_k| > C$ and $[s]_q! \geq s!$ for arbitrary s. This implies that the series (3.4) is uniformly convergent on the given above area. ∎

Now we are ready to describe the asymptotics of $L_n(x)$.

We will use a lemma about coefficients of the power series of functions holomorphic in the neighbourhood of 0. (see [GR]).

LEMMA 3.4. *Let $f(r) = \sum_{n=0}^{\infty} a_n r^n$ be a meromorphic function with a finite number of poles, holomorphic around zero. Let $r_1, \ldots, r_k$ be critical points closest to zero. Let the critical points $r_1, \ldots, r_k$ be the poles of the first degree. Then*

$$\left| a_n - \sum_{i=1}^{k} -(r_i)^{-(n+1)} \mathrm{Res}(f, r_i) \right| = o(|r_1|^{-n}). \tag{3.7}$$

As a consequence we get:

THEOREM 3.2. *Let $L_n(x) = Q_n(x)/[n]_q!$. Then*

$$L_\infty(x) = \lim_{n\to\infty} L_n(x) = \sum_{s=0}^{\infty} \frac{(-x)^s}{[s]_q!(q^{-1}; q^{-1})_s} q^{-\binom{s}{2}}. \tag{3.8}$$

Proof. From lemma 3.3 we know that

$$\sum_{n=0}^{\infty} r^n L_n(x) = \sum_{s=0}^{\infty} \frac{(-rx)^s}{[s]_q!(r; q^{-1})_{s+1}} q^{-\binom{s}{2}} = f(r).$$

The function $f(r)$ has a pole at $r = 1$. The remaining poles are at points $r_k = q^k > 1$. Additionally

$$Res(f, 1) = \lim_{r\to 1}(r-1)f(r) = -\sum_{s=0}^{\infty} \frac{(-x)^s}{[s]_q!(q^{-1}; q^{-1})_s} q^{-\binom{s}{2}}.$$

Application of Lemma 3.4 ends the proof. ∎

REMARK 3.1. The polynomials L_n are connected with the q-Bessel function:

$$J_0(x, q^{-1}) = \sum_{s=0}^{\infty} \frac{(-1)^s q^{-s^2} \left(\frac{x}{2}\right)^{2s}}{[s]_{\frac{1}{q}}! [s]_{\frac{1}{q}}!}.$$

Also

$$L_\infty(x) = J_0\Big(2q\sqrt{\frac{x}{q-1}}, q^{-1}\Big).$$

4. Two special radial measures. The problem of finding all measures which satisfy $\int_0^\infty x^k d\nu = m_k$ for a given sequence of numbers m_k is called the Stieltjes moment problem. In our case we have $m_k = [k]_q!$. In section 2 we prove that the problem has infinitely many solutions. Instead of considering the Stieltjes moment problem we will consider the problem of finding symmetric measures which satisfy: $\int_{-\infty}^\infty x^k = [k]_q!$. We will apply the general theory to our case. With every moment problem we can associate two sequences of polynomials $P_n(x)$, $R_n(x)$. They are solutions of the following recurrence relation:

$$\omega_{n+1}(x) = (x - \alpha_{n+1})\omega_n(x) - \beta_n \omega_{n-1}(x),$$

with initial conditions:

$$R_0(x) = 1, \qquad R_1(x) = x - \alpha_1, \qquad P_0(x) = 0, \qquad P_1(x) = 1,$$

where $\alpha_n \in \mathbf{R}$ for $n > 0$ and $\beta_n \geq 0$ for $n > 0$.

The polynomials $R_n(x)$ generated in this way are orthogonal with respect to the measure which solves the problem. Moreover α_n, β_n are uniquely determined by the measure. Conversely, arbitrary pair of sequences $(\alpha_n \in \mathbf{R})$ and $(\beta_n \geq 0)$ correspond to some measure with moments uniquely determined by (α_n) and (β_n). Additionally $\alpha_n = 0$ for every n if and only if $m_{2s+1} = 0$ for each s (see [A] for details).

In our case as we show in lemmas 3.1 and 3.2 that

$$R_n(x) = \sum_{k=0}^{n} \binom{n}{k}_q (-1)^{n+k} q^{\frac{k(k+1-2n)}{2}} q^{\binom{k}{2}} \frac{[n]_q!}{[k]_q!} x^k.$$

Our moment problem $\int_0^\infty x^k d\nu = [k]_q!$ is the Stieltjes moment problem.

