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ASYMPTOTIC SPECTRAL ANALYSIS OF GENERALIZED ERDŐS–RÉNYI RANDOM GRAPHS

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Abstract. Motivated by the Watts–Strogatz model for a complex network, we introduce a generalization of the Erdős–Rényi random graph. We derive a combinatorial formula for the moment sequence of its spectral distribution in the sparse limit.

1. Introduction. Since the epoch-making papers by Watts–Strogatz [13] in 1998 and by Barabási–Albert [1] in 1999 the *network science* has become one of the most fashionable interdisciplinary research areas in current years. Various network models proposed so far are very interesting from the mathematical point of view too. We aim at exploring spectral properties of complex networks with mathematical rigor, along with the quantum probabilistic techniques (see Hora–Obata [11] and references cited therein).

As characteristics of real world complex networks, simple statistics such as the degree distribution, the mean distance of vertices, and the cluster coefficient have been discussed in many papers. To go into further detailed structure, spectral analysis is expected to be one of the promising directions, as was indicated by Dorogovtsev–Goltsev–Mendes–Samukhin [7], Dorogovtsev–Mendes [8], Farkas–Derényi–Barabási–Vicsek [9], Rodgers–Austin–Kahng–Kim [12], and others. For recent relevant works see Chung–Lu [5] and references cited therein.

In this paper, motivated by the Watts–Strogatz model [13], we propose a model for a complex network and study its spectra in the sparse limit. Although the Watts–Strogatz

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model is described in terms of a simple algorithm and is suitable for computer simulation, its mathematical analysis seems very difficult because as a stochastic process it requires to remember long history. Our model is obtained by simplifying this point and, as a result, shares a common spirit with the famous Erdős–Rényi random graph $\mathcal{G}(n,p)$, which is a random graph with n vertices where an edge between two vertices occurs with probability p and is independent of other edges, see Bollobás [3] for a comprehensive survey. Our model $\mathcal{G}(n,R;p,p')$ is, in a strict sense, not a random "graph" but a random "network." In fact, in terms of the adjacency matrix $A=(A_{ij})$, the matrix element A_{ij} is a random variable taking values in $\{0,1/2,1\}$. Moreover, the distribution of A_{ij} varies according to a "geometric distance" between two vertices i and j, defined by a relation R among vertices.

This paper is organized as follows. In Section 2 we assemble some basic notions and notations. In Section 3 we recall a combinatorial formula (Theorem 3.3) for computing the moments of eigenvalue distribution of a random matrix. In Section 4 we define a generalized Erdős–Rényi random graph $\mathcal{G}(n,R;p,p')$ and derive a combinatorial formula for the moments of its mean spectral distribution (Theorem 4.3). In Section 5 we propose two concrete models by specifying a relation R in our model $\mathcal{G}(n,R;p,p')$. In both cases the mean degree is a linear combination of p,p',np' so we are interested in the sparse limit taken as

$$n\to\infty, \quad p'\to 0, \quad np'\to\lambda \text{ (constant)}.$$

We derive formulae for the moments of their mean spectral distributions in the sparse limit in terms of graph-geometric characteristics (Theorems 5.8 and 5.10). Section 6 contains some remarks on the Erdős–Rényi random graphs.

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2. Preliminaries

2.1. *Spectral distribution of a finite graph.* Let $G=(V,E)$ be a finite graph, where V is the vertex set (a non-empty finite set) and E the edge set (a subset of the two-point subsets of V). Let $A=(A_{ij})_{i,j\in V}$ be the *adjacency matrix* of G defined by

$$A_{ij}=\begin{cases}1, & \text{if } \{i,j\}\in E,\\ 0, & \text{otherwise.}\end{cases}$$

Clearly, A is a symmetric matrix with zero diagonal elements and off-diagonal elements taking values in $\{0,1\}$. Let $\lambda_1<\lambda_2<\cdots<\lambda_s$ be the eigenvalues of A with multiplicities $m_1,m_2,\dots,m_s$. These data are often referred to as the *spectrum* of G, for generalities see e.g., Chung [4], Cvetković–Doob–Sachs [6], Hora–Obata [11]. We associate a probability distribution μ_G on $\mathbb{R}$ defined by

$$\mu_G(dx)=\frac{1}{|V|}\sum_k m_k\delta(x-\lambda_k)dx,$$

which we call the *spectral distribution* of G (or of A). Note that μ_G is characterized by its moment sequence:

$$(2.1) \qquad M_m(\mu_G) = \int_{-\infty}^{+\infty} x^m \mu_G(dx) = \frac{1}{|V|} \operatorname{Tr}(A^m), \qquad m = 1, 2, \ldots.$$

