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A MOMENT SEQUENCE IN THE q-WORLD

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Abstract. The aim of the paper is to present some initial results about a possible generalization of moment sequences to a so-called q-calculus. A characterization of such a q-analogue in terms of appropriate positivity conditions is also investigated. Using the result due to Maserick and Szafraniec, we adapt a classical description of Hausdorff moment sequences in terms of positive definiteness and complete monotonicity to the q-situation. This makes a link between q-positive definiteness and q-complete monotonicity.

The classical results due to Stieltjes, Hamburger and Hausdorff allow one to characterize moment sequences via positive definiteness and complete monotonicity (depending on the support of measure). The situation changes in the q-world. Although one of the possible modifications of the positivity condition has already been introduced by Ota and Szafraniec (cf. [11]), there is no clear idea how to define q-analogues of complete monotonicity, or of moment sequences. This gives the following picture:

$$\begin{array}{llllll} \text{classic:} & \text{MP} & \sim & \text{PD} & \& & \text{CM} \\ & \downarrow & \downarrow & \downarrow & & \downarrow \\ q\text{-world:} & ? & ? & q\text{PD} & & ? \end{array}$$

Moreover, in the classical case a nice description of completely monotonic sequences is known: a sequence $\{a_n\}_n$ is completely monotonic if and only if both $\{a_n\}_n$ and $\{a_n - a_{n+1}\}_n$ are positive definite. We therefore ask whether some similar characterization exists in the q-deformed situation.

The inspiration to deal with q-analogues of complete monotonicity and moment sequences comes from [10]. The authors gave the proof of the aforesaid classical description of completely monotonic sequence without referring to their integral representation. The same method gives the opportunity to characterize q-completely monotonic sequences

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in terms of q-positive definiteness without specifying the integral representation of q-moments. This is the way we try to proceed. Further studies of the q-analogues give the idea how to define the q-moment sequence.

The paper is a written version of the talk at the 9th Workshop "Non-Commutative Harmonic Analysis with Applications to Non-Commutative Probability" in Będlewo '2006. That is the reason why we concentrate on results rather than on proofs (which are just sketched). For more details, we refer the reader to [7] which is an extended version of the talk.

The paper is organized as follows. Section 1 contains introductory information about q-calculus. In Section 2 we collect the basic definitions and results that will be used later. Definitions of q-positive definite and q-completely monotonic sequences are given and discussed in Sections 3 and 4. In particular, the main result (Theorem 4.3) gives a characterization of q-completely monotonic sequences in terms of q-positive definiteness. In Section 5 we investigate relations between the classical properties and their q-analogues and define (possibly one of) the q-analogue of a moment sequence.

In the following we set $q \in (0,1)$. All sequences appearing below have indices ranging from 0 to $+\infty$. $\mathbb{N} = \{0,1,2,\ldots\}$.

1. What is q-calculus? The q-calculus is a kind of "calculus without taking limits". It considers q-objects which are generalizations of some mathematical objects, parameterized by a quantity q. Moreover, the q-objects become the classical ones for $q \to 1$.

The development of q-analysis dates back to the 18th century, to Euler, Jacobi and Gauss (for more information about the history of the q-calculus see [3] and the references given there). The basis of the q-analysis was the q-binomial identity invented probably by Gauss: if x and y are elements of an associative algebra satisfying $xy = qyx$ (q-commutativity), then

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^{n-k} x^k, \text{ where } \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q! [k]_q!}$$

with

$$[n]_q = \frac{q^n - 1}{q - 1}\ (n \in \mathbb{N}) \quad \text{and} \quad [0]_q! = 1, \quad [n]_q! = [n]_q \cdot [n-1]_q!\ (n \geq 1).$$

Gauss also invented the hypergeometric series which were generalized and investigated by E. Heine in the mid-19th century. The first to develop q-calculus in a systematic way was F. H. Jackson at the beginning of the 20th century. He introduced the q-derivatives and q-integrals, the latter are currently known as Jackson's integrals. Since then a lot of nice identities were proved and the theory of q-special functions, q-hypergeometric functions and q-orthogonal polynomials was enriched thanks to S. I. Ramuanujan, J. Cigler, G. E. Andrews, R. Askey, G. Gasper, T. Koornwinder and many others (cf. [6]).