PROPOSITION 4.1. *There is one to one correspondence between the solutions of the problem $\int_0^\infty x^k d\nu = m_k$ and the symmetric solutions of the problem:*

$$\int_{-\infty}^{\infty} x^s d\mu = \begin{cases} 0 & \text{if } s = 2k+1, \\ m_k & \text{if } s = 2k, \end{cases} \tag{4.1}$$

The problem (4.1) for the sequence $m_k = [k]_q!$ does not have a unique solution. The following classical theorem is true (see [A]).

THEOREM 4.1. *If the moment problem $\int_{-\infty}^\infty x^k d\nu = m_k$ does not have a unique solution then the set of measures $\Theta(dx, \sigma)$ which solve the problem is indexed by the functions $\sigma(z)$, which are analytic in $\mathbf{C}^+$ and satisfy $Im\sigma(z) \leq 0$ for $Imz > 0$. Additionally, there exist entire functions $A(z), B(z), C(z), D(z)$ such that*

$$\int_{-\infty}^{\infty} \frac{\Theta(dx, \sigma)}{z - x} = \frac{A(z) - \sigma(z)C(z)}{B(z) - \sigma(z)D(z)}.$$

A, B, C, D are almost uniform limits of A_n, B_n, C_n, D_n, where

$$\begin{aligned}
A_{n+1} &= [P_{n+1}(z)P_n(0) - P_{n+1}(0)P_n(z)]c_n^{-1},\\
B_{n+1} &= [R_{n+1}(z)P_n(0) - P_{n+1}(0)R_n(z)]c_n^{-1},\\
C_{n+1} &= [P_{n+1}(z)R_n(0) - R_{n+1}(0)P_n(z)]c_n^{-1},\\
D_{n+1} &= [R_{n+1}(z)R_n(0) - R_{n+1}(0)R_n(z)]c_n^{-1},
\end{aligned}$$

where

$$c_n = (\beta_1 \cdots \beta_n) = \int_{-\infty}^{\infty} |R_n(x)|^2 \Theta(dx).$$

Additionally $\sigma(z)$ is uniquely determined by $\Theta(dx, \sigma)$.

There are some special solutions.

THEOREM 4.2. *Let $\Theta(dx, \varrho)$ be the solution to our moment problem, which corresponds to the function $\sigma(z) = const. = \varrho$, where $\varrho \in \mathbf{R}$. Then the support of the measure $\Theta(dx, \varrho)$ coincides with zeros of the function $B(z) - \varrho D(z)$, $z \in \mathbf{R}$. The zeros $x_j(\varrho_1)_{j=0}^{\infty}$ interlace with the zeros $x_j(\varrho_2)_{j=0}^{\infty}$ for $\varrho_1 \neq \varrho_2$. Also, for arbitrary $t \in \mathbf{R}$, there exists a unique ϱ_0, such that $t \in supp\, \Theta(dx, \varrho_0)$. Additionally $x_j(\varrho)$ is monotonic and continuous in ϱ.*

REMARK 4.1. The solutions $\Theta(dx, \varrho)$ described in the above theorem are called N-extremal. For such measures polynomials are dense in $L^2(\Theta(dx, \sigma))$.

The symmetric moment problem has only two symmetric N-extremal solutions. They correspond to the cases $\varrho = 0$ and $\varrho = +\infty$. They correspond to N-extremal solutions of the corresponding Stieltjes moment problem (see Proposition 4.1). The case $\sigma(z) = 0$ corresponds to N-extremal solution with support which has the longest distance to zero from all solutions. The case $\sigma(z) = +\infty$ corresponds to N-extremal measure with support which contains zero.

The supports of these measures are limit points of zeros of orthogonal polynomials of even and odd degree.

LEMMA 4.1. *For the symmetric moment problem we have:*

$$B(z) = \lim_{n\to\infty} a_n R_{2n}(z), \qquad D(z) = \lim_{n\to\infty} b_n R_{2n+1}(z),$$

for some sequences a_n, b_n, where $B(z), D(z)$ are as in Theorem 4.1.