2.2. *Mean spectral distribution of a random graph.* In general, by a *random graph* we mean a probability space $\mathcal{G}$ whose sample space consists of graphs. For a random graph G the spectral distribution μ_G becomes a random measure on $\mathbb{R}$ so we are interested in the mean spectral distribution $\mu = \mathbf{E}(\mu_G)$ which is again a probability measure on $\mathbb{R}$.

In this paper, we restrict ourselves to a random graph whose vertex set is common for all sample graphs. More precisely, let V be a finite set, say $V = \{0, 1, 2, \ldots, n-1\}$, and Ω the set of all graphs with vertex set V. A probability space (Ω, P), where P is a probability measure on Ω, is our random graph and denoted by $\mathcal{G}(n, P)$. Here we note that Ω is not a set of equivalence classes determined by graph-isomorphisms, but consists of $2^{\binom{n}{2}}$ sample graphs.

PROPOSITION 2.1. *The mean spectral distribution μ of a random graph $\mathcal{G}(n, P)$ is characterized by its moment sequence:*

$$(2.2) \qquad M_m(\mu) = \frac{1}{n} \mathbf{E}(\operatorname{Tr}(A^m)), \qquad m = 1, 2, \ldots.$$

Proof. Relation (2.2) follows by taking the expectation of (2.1). Since $\mu = \mathbf{E}(\mu_G)$ is supported by a finite subset of $\mathbb{R}$, we see by Carleman's moment test (e.g., Hora–Obata [11, Chapter 1]) that μ is uniquely determined by its moment sequence. ∎

2.3. *A random graph with independent edges.* Given a random graph $\mathcal{G}(n, P)$, the adjacency matrix $A = (A_{ij})_{i,j \in V}$ becomes a random matrix satisfying the following conditions:

(A0) A_{ij} is a $\{0, 1\}$-valued random variable;
(A1) the diagonal elements vanish, i.e., $A_{ii} = 0$ for all $i \in V$;
(A2) A is symmetric, i.e., $A_{ij} = A_{ji}$ for all $i, j \in V$.

Conversely, a random matrix A with index set V satisfying conditions (A0)–(A2) determines a random graph $\mathcal{G}(n, P)$ whose adjacency matrix is A.

Our concern in this paper is a random graph with independent edges, which means that the occurrence of an edge is independent of other edges. In terms of the adjacency matrix, this condition is equivalent to the following

(A3) the random variables $\{A_{ij}\,;\, 0 \le i < j \le n-1\}$ are independent.

Thus, it follows from Proposition 2.1 that the mean spectral distribution of a random graph with independent edges is reduced to computation of $\mathbf{E}(\operatorname{Tr}(A^m))$ for a random matrix A satisfying (A0)–(A3). A useful formula will be derived in the next section.

3. A combinatorial formula for computing moments

3.1. *Moments of a random matrix.* We consider a random matrix A with index set $V=\{0,1,2,\dots,n-1\}$ satisfying conditions (A1)–(A3) and, instead of (A0) we assume

(A0$'$) A_{ij} is a real-valued random variable having finite moments of all orders.

Let $\mathcal{D}(A)$ denote the set of distributions of the matrix elements of A.

Our goal in this section is to derive a combinatorial formula for computing $\mathbf{E}((A^m)_{00})$. For $m=1$ we see trivially from (A1) that

$$\mathbf{E}(A_{00})=0. \tag{3.1}$$

For $m\ge 2$ we start with the obvious expression:

$$\mathbf{E}((A^m)_{00})=\sum_{0\neq i_1\neq i_2\neq\cdots\neq i_{m-1}\neq 0}\mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}), \tag{3.2}$$

where condition (A1) is taken into account. The case of $m=1$ may be considered as a special case of (3.2) on the understanding that the sum over an empty set is zero. Remarks of this kind will be omitted below.