The notion of q-analogues ranges from the very simple to the very deep. We start with basic examples: q-integers $[n]_q$ and q-factorials $[n]_q!$ as defined above. Next we develop a one-dimensional q-analysis where q-objects (q-derivative, q-logarithm, etc.) are related to functions in one variable x. In this case q should be seen as a deformation parameter and x can be considered as a number. The problem to solve when searching for q-objects is where to put the q in the formula in question to obtain the desired limit relation, or rather: how to modify the classical notion in order to obtain an interesting q-object. There are generally several possibilities to do it and there may exist several q-analogous for one classical object (q-exponential functions, for instance).

The situation is even more complicated in 2-dimensional case, where the real noncommutative nature of q-calculus appears. We take two generators x and y which q-commute and consider the complex associative unital algebra generated by x and y. It is obvious that in this case x and y can no longer be realized by numbers but rather by operators. Such non-commuting variables appear in quantum groups which are the q-deformations of appropriate classical groups (for more on quantum groups see [4], [5] or [9]). Another way to get the q-commutative variables is the braided approach due to Majid [8].

2. Preliminaries. We start by recalling results on classical moment problems (cf. [12], [15]) and the presentation of basic definitions and results from [10].

A sequence $\{\varphi(n)\}_n$ is called:

- *positive definite* (PD) if for every $n \in \mathbb{N}$ and any real scalars $\alpha_1, \dots, \alpha_n$
$$\sum_{i,j=0}^{n} \alpha_i \alpha_j \varphi(i+j) \geq 0, \tag{1}$$
- *completely monotonic* (CM) if for all $k, n \in \mathbb{N}$
$$\sum_{m=0}^{k} (-1)^{m+k} \binom{k}{m} \varphi(n+k-m) \geq 0,$$
- *a (Hamburger) moment sequence* if there exists a Borel measure μ on $\mathbb{R}$ such that we have
$$\varphi(n) = \int_{\mathbb{R}} t^n d\mu(t), \quad n \in \mathbb{N}. \tag{2}$$

The measure μ is often called the *representing measure* for the sequence $\{\varphi(n)\}_n$. If the measure is concentrated on the interval $[0, +\infty)$ (respectively, on $[0,1]$), then the sequence is called a *Stieltjes* (resp. *Hausdorff*) *moment sequence.*

Note that the definition of positive definiteness is usually formulated in a more general context, i.e. for a function on a semigroup with involution. This is done in the following way (cf. [1]): Let $(G, \circ, ^*)$ be a semigroup with involution. A function $\varphi : G \to \mathbb{C}$ is called positive definite if
$$\sum_{i,j=0}^{n} \alpha_i \overline{\alpha}_j \varphi(s_i^* \circ s_j) \geq 0, \tag{3}$$
where $s_0, \dots, s_n \in G$, $\alpha_0, \dots, \alpha_n \in \mathbb{C}$. In this paper, however, we consider only the additive semigroup of positive integers $(\mathbb{N}, +)$ with identity involution. Then the mapping $\mathbb{N} \times \mathbb{N} \ni (i,j) \to \varphi(i+j) \in \mathbb{R}$ is symmetric and thus
$$\sum_{i,j=0}^{n} \alpha_i \overline{\alpha}_j \varphi(i+j) = \sum_{i,j=0}^{n} a_i a_j \varphi(i+j) + \sum_{i,j=0}^{n} b_i b_j \varphi(i+j),$$

where $a_i, b_i \in \mathbb{R}$, $a_i = \mathfrak{Re}\, \alpha_i$, $b_i = \mathfrak{Im}\, \alpha_i$ (see [1], 1.6). Hence the positivity condition (3) with complex α's is equivalent to the condition (3) with real scalars as stated in (1).

THEOREM 2.1 (cf. [1]). *A sequence $\{\varphi(n)\}_n$ is completely monotonic if and only if its (classical) m-th differences, i.e.*

$$\Delta_0\varphi(n_0) = \varphi(n_0),$$
$$\Delta_{m+1}\varphi(n_0; n_1, \ldots, n_{m+1}) = \Delta_m\varphi(n_0; n_1, \ldots, n_m) - \Delta_m\varphi(n_0 + n_{m+1}; n_1, \ldots, n_m),$$

are nonnegative for all $m \in \mathbb{N}$ and $n_0, \ldots, n_m \in \mathbb{N}$.