Proof. For the symmetric moment problem, P_n and R_n satisfy the following recurrence formula:

$$\omega_{n+1}(x) = x\omega_n(x) - \beta_n \omega_{n-1}(x), \qquad \beta_n > 0.$$

Therefore $P_0(0) = 0$ i $R_1(0) = 0$ implies that $P_{2s}(0) = 0$ and $R_{2s+1}(0) = 0$ for every natural s. Applying this to theorem 4.1 we get the assertion. ∎

We can calculate explicitly orthogonal polynomials for our symmetric moment problem.

LEMMA 4.2. *Let*

$$\int_0^{\infty} x^k d\Theta_1 = m_k \quad \text{and} \quad \int_{-\infty}^{\infty} x^s d\Theta_2 = \begin{cases} 0 & \text{if } s = 2k+1,\\ m_k & \text{if } s = 2k. \end{cases}$$

Let $R_n^{(1)}$ and $R_n^{(2)}$ be sequences of orthogonal monomials with respect to Θ_1 and Θ_2 respectively. Then

$$R_{2n}^{(1)}(x) = R_n^{(2)}(x^2) \quad \text{for } x \in \mathbf{R}.$$

We know that $R_{2k}(x) = (-1)^k q^{\binom{k}{2}} Q_k(x^2)$. We also know that $xR_{2k+1} = R_{2k+2} + \gamma_k R_{2k}$. Since $R_{2k+2} + \gamma_k R_{2k}(0) = 0$ we have $\gamma_k = [k+1]q^k$. Further

$$\begin{aligned} xR_{2k+1} &= x^{2k+2} + \sum_{s=1}^{k} (-1)^{s+k+1} \frac{[k+1]_q!}{[s]_q!} q^{\binom{k+1}{2}} q^{\frac{s(s+1-2k-2)}{2}} \left(\binom{k+1}{s}_q - q^s \binom{k}{s}_q \right) x^{2s} \\ &= \sum_{s=1}^{k+1} (-1)^{s+k+1} \frac{[k+1]_q!}{[s]_q!} q^{\binom{k+1}{2}} q^{\frac{s(s+1-2k-2)}{2}} \binom{k}{s-1}_q x^{2s}. \end{aligned}$$

Therefore

$$R_{2k+1} = x \sum_{s=0}^{k} (-1)^{s+k} \frac{[k+1]_q!}{[s+1]_q!} q^{\binom{k+1}{2}} q^{\frac{(s+1)(s-2k)}{2}} \binom{k}{s}_q x^{2s}.$$

Analogously as in case of $L_n(x)$ we can estimate the asymptotics of $R_{2k+1}(x)$ as $k \to \infty$. First we will calculate the generating function.

LEMMA 4.3. *Let $M_n(x) = \frac{R_{2n+1}(x)}{[n+1]_q!} q^{-\binom{n}{2}}$. Then*

$$\sum_{n=0}^{\infty} r^n M_n(x) = x \sum_{s=0}^{\infty} \frac{(rx^2)^s}{[s+1]_q!(-r;q^{-1})_{s+1}} q^{-\binom{s}{2}}.$$

The application of the Darboux method [see lemma 3.4] finally gives:

THEOREM 4.3. *Let*

$$M_n(x) = \frac{R_{2n+1}(x)}{[n+1]_q!} q^{-\binom{n}{2}}.$$

Then

$$\lim_{n\to\infty} M_n(x) = M_\infty(x) = x \sum_{s=0}^{\infty} \frac{(x)^{2s}(-1)^s}{[s+1]_q!(q^{-1};q^{-1})_s} q^{-\binom{s}{2}}.$$

Additionally the polynomials M_n are connected with the q-Bessel function:

$$M_\infty(x) = J_1\left(2q\sqrt{\frac{x^2}{q-1}}, q^{-1}\right) \frac{\sqrt{q-1}}{q},$$

where

$$J_1(x, q^{-1}) = \sum_{s=0}^{\infty} \frac{(-1)^s q^{-s^2-s} \left(\frac{x}{2}\right)^{2s+1}}{[s+1]_{q^{-1}}! [s]_{\frac{1}{q}}!}.$$

The proof is the same as in case of $L_n(x)$ and therefore we omit it.

We have the following corollary.