We need some notation. For $m\ge 2$ let $\mathcal{W}(V,m)$ be the set of sequences of elements in V of the form:

$$[i]:(i_0\equiv)\ 0\neq i_1\neq i_2\neq\cdots\neq i_{m-1}\neq 0\ (\equiv i_m). \tag{3.3}$$

Given $[i]\in\mathcal{W}(V,m)$ as in (3.3), let $G[i]$ denote the underlying graph. Namely, its vertex set $V(G[i])$ is defined to be the set of elements appearing in the sequence $[i]$ (including 0). Two distinct vertices $j,j'\in V(G[i])$ are adjacent by definition if there exists $0\le s\le m-1$ such that $\{i_s,i_{s+1}\}=\{j,j'\}$. Thus the edge set $E(G[i])$ is defined. It is then obvious that $[i]$ becomes an m-step walk in the graph $G[i]$ starting from and terminating at 0 and passing through all the edges.

We will assign a label to every edge of $G[i]$. For $e=\{j,j'\}\in E(G[i])$ define

$$\nu(e)=\text{the distribution of } A_{jj'}=A_{j'j}, \tag{3.4}$$

$$\kappa(e)=|\{0\le s\le m-1\,;\,\{i_s,i_{s+1}\}=\{j,j'\}\}|. \tag{3.5}$$

Note that $\kappa(e)$ is the number of how many times the walk $[i]$ passes through the edge e. Thus, each edge $e\in E(G[i])$ is given a label $(\nu(e),\kappa(e))\in\mathcal{D}(A)\times\{1,2,\dots,m\}$.

LEMMA 3.1. *For $m=1,2,\dots$ we have*

$$\mathbf{E}((A^m)_{00})=\sum_{[i]\in\mathcal{W}(V,m)}\prod_{e\in E(G[i])}M_{\kappa(e)}(\nu(e)), \tag{3.6}$$

where $M_\kappa(\nu)$ stands for the κ-th moment of ν.

Proof. Let $m\ge 2$ and consider a general term in (3.2):

$$\mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}),\qquad [i]\in\mathcal{W}(V,m).$$

On computing the above expectation we need to note that $A_{jj'}=A_{j'j}$ appears with multiplicities inside the bracket. So, writing

$$A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}=\prod_{0\le j<j'\le n-1}A_{jj'}^{s_{jj'}},\qquad s_{jj'}=0,1,2,\dots,$$

we apply independence condition (A3) to obtain the factorization:

$$\mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}) = \prod_{0\le j<j'\le n-1} \mathbf{E}(A_{jj'}^{s_{jj'}}).$$

Obviously, $s_{jj'} \geq 1$ occurs only when $\{j,j'\} \in E(G[i])$ and $s_{jj'} = \kappa(\{j,j'\})$. In this case,

$$\mathbf{E}(A_{jj'}^{s_{jj'}}) = M_{\kappa(\{j,j'\})}(\nu(\{j,j'\})).$$

Consequently,

$$\mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}) = \prod_{e\in E(G[i])} M_{\kappa(e)}(\nu(e))$$

and, taking a sum over $[i] \in \mathcal{W}(V,m)$, we obtain (3.6). ∎

3.2. *A labeled rooted graph.* We proceed to compute the right hand side of (3.6). Let $m \geq 2$ and $\mathcal{D}$ a finite set. A *$\mathcal{D}$-labeled rooted graph of size m*, denoted by $\mathcal{L} = (\mathcal{V},\mathcal{E},o,\nu,\kappa)$, consists of

(L1) a connected graph $(\mathcal{V},\mathcal{E})$ with $2 \leq |\mathcal{V}| \leq m$;
(L2) a distinguished vertex $o \in \mathcal{V}$ which is called the root;
(L3) a map $\nu : \mathcal{E} \to \mathcal{D}$;
(L4) a map $\kappa : \mathcal{E} \to \{1,2,\ldots,m\}$ such that $\sum_{e\in\mathcal{E}} \kappa(e) = m$.

The pair (ν,κ) is called the *label* of $\mathcal{L}$. We write $\mathcal{V} = V(\mathcal{L})$ and $\mathcal{E} = E(\mathcal{L})$. We note an obvious inequality:

$$|\mathcal{V}| - 1 \leq |\mathcal{E}| \leq m, \tag{3.7}$$

where the first one follows by connectivity of $(\mathcal{V},\mathcal{E})$ and the second from (L4), see also Proposition 5.4 below.