THEOREM 2.2 (Hamburger). *A necessary and sufficient condition for $\{\varphi(n)\}_n$ to be a Hamburger moment sequence is that it is positive definite.*

THEOREM 2.3 (Stieltjes). *A sequence $\{\varphi(n)\}_n$ admits an integral representation* (2) *with the measure concentrated on the interval* $[0, +\infty)$ *if and only if it is positive definite and the sequence $\{\varphi(n+1)\}_n$ is also positive definite.*

THEOREM 2.4 (Hausdorff). *A sequence $\{\varphi(n)\}_n$ admits an integral representation* (2) *with the measure concentrated on the interval* $[0, 1]$ *if and only if it is completely monotonic.*

Let $\mathcal{R}$ be a commutative algebra with identity 1 and involution *. Call a subset $\tau \subset \mathcal{R}$ *admissible* if the following conditions are satisfied:

1. $x^* = x$ for all $x \in \tau$;
2. $1 - x \in \mathrm{Alg}^+(\tau)$ for all $x \in \tau$, where $\mathrm{Alg}^+(\tau)$ is the set of all nonnegative combinations of (finite) products of members of τ;
3. $\mathcal{R} = \mathrm{Alg}(\tau)$, *i.e.* every $x \in \mathcal{R}$ is a combination of (finite) products of members of τ.

A linear functional f on $\mathcal{R}$ is called *τ-positive* if τ is admissible and $f(x) \geq 0$ for all $x \in \mathrm{Alg}^+(\tau)$. Following standard conventions, f is called *positive* if $f(x^*x) \geq 0$ for all $x \in \mathcal{R}$. If f is positive then we set

$$|x|_f^2 = \sup_{y \in \mathcal{R}} \frac{f(x^*xy^*y)}{f(y^*y)}$$

($\frac{0}{0} = 0$) and we call f *bounded* whenever $|x|_f < \infty$ for all $x \in \mathcal{R}$.

For all $x \in \mathcal{R}$ define the *shift operator* E_x on the set of all linear functionals on $\mathcal{R}$ by

$$E_x f(y) = f(xy), \ y \in \mathcal{R}.$$

THEOREM 2.5 (Maserick, Szafraniec [10]). (1) *Let f be a bounded positive linear functional on $\mathcal{R}$. If τ is admissible and $E_x f$ is positive for all $x \in \tau$, then f is τ-positive.*
(2) *If f is τ-positive for an admissible τ, then f is positive and bounded and $E_x f$ is positive for all $x \in \tau$.*

COROLLARY 2.6. *If $\{\varphi(n)\}_n$ is completely monotonic then $\{\varphi(n+k)\}_n$ and $\{\varphi(n) - \varphi(n+k)\}_n$ (for all $k \in \mathbb{N}$) are positive definite.*

Proof. Apply Theorem 2.5 to $\mathcal{R} = \{S_k; k \in \mathbb{N}\}$ and $\tau = \{S_k, I - S_k; k \in \mathbb{N}\}$, where S_k is defined by $(S_k\mu)(n) := \varphi(n+k)$ for a sequence $\{\varphi(n)\}_n$. See [10] for details. ∎

3. q-positive definite sequences

Definition 1. The sequence $\{\varphi(n)\}_n$ is called *q-positive definite* (qPD) if for all $n \in \mathbb{N}$ and all real scalars $\alpha_1, \dots, \alpha_n$

$$\sum_{i,j=0}^{n} q^{-ij} \alpha_i \alpha_j \varphi(i+j) \geq 0.$$

The motivation for such a definition comes from the theory of q-deformed normal operators (see [11]). Note that a sequence $\{\varphi(n)\}_n$ is q-positive definite in the sense of the definition given above if and only if it is q^{-1}-positive definite in the sense of the definition given by Ota and Szafraniec.

A good way to analyze q-positivity of a sequence is to relate it to some functionals. For this, consider a linear space $\mathcal{F}$ of all real sequences with the identity involution $\{\varphi(n)\}_n^* = \{\varphi(n)\}_n$ and define the linear map $F_m : \mathcal{F} \to \mathcal{F}$, called a *$q$-shift*, by the formula

$$F_m \varphi(k) := q^{-mk} \varphi(k+m), \quad \{\varphi(n)\}_n \in \mathcal{F}.$$

It is clear that the set $\mathcal{R} = \mathrm{Lin}\{F_m; m \in \mathbb{N}\}$ is a commutative algebra with identity $I = F_0$ and involution $F_i^* = F_i$. Moreover, $F_m F_n = q^{-nm} F_{m+n}$ and since $F_m = q^{m(m-1)/2} F_1^m$, the set $\tau = \{F_1, I - F_m; m \in \mathbb{N}\}$ is admissible.