COROLLARY 4.1. *Symmetric N-extremal measures of the symmetric problem:*

$$\int_{-\infty}^{\infty} x^s d\mu = \begin{cases} 0 & \text{if } s = 2k+1, \\ [k]_q! & \text{if } s = 2k, \end{cases}$$

have their supports concentrated at zeros of the functions:

$$xJ_1\left(2\sqrt{\frac{x^2}{q-1}}, q^{-1}\right) \quad \textit{and} \quad J_0\left(2\sqrt{\frac{x^2}{q-1}}, q^{-1}\right)$$

respectively, where J_0, J_1 *are the q-Bessel functions of degree zero and one.*

As in case of classical Bessel function we know only approximate value of the zeros of $J_0(z, q^{-1})$ and $J_1(z, q^{-1})$. Details can be found in [IM].

5. Characterization of non-radial measures. This section is devoted to the description of non-radial solutions of our initial complex moment problem. It turns out that the problem of the existence of such measures is closely connected with the number of solutions of the real problem $\int_0^\infty x^k d\nu = m_k$.

THEOREM 5.1. *A non-radial measure which satisfies*

$$\int_{\mathbf{C}} z^n \overline{z}^k \mu(dz) = \delta_{k,n} m_n \tag{$\diamond\diamond\diamond$}$$

exists if and only if there exists a measure ν *on* $\mathbf{R}_+$ *which has the following properties:*

$$\int_0^\infty x^{2k} \nu(dx) = m_k$$

and there exists $f(x) \neq 0$ *v-a.e. such that*

$$\int_0^\infty f(x) x^{2k} \nu(dx) = 0 \textit{ for } k \in \mathbf{N} \textit{ and } |f(x)| \leq x^N \textit{ on } \mathbf{R}_+ \textit{ for some natural } N.$$

Proof. Necessity. Suppose that $d\mu(z) = d\mu_{(r)}(\varphi) d\nu(r)$, where $\int_0^{2\pi} \mu_x(\varphi) = 1$ is a non-radial measure with the property:

$$\int_{\mathbf{C}} z^n \overline{z}^k d\mu(z) = \delta_{k,n} m_n.$$

Then for some $n > 0$ $f(x) := x^n \int_0^{2\pi} e^{in\varphi} d\mu_x(\varphi) \neq 0$.

Sufficiency—construction. Let us define

$$\mu'_x(\varphi) = \begin{cases} \dfrac{|f(x)|}{\pi x^N} & \text{for } \dfrac{2\pi k}{N} \leq \varphi < \dfrac{2\pi k + \pi}{N} \quad \text{if } f(x) > 0, \\[2ex] \dfrac{|f(x)|}{\pi x^N} & \text{for } \dfrac{2\pi k + \pi}{N} \leq \varphi < \dfrac{2\pi(k+1)}{N} \quad \text{if } f(x) < 0, \\[2ex] 0 & \text{otherwise,} \end{cases}$$

and $d\mu(z) = (d\mu'_x(\varphi) + d\mu''_x(\varphi)) d\nu(x)$ and $\mu''_x(d\varphi) = \frac{1}{2\pi}(1 - \frac{|f(x)|}{x^N}) d\varphi$.

We will show that the measure defined above fulfills all the requirements of the theorem.

First, for $s \in \mathbf{N}$,

$$\int_{\mathbf{C}} \overline{z}^l z^{l+(2s+1)N} d\mu(z) = \int_0^\infty x^{2(l+sN)+N} \int_0^{2\pi} e^{i(2s+1)\varphi} \mu_x'(\varphi) d\varphi \nu(x)$$

$$= [\theta = N\varphi] = \int_0^\infty x^{2l+2sN+N} \frac{1}{N} \int_0^{2\pi N} e^{i(2s+1)\theta} \mu_x'(\frac{\theta}{N}) d\theta \nu(x)$$

$$= \int_0^\infty x^{2l+2sN+N} \frac{1}{N} N \int_0^{2\pi} e^{i(2s+1)\theta} \mu_x'(\frac{\theta}{N}) d\theta \nu(x)$$

$$= \int_0^\infty x^{2l+2sN+N} \left[\int_0^{\pi} e^{i(2s+1)\theta} \mu_x'(\frac{\theta}{N}) d\theta \nu(x) + \int_\pi^{2\pi} e^{i(2s+1)\theta} \mu_x'(\frac{\theta}{N}) d\theta \nu(x) \right]$$

$$= \int_0^\infty x^{2l+2sN+N} \left[\frac{|f(x)| + f(x)}{2\pi x^N} \int_0^{\pi} e^{i(2s+1)\theta} d\theta - \frac{|f(x)| - f(x)}{2\pi x^N} \int_0^{\pi} e^{i(2s+1)\theta} d\theta \right] d\nu(x)$$

$$= \int_0^\pi e^{i(2s+1)\theta} d\theta \frac{1}{\pi} \int_0^\infty x^{2l+2sN} f(x) d\nu(x) = 0.$$