Two labeled rooted graphs are called *isomorphic* if there exists a graph-isomorphism preserving the root and label. Let $\Lambda_m(\mathcal{D})$ denote the complete set of representatives of $\mathcal{D}$-labeled rooted graphs of size m up to isomorphisms.

For $[i] \in \mathcal{W}(V,m)$, $m \geq 2$, the underlying graph $G[i]$ is naturally equipped with structure of a $\mathcal{D}(A)$-labeled rooted graph of size m, which is denoted by $\mathcal{L}[i] = (G[i],0,\nu,\kappa)$, where the label (ν,κ) is defined in (3.4) and (3.5). Noting that the product factor in (3.6) is constant on $[i]$'s generating isomorphic $\mathcal{L}[i]$'s, we obtain the following

Lemma 3.2. *For $m = 1,2,\ldots$ we have*

$$\mathbf{E}((A^m)_{00}) = \sum_{\mathcal{L}\in\Lambda_m(\mathcal{D}(A))} |\{[i] \in \mathcal{W}(V,m)\,;\, \mathcal{L}[i] \cong \mathcal{L}\}| \prod_{e\in E(\mathcal{L})} M_{\kappa(e)}(\nu(e)). \tag{3.8}$$

Finally, we study the combinatorial number appearing in the above formula. We need further notation. A *unicursal walk* on $\mathcal{L} = (\mathcal{V},\mathcal{E},o,\nu,\kappa) \in \Lambda_m(\mathcal{D})$ is a walk on the graph $(\mathcal{V},\mathcal{E})$ from the root o to itself such that every edge $e \in \mathcal{E}$ is passed through as many times as $\kappa(e)$. It follows from (L4) that a unicursal walk is necessarily of m-step. Let $u(\mathcal{L})$ denote the number of unicursal walks in $\mathcal{L}$. Obviously, $u(\mathcal{L})$ is independent of the ν-label of $\mathcal{L} = (\mathcal{V},\mathcal{E},o,\nu,\kappa)$.

An *A-admissible embedding* of $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa) \in \Lambda_m(\mathcal{D}(A))$ is an injective map $\varphi : \mathcal{V} \to \{0, 1, 2, \dots, n-1\}$ such that $\varphi(o) = 0$ and for every $\{v, v'\} \in \mathcal{E}$, $\nu(\{v, v'\})$ coincides with the distribution of $A_{\varphi(v)\varphi(v')}$. Let $t(\mathcal{L}; n)$ denote the number of A-admissible embeddings. The number is irrelevant to the κ-label of $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa)$.

THEOREM 3.3. *Let A be a random matrix indexed by $V = \{0, 1, 2, \dots, n-1\}$ satisfying conditions (A0′), (A1)–(A3). Then, for $m = 1, 2, \dots$ we have*

$$\mathbf{E}((A^m)_{00}) = \sum_{\mathcal{L} \in \Lambda_m(\mathcal{D}(A))} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n) \prod_{e \in E(\mathcal{L})} M_{\kappa(e)}(\nu(e)). \tag{3.9}$$

Proof. By Lemma 3.2 we need only to show that

$$|\{[i] \in \mathcal{W}(V, m)\,;\, \mathcal{L}[i] \cong \mathcal{L}\}| = |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n), \qquad \mathcal{L} \in \Lambda_m(\mathcal{D}(A)).$$

Let $\mathcal{L} \in \Lambda_m(\mathcal{D}(A)) = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa)$ be fixed. Let Ξ be the set of unicursal walks on $\mathcal{L}$ and Φ the set of A-admissible embeddings of $\mathcal{L}$. We first introduce a map $\Phi \times \Xi \to \mathcal{W}(V, m)$. For $\varphi \in \Phi$ and a unicursal walk $[\xi] \in \Xi$, which is of the form:

$$[\xi] : o \sim \xi_1 \sim \xi_2 \sim \cdots \sim \xi_{m-1} \sim o,$$

we define $\varphi([\xi])$ to be just the image of the above sequence. Then, since φ is injective and $\varphi(o) = 0$, we have

$$\varphi([\xi]) : 0 \neq \varphi(\xi_1) \neq \varphi(\xi_2) \neq \cdots \neq \varphi(\xi_{m-1}) \neq 0.$$