Now, any sequence $\{\varphi(n)\}_n \in \mathcal{F}$ can uniquely define (and thus be identified with) a linear functional f on $\mathcal{R}$. Indeed, the condition

$$f(F_n) = \varphi(n)$$

defines the values of f on the basis $\{F_m; m \in \mathbb{N}\}$ and f extends by linearity on the whole $\mathcal{R}$.

A natural question arises here: what are the linear functionals corresponding to the qPD sequences? Observe that for $p = \sum \alpha_i F_i \in \mathcal{R}$ we have

$$f(p^* p) = \sum_{i,j=0}^{n} \alpha_i \alpha_j f(F_i^* F_j) = \sum_{i,j=0}^{n} \alpha_i \alpha_j F_i F_j \varphi(0) = \sum_{i,j=0}^{n} \alpha_i \alpha_j \, q^{-ij} \varphi(i+j).$$

Thus the positivity of the left hand side is equivalent to the positivity of the right hand side. What we have proved is the following result.

Proposition 3.1. *The sequence $\{\varphi(n)\}_n$ is q-positive definite if and only if the corresponding linear functional f on $\mathcal{R}$ is positive.*

4. q-complete monotonicity. For a sequence $\{\varphi(n)\}_n$ we define (the q-generalization of) its *m-th differences* by the formula

$$\begin{aligned}
\Delta_0 \varphi(n_0) = {} & \Delta_0^{(q)} \varphi(n_0) = \varphi(n_0),\\
\Delta_{m+1} \varphi(n_0; n_1, \dots, n_{m+1}) &= \Delta_{m+1}^{(q)} \varphi(n_0; n_1, \dots, n_{m+1})\\
&= \Delta_m \varphi(n_0; n_1, \dots, n_m) - q^{-n_0 n_{m+1}} \Delta_m \varphi(n_0 + n_{m+1}; n_1, \dots, n_m).
\end{aligned}$$

Definition 2. The sequence $\{\varphi(n)\}_n$ is called *q-completely monotonic* (qCM) if $\Delta_m \varphi(n_0; n_1, \dots, n_m) \geq 0$ for arbitrary $m \in \mathbb{N}$ and $n_0, \dots, n_m \in \mathbb{N}$.

For $q \to 1$ the definition above leads to the classical one. The modification is justified by the following result that can be easily proved by induction with respect to m.

PROPOSITION 4.1.

$$\Delta_m \varphi(n_0; n_1, \ldots, n_m) = F_{n_0} \prod_{k=1}^{m} (I - F_{n_k}) \varphi(0), \quad \textit{for all } m, n_0, \ldots, n_m \in \mathbb{N}.$$

This formula, which is the q-analogue of the formula in the classical case (see [10]), provides a description of the linear functionals corresponding to the qCM sequences.

PROPOSITION 4.2. *The sequence* $\{\varphi(n)\}_n$ *is q-completely monotonic if and only if the corresponding functional* f *is* τ*-positive with respect to the set* $\tau = \{F_1, I - F_m; m \in \mathbb{N}\}$.

Proof. Observe that all elements in $\mathrm{Alg}^+(\tau)$ are of the form

$$x = \sum_{i=1}^{n} \alpha_i x_i, \text{ where } \alpha_i \geq 0,\ x_i = F_1^{n_{0,i}} \prod_{k=1}^{m_i} (I - F_{n_{k,i}})$$

and use the fact that

$$\Delta_m \varphi(n_0; n_1, \ldots, n_m) = q^{n_0(n_0-1)/2} f(F_1^{n_0} \prod_{k=1}^{m} (I - F_{n_k})). \ \blacksquare$$

Now we are ready to present the main theorem which gives a characterization of q-completely monotonic sequences in terms of q-positive definiteness.

THEOREM 4.3. *The sequence* $\{\varphi(n)\}_n$ *is* qCM *if and only if the following conditions are satisfied*

$$\{\varphi(n)\}_n \textit{ is } q\,PD, \tag{qCM1}$$

$$\{q^{-n}\varphi(n+1)\}_n \textit{ is } q\,PD, \tag{qCM2}$$

$$\forall_{m \in \mathbb{N}}\ \{\varphi(n) - q^{-nm}\varphi(n+m)\}_n \textit{ is } q\,PD. \tag{qCM3}$$

For the proof, let us fix the sequence $\{\varphi(n)\}_n$ and take a corresponding functional f on $\mathcal{R}$ given by

$$f(F_n) = F_n \varphi(0) = \varphi(n).$$

We want to apply Theorem 2.5 and it is convenient to divide the proof into several independent lemmas each of which can be proved by a direct calculation.