Next, for $s \in \mathbf{N}_+$,

$$\int_{\mathbf{C}} \overline{z}^l z^{l+2sN} d\mu(z) = \int_0^\infty x^{2(l+sN)} \int_0^{2\pi} e^{i(2s+1)N\varphi} \mu_x'(\varphi) d\varphi \nu(x)$$

$$= [\theta = N\varphi] = \int_0^\infty x^{2l+2sN} \frac{1}{N} \int_0^{2\pi N} e^{i(2s)\theta} \mu_x' \left(\frac{\theta}{N} \right) d\theta \nu(x)$$

$$= \int_0^\infty x^{2l+2sN} \int_0^{2\pi} e^{i2s\theta} \mu_x' \left(\frac{\theta}{N} \right) d\theta d\nu$$

$[2\theta = \alpha]$

$$= \int_0^\infty x^{2l+2sN} \frac{1}{2} \int_0^{4\pi} e^{is\alpha} \mu_x' \left(\frac{\alpha}{2N} \right) d\alpha \nu(x)$$

$$= \int_0^\infty x^{2l+2sn} \frac{1}{2} \left[\int_0^{2\pi} e^{is\alpha} \mu_x' \left(\frac{\alpha}{2N} \right) d\alpha \nu(x) + \int_{2\pi}^{4\pi} e^{is\alpha} \mu_x' \left(\frac{\alpha}{2n} \right) d\alpha \nu(x) \right]$$

$[\mu_x'(\alpha/2N)$ on $[0, 2\pi]$ and on $[2\pi, 4\pi]$ does not depend on $\alpha]$

$$= \frac{1}{2} \int_0^\infty x^{2l+2sN} \left[C_o \int_0^{2\pi} e^{is\alpha} d\alpha \nu(x) + C_1 \int_{2\pi}^{4\pi} e^{is\alpha} d\alpha \nu(x) \right] = 0.$$

Finally, for $m \in \mathbf{N},\ m \neq kN$,

$$= \int_{\mathbf{C}} \overline{z}^l z^{l+m} d\mu(z) = \int_0^\infty x^{2l+m} \int_0^{2\pi} e^{im\varphi} \mu_x'(\varphi) d\varphi \nu(x)$$

$$= \int_0^\infty x^{2l+m} \sum_{s=0}^{N-1} \int_{2\pi s \frac{1}{N}}^{2\pi(s+1)\frac{1}{N}} e^{im\varphi} \mu_x'(\varphi) d\varphi d\nu$$

[the construction]

$$= \int_0^\infty x^{2l+m} \sum_{s=0}^{N-1} \int_0^{2\pi \frac{1}{N}} e^{im(\varphi + k\frac{2\pi}{N})} \mu_x'(\varphi) d\varphi d\nu$$

$$= \int_0^\infty x^{2l+m} \int_0^{2\pi \frac{1}{N}} e^{im\varphi} \mu_x'(\varphi) d\varphi d\nu \sum_{s=0}^{N-1} e^{ik\frac{2\pi m}{N}} = 0.$$

We can apply this theorem to an arbitrary moment problem which does not have a unique solution.

COROLLARY 5.1. *Let ν_1 and ν_2 be two different solutions of the problem:*

$$\int_0^\infty x^{2k} d\nu(x) = m_k.$$

Then, also $\nu(x) = \frac{\nu_1(x)+\nu_2(x)}{2}$ is a solution to the problem. Let $f(x) = \frac{\nu_1-\nu_2}{\nu_1+\nu_2} x^2$. Then we have $|f(x)| \le 2x^2$ and $\int_0^\infty f(x)x^{2n}\nu(dx) = 0$ for all natural n.

Now we can apply theorem 5.1 and get non-radial measures which define the scalar product in the Bargmann representation for $q > 1$.

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