Indeed, $\varphi([\xi]) \in \mathcal{W}(V, m)$ so that $\mathcal{L}(\varphi([\xi])) \in \Lambda_m(\mathcal{D}(A))$ is defined. It is obvious that $\mathcal{L}(\varphi([\xi])) \cong \mathcal{L}$ by the properties of φ and $[\xi]$. Conversely, every $[i] \in \mathcal{W}(V, m)$ such that $\mathcal{L}[i] \cong \mathcal{L}$ is of the form $[i] = \varphi([\xi])$. In fact, an isomorphism $\psi : \mathcal{L}[i] \to \mathcal{L}$ induces a bijection from the vertex set of $G[i]$ onto $\mathcal{V}$. Let φ be the inverse of this bijection. It is obvious that $[\xi] = \psi([i])$ is a unicursal walk on $\mathcal{L}$ and $\varphi([\xi]) = [i]$. Thus,

$$\{[i] \in \mathcal{W}(V, m)\,;\, \mathcal{L}[i] \cong \mathcal{L}\} = \{\varphi([\xi])\,;\, (\varphi, [\xi]) \in \Phi \times \Xi\}.$$

We next need to examine condition for $\varphi([\xi]) = \varphi'([\xi'])$. We see easily that $\alpha = \varphi^{-1}\varphi'$ induces an automorphism of $\mathcal{L}$. In this case we have $(\varphi', [\xi']) = (\varphi \circ \alpha, \alpha^{-1}([\xi]))$. Conversely, if $(\varphi', [\xi'])$ is obtained from $(\varphi, [\xi])$ by an automorphism $\alpha \in \mathrm{Aut}\,(\mathcal{L})$ in this manner, we have $\varphi([\xi]) = \varphi'([\xi'])$. Since the action of $\alpha \in \mathrm{Aut}\,(\mathcal{L})$ is apparently faithful, there is a one-to-one correspondence between $\{[i] \in \mathcal{W}(V, m)\,;\, \mathcal{L}[i] \cong \mathcal{L}\}$ and $(\Phi \times \Xi)/\mathrm{Aut}\,(\mathcal{L})$. This completes the proof. ∎

Formulae equivalent to (3.9) have been implicitly or explicitly used in computation of moments of a random matrix, see e.g., Bauer–Golinelli [2], Hiai–Petz [10], Wigner [14, 15, 16].

4. Generalized Erdős–Rényi random graphs

4.1. *Construction.* As in the previous section we maintain $V = \{0, 1, 2, \dots, n-1\}$, $n \geq 1$, as a fixed vertex set and take two constant numbers $0 < p < 1$ and $0 < p' < 1$. Furthermore, we choose a subset

$$R \subset \{(i, j) \in V \times V\,;\, i \neq j\},$$

which will define a "geometric distance" among vertices. For $i, j \in V$ with $i \neq j$, let X_{ij} be a Bernoulli random variable such that

$$\begin{aligned} &P(X_{ij} = 1) = p, \quad P(X_{ij} = 0) = 1 - p, \quad \text{if } (i,j) \in R, \\ &P(X_{ij} = 1) = p', \quad P(X_{ij} = 0) = 1 - p', \quad \text{otherwise.} \end{aligned}$$

Moreover, we assume that the random variables $\{X_{ij}\,; i, j \in V,\, i \neq j\}$ are independent. We do not define X_{ii} though one may set it to be 0. Obviously,

$$\mathbf{E}(X_{ij}) = \begin{cases} p, & \text{if } (i,j) \in R, \\ p', & \text{otherwise.} \end{cases} \tag{4.1}$$

Now, for every pair $i, j \in V$ we set

$$A_{ij} = \begin{cases} \dfrac{1}{2}\,(X_{ij} + X_{ji}), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases} \tag{4.2}$$

Then, $A = (A_{ij})_{i,j \in V}$ becomes a random matrix satisfying (A0′), (A1)–(A3).

In a strict sense A does not represent a random "graph" but a slightly generalized one, which distinguishes a "grade" of connection, say, tight connection ($A_{ij} = 1$), loose connection ($A_{ij} = 1/2$) and no connection ($A_{ij} = 0$). The probability space of our generalized random graph is denoted by $\mathcal{G}(n, R; p, p')$.

Remark. One might prefer to the term "network," which means a graph with weighted edges. Then, $\mathcal{G}(n, R; p, p')$ is a random network. In this sense, $\mathcal{G}(n, R; p, p')$ is not a direct generalization of the Erdős–Rényi random graph, see Section 6.