LEMMA 4.4. $\{q^{-n}\varphi(n+1)\}_{n\in\mathbb{N}}$ *is qPD if and only if* $E_{F_1} f$ *is positive.*

Proof. For $y = \sum_{i=0}^{n} \alpha_i F_i \in \mathcal{R}$ we have

$$E_{F_1} f(yy^*) = \sum_{i,j=0}^{n} q^{-ij} \alpha_i \alpha_j [q^{-(i+j)} \varphi(i+j+1)]. \ \blacksquare$$

LEMMA 4.5. $\{\varphi(n) - q^{-nm}\varphi(n+m)\}_{n\in\mathbb{N}}$ *is qPD if and only if* $E_{I-F_m} f$ *is positive.*

Proof. For $m \in \mathbb{N}$ and $y = \sum_{i=0}^{n} \alpha_i F_i \in \mathcal{R}$ we have

$$E_{(I-F_m)} f(yy^*) = \sum_{i,j=0}^{n} q^{-ij} \alpha_i \alpha_j [\varphi(i+j) - q^{-m(i+j)} \varphi(i+j+m)]. \ \blacksquare$$

LEMMA 4.6. *If f, $E_{F_1}f$ and $E_{F_m}f$ (for all $m \in \mathbb{N}$) are positive (or equivalently, the conditions* (qCM1)-(qCM3) *are satisfied), then f is bounded.*

Proof. Taking appropriate scalars in the positivity conditions, we get that $\varphi(2m) \geq 0$, $\varphi(2m+1) \geq 0$ and $\varphi(m) \leq \varphi(0)$ for all $m \in \mathbb{N}$. Thus $|f(F_m)| = |\varphi(m)| \leq \varphi(0)$, *i.e.* f is bounded. ∎

Proof of the main theorem. Suppose the sequence $\{\varphi(n)\}_n$ is qCM. It follows from the Proposition 4.2 that the functional f on $\mathcal{R}$ given by

$$f(F_n) = F_n\varphi(0) = \varphi(n)$$

is τ-positive with respect to the admissible set $\tau = \{F_1, I - F_m; m \in \mathbb{N}\}$. Then Theorem 2.5 states that f is positive and bounded and $E_x f$ is positive for every $x \in \tau$. According to Proposition 3.1 and Lemmas 4.4 and 4.5 this means that the conditions (qCM1), (qCM2) and (qCM3) are satisfied.

Suppose the converse. The condition (qCM1) implies that f is positive, while the next two conditions imply positivity of $E_x f$ for every $x \in \tau$. Moreover, according to Lemma 4.6, f is bounded. Theorem 2.5 implies that f is τ-positive, which is equivalent to the fact that $\{\varphi(n)\}_n$ is qCM. ∎

5. q-moment sequences. In this section we investigate the relation between the classical and the q-properties and find that a description of the class of q-positive definite sequences in terms of some integral representation can be easily obtained due to the Hamburger theorem. This allows us to state a definition of q-moment sequence. We start with an easy observation.

PROPOSITION 5.1. *A sequence $\{\varphi_n\}_n$ is qPD if and only if the sequence $\{\mu_n\}_n$, where $\mu_n = q^{-\frac{n(n-1)}{2}}\varphi_n$, is PD.*

Proof.

$$\sum_{n,m=0}^{N} a_n a_m \mu_{m+n} = \sum_{n,m=0}^{N} a_n a_m q^{-\frac{(m+n)(m+n-1)}{2}}\varphi_{m+n}$$
$$= \sum_{n,m=0}^{N} (q^{-\frac{n(n-1)}{2}}a_n)(q^{-\frac{m(m-1)}{2}}a_m)q^{-mn}\varphi_{m+n} = \sum_{n,m=0}^{N} b_n b_m q^{-mn}\varphi_{m+n},$$

where $N \in \mathbb{N}$ and $b_n = q^{-\frac{n(n-1)}{2}}a_n$. Thus the positivity of the left hand side is equivalent to the positivity of the right hand side. ∎

This Proposition together with the Hamburger theorem 2.2 gives us a description of the class of q-positive definite sequences.