We see from (4.2) that the distribution of A_{ij} ($i \neq j$) is essentially the convolution of two Bernoulli distributions coming from X_{ij} and X_{ji}. Therefore, A_{ij} obeys one of the three distributions according to the "geometric distance" of i, j given by R. More precisely, for $(i,j) \in R \cap R^t$, $R^t = \{(i,j)\,; (j,i) \in R\}$,

$$P(A_{ij} = 1) = p^2, \quad P(A_{ij} = 1/2) = 2p(1-p), \quad P(A_{ij} = 0) = (1-p)^2,$$

for $(i,j) \in (R \cup R^t) \backslash (R \cap R^t)$,

$$P(A_{ij} = 1) = pp', \quad P(A_{ij} = 1/2) = p + p' - 2pp', \quad P(A_{ij} = 0) = (1-p)(1-p'),$$

and otherwise,

$$P(A_{ij} = 1) = p'^2, \quad P(A_{ij} = 1/2) = 2p'(1-p'), \quad P(A_{ij} = 0) = (1-p')^2.$$

These distributions are denoted by α, β, γ, respectively. Thus $\mathcal{D}(A) = \{\alpha, \beta, \gamma\}$.

We say that a generalized random graph $\mathcal{G}(n, R; p, p')$ is *symmetric* if for any $i_0 \in V$ there exists a permutation σ on V such that $\sigma(i_0) = 0$ and for all $i, j \in V$, the distributions of A_{ij} and $A_{\sigma(i)\sigma(j)}$ coincide.

Proposition 4.1. *A generalized random graph $\mathcal{G}(n, R; p, p')$ is symmetric if and only if for any $i_0 \in V$ there exists a permutation σ on V such that*

$$\sigma(i_0) = 0, \quad \sigma(R \cap R^t) = R \cap R^t, \quad \sigma(R \cup R^t) = R \cup R^t.$$

Proof. The proof is straightforward. We only need to note that a permutation σ on V induces a bijection from $V \times V$ onto itself. ■

4.2. *Mean degree and mean spectral distribution.* For a vertex i of a graph G the degree $\deg_G(i)$ is defined to be the number of edges whose end vertices are i. In terms of the adjacency matrix we have

$$\deg_G(i) = \sum_{j \neq i} A_{ij} \,.$$

This identity can be used to define the degree of a vertex of our generalized random graph $\mathcal{G}(n, R; p, p')$. Then the mean degree is defined by

$$\bar{d}(\mathcal{G}(n, R; p, p')) = \frac{1}{n} \sum_{i \in V} \mathbf{E}(\deg_G(i)) = \frac{1}{n} \sum_{i \in V} \sum_{j \neq i} \mathbf{E}(A_{ij}).$$

THEOREM 4.2. *If $\mathcal{G}(n, R; p, p')$ is symmetric, its mean degree is given by*

$$\bar{d}(\mathcal{G}(n, R; p, p')) = pR_2 + \frac{p+p'}{2} R_1 + p'R_0, \tag{4.3}$$

where

$$\begin{aligned} R_2 &= |\{j \in V\,;\, (0,j) \in R \cap R^t\}|, \\ R_1 &= |\{j \in V\,;\, (0,j) \in R \cup R^t \textit{ but } \notin R \cap R^t\}|, \\ R_0 &= |\{j \in V\,;\, j \neq 0,\, (0,j) \notin R \cup R^t\}|. \end{aligned}$$

Moreover,

$$R_0 + R_1 + R_2 = n - 1. \tag{4.4}$$

Proof. Since the distribution of $\deg_G(i)$ coincides with that of $\deg_G(0)$ by symmetry, we have

$$\bar{d}(\mathcal{G}(n, R; p, p')) = \mathbf{E}(\deg_G(0)) = \sum_{j \neq 0} \mathbf{E}(A_{0j}).$$