COROLLARY 5.2. *Any q-positive definite sequence can be represented in the form*

$$\varphi_n = \int_{\mathbb{R}} q^{\frac{n(n-1)}{2}} t^n \, d\mu(t), \; n \in \mathbb{N},$$

where μ is a representing measure for the sequence $\{q^{-\frac{n(n-1)}{2}}\varphi_n\}_n$.

The result above suggests the following definition of q-moment sequences.

DEFINITION 3. Call $\{\varphi_n\}_n$ a *q-moment sequence* if there exists a Borel measure μ on some set $X \subset \mathbb{R}$ such that

$$\varphi_n = \int_X q^{\frac{n(n-1)}{2}} t^n d\mu(t), \quad n \in \mathbb{N}.$$

As in the classical case, we may talk about Hamburger, Stieltjes or Hausdorff q-moment sequences depending on the support of the representing measure (respectively: $X = \operatorname{supp}\mu = \mathbb{R}, [0, +\infty), [0, 1]$). Corollary 5.2 states that q-positive definiteness is a necessary and sufficient condition for a sequence to be a Hamburger q-moment sequence.

The natural question to ask in this case is whether a similar description is true for q-moments with measure concentrated on other two intervals. Actually, the answer is positive and for the case $\operatorname{supp}\mu = [0, +\infty)$ can obtained by a direct calculation. For $\operatorname{supp}\mu = [0, 1]$ one of the implications may also be easily shown.

PROPOSITION 5.3. *A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ is a Stieltjes q-moment sequence with the measure μ on $[0, +\infty)$ if and only if $\{\varphi_n\}_{n\in\mathbb{N}}\}$ is $q\,PD$ and $\{q^{-n}\varphi_{n+1})\}_{n\in\mathbb{N}}$ is $q\,PD$.*

Proof. The assertion follows from the Stieltjes Theorem 2.3 and fact that q-positive definiteness of the sequence $\{q^{-n}\varphi_{n+1}\}_n$ is equivalent to positive definiteness of the sequence $\{\mu_{n+1}\}_n$. But this is obvious because of the following calculation.

$$\begin{aligned}\sum_{n,m=0}^{N} a_n a_m \mu_{m+n+1} &= \sum_{n,m=0}^{N} a_n a_m q^{-\frac{(m+n+1)(m+n)}{2}} \varphi_{m+n+1} \\ &= \sum_{n,m=0}^{N} b_n b_m q^{-mn} q^{-(m+n)} \varphi_{m+n+1},\end{aligned}$$

where $b_n = q^{-\frac{n(n-1)}{2}} a_n$. ■

PROPOSITION 5.4. *If a sequence $\{\mu_n\}_n$ is CM, then $\{q^{\frac{n(n-1)}{2}}\mu_n\}_n$ is qCM.*

Proof. Take a completely monotonic sequence $\{\mu_n\}_n$ and define $\varphi_n = q^{\frac{n(n-1)}{2}}\mu_n$. According to Theorem 4.3 we need to show that $\{\varphi_n\}_n$ satisfies (qCM1), (qCM2) and (qCM3) conditions.

From the classical theory of moment sequences we know that $\{\mu_n\}_n$ and $\{\mu_n - \mu_{n+1}\}_n$ are PD. Moreover, from the Corollary 2.6 we conclude that for every $k \in \mathbb{N}$ the sequences $\{\mu_{n+k}\}_n$ and $\{\mu_n - \mu_{n+k}\}_n$ are PD as well. Now, apply Propositions 5.1 and 5.3 to ensure that (qCM1) and (qCM2) hold.

Next, observe that for an arbitrary $k \in \mathbb{N}$

$$\sum_{n,m=0}^{N} b_n b_m q^{-mn} q^{-(m+n)k} \varphi_{m+n+k} = q^{\frac{k(k-1)}{2}} \sum_{n,m=0}^{N} a_n a_m \mu_{m+n+k} \geq 0, \tag{4}$$

where $N \in \mathbb{N}$ and $b_n = q^{-\frac{n(n-1)}{2}} a_n$, providing the sequence $\{\mu_n\}_n$ is completely monotonic.