The distribution of A_{0j} is α if $(0,j) \in R \cap R^t$, β if $(0,j) \in (R \cup R^t) \backslash (R \cap R^t)$, and γ if $(0,j) \neq R \cup R^t$. The mean values of α, β and γ are p, $(p+p')/2$ and p', respectively. Hence

$$\sum_{j \neq 0} \mathbf{E}(A_{0j}) = pR_2 + \frac{p+p'}{2} R_1 + p'R_0,$$

which proves the assertion. ■

THEOREM 4.3. *If $\mathcal{G}(n, R; p, p')$ is symmetric, its mean spectral distribution is characterized by the moment sequence:*

$$\begin{aligned} &M_m(n, R; p, p') \\ &\quad = \sum_{\mathcal{L} \in \Lambda_m(\{\alpha,\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n) \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e) = \alpha}} \alpha_{\kappa(e)} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e) = \beta}} \beta_{\kappa(e)} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e) = \gamma}} \gamma_{\kappa(e)}, \end{aligned}$$

where α_κ, β_κ, γ_κ are the κ-th moments of the distributions α, β, γ, respectively. Namely,

$$\alpha_\kappa = p^2 + \frac{2p(1-p)}{2^\kappa}, \quad \beta_\kappa = pp' + \frac{p+p'-2pp'}{2^\kappa}, \quad \gamma_\kappa = p'^2 + \frac{2p'(1-p')}{2^\kappa}\,. \tag{4.5}$$

Proof. Since $\mathcal{G}(n, R; p, p')$ is symmetric, $\mathbf{E}((A^m)_{ii})$ does not depend on the choice of $i \in V$ so the moment of the mean spectral distribution coincides with $\mathbf{E}((A^m)_{00})$. Then, as a direct consequence of Theorem 3.3 we arrive at the assertion. (4.5) is obtained directly from the definition. ■

For example,

$$\begin{aligned} M_2(n, R; p, p') &= R_2\alpha_2 + R_1\beta_2 + R_0\gamma_2 \\ &= \frac{1}{2}\,\bar{d}(\mathcal{G}(n, R; p, p')) + \frac{1}{2}\,(R_2 p^2 + R_1 pp' + R_0 p'^2). \end{aligned}$$

We are interested in asymptotic behavior of our model $\mathcal{G}(n, R; p, p')$ in the sparse limit, that is, as $n \to \infty$ while $\bar{d}(\mathcal{G}(n, R; p, p'))$ tends to a finite constant. We note from (4.3) and (4.4) that the sparse limit shares a similar spirit with the Poisson limit. In the next section we will study two concrete models.

5. Concrete models and their sparse limits

5.1. *Model I.* For convenience, for $i \in V$ we understand $i \pm 1$ to be a vertex in V uniquely determined by addition modulo n. Let $\mathcal{G}_I(n, p, p') = \mathcal{G}(n, R; p, p')$ be a generalized random graph with R defined by

$$R = \{(i, i+1)\,;\, i \in V\}.$$

It is obvious by Proposition 4.1 that $\mathcal{G}_I(n, p, p')$ is symmetric. Moreover, since $R \cap R^t = \emptyset$, there is no matrix element A_{ij} which obeys the distribution α. Namely, we have $\mathcal{D}(A) = \{\beta, \gamma\}$.

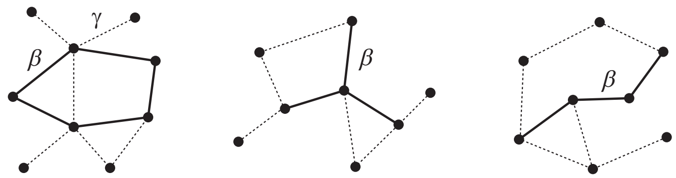

Fig. 1. β-cycle, β-branch, β-segment

Before applying Theorem 4.3 we note some obstruction to an A-admissible embedding. Let $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa) \in \Lambda_m(\{\beta, \gamma\})$ and consider an A-admissible embedding $\varphi : \mathcal{V} \to \{0, 1, 2, \dots, n-1\}$. If $e = \{j, j'\} \in \mathcal{E}$ is an β-*edge*, that is, an edge whose ν-label is β, $A_{\varphi(j)\varphi(j')}$ obeys the distribution β so that $\varphi(j) = \varphi(j') \pm 1$ by our relation R. Therefore, for a large n, there is no A-admissible embedding if (i) $\mathcal{L}$ contains an β-*cycle*, i.e., a cycle consisting of β-edges; or if (ii) $\mathcal{L}$ contains an β-*branch*, i.e., a vertex with three or more β-edges, see Figure 1. Here the assumption "for a large n" applies only to avoid a trivial case for (i). Now we set

$$\Lambda_m^*(\{\beta, \gamma\}) = \left\{ \mathcal{L} \in \Lambda_m(\{\beta, \gamma\})\,;\, \begin{array}{l} \text{(i) contains no } \beta\