Finally, observe that we have

$$\sum_{n,m=0}^{N} b_n b_m q^{-mn} \varphi_{m+n} = \sum_{n,m}^{N} a_n a_m \mu_{m+n} \geq \sum_{n,m}^{N} a_n a_m \mu_{m+n+k}$$
$$= q^{-\frac{k(k-1)}{2}} \sum_{n,m}^{N} b_n b_m q^{-mn} q^{-k(m+n)} \varphi_{m+n+k} \geq \sum_{n,m}^{N} b_n b_m q^{-mn} q^{-k(m+n)} \varphi_{m+n+k}.$$

The last inequality follows from (4) and the fact that $q^{-\frac{k(k-1)}{2}} \geq 1$ for $q \in (0,1)$. It means that $\{\varphi_n - q^{-nm}\varphi_{n+m}\}_n$ is qPD which finishes the proof. ∎

To prove the opposite implication, we need some more advanced arguments: the RKHS technique used as in [11] and [13] (for more on this subject see [14]).

THEOREM 5.5. *If a sequence $\{\varphi_n\}_n$ is qCM, then there exists a measure μ on $[0,1]$ such that*

$$\varphi_n = \int_{[0,1]} q^{\frac{n(n-1)}{2}} t^n d\mu(t), \quad n \in \mathbb{N}.$$

Proof. Let $\{\varphi(n)\}_n$ be a q-completely monotonic sequence, *i.e.* (by Theorem 4.3) the sequence satisfying conditions (qCM1), (qCM2) and (qCM3). Define the (positive definite) kernel on $\mathbb{N}$ by the formula

$$K(n,m) := q^{-mn}\varphi_{n+m}, \quad n, m \in \mathbb{N},$$

Now, the factorization theorem of Aronszajn (cf. [14], for example) implies that there exists a Hilbert space $\mathcal{H}$ and a mapping $\mathbb{N} \ni n \mapsto \gamma_n \in \mathcal{H}$ such that

$$\mathcal{H} = \overline{\mathrm{Lin}}\,\{\gamma_n; n \in \mathbb{N}\}, \quad K(n,m) = \langle \gamma_n, \gamma_m \rangle.$$

Next, we set

$$\mathcal{D} := \mathrm{Lin}\,\{\gamma_n; n \in \mathbb{N}\}, \quad T : \mathcal{D} \ni \sum_n \alpha_n \gamma_n \mapsto \sum_n \alpha_n q^{-n} \gamma_{n+1} \in \mathcal{D}.$$

Observe that the operator $\bar{T}$ (the closure of T) is closed, symmetric and has a cyclic vector, thus it admits a self-adjoint extension S in the same space $\mathcal{H}$ (cf. [2]) and by spectral theorem for self-adjoint operators (cf. [2]) there exists a spectral measure E such that

$$S = \int_{\mathbb{R}} t dE(t).$$

The representing measure for the sequence $\{\varphi(n)\}_n$ is given by $\mu(\sigma) := \langle E(\sigma)\gamma_0, \gamma_0 \rangle$ for all Borel sets $\sigma \subset \mathbb{R}$.

Finally, standard calculations yield the inequality $0 \leq \langle Su, u \rangle \leq \langle u, u \rangle$. But this is equivalent to the fact that the measure μ is concentrated on the interval $[0,1]$ and thus the assertion follows. ∎

COROLLARY 5.6. *For a sequence $\{\varphi_n\}_n$ the following conditions are equivalent:*

1. *$\{\varphi_n\}_n$ is qCM,*
2. *$\{q^{-\frac{n(n-1)}{2}}\varphi_n\}_n$ is CM,*

3. $\{\varphi_n\}_n$ *is a Hausdorff q-moment sequence. i.e. there exists a measure* μ *on* $[0,1]$ *such that*

$$\varphi_n = \int_{[0,1]} q^{\frac{n(n-1)}{2}} t^n d\mu(t), \quad n \in \mathbb{N}.$$

Proof. Implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) follow from Proposition 5.4 and Theorem 5.5. Implication (3) $\Rightarrow$ (2) is a consequence of the Hausdorff Theorem 2.4. ∎

Final remarks. The results presented above show that the so defined q-moment sequences can be viewed as the weighted (classical) moment sequences as long as $0 < q < 1$. A natural question is whether this is still the case for $q > 1$. The answer is negative in general, but some partial results remain true. More details can be found in [7].

The reader should also notice that we considered here only the one-dimensional sequences whereas many interesting features of the q-calculus appear while looking on the algebra generated by two q-commuting variables. This may motivate the investigation of the two-dimensional moment sequences which we intend to do elsewhere.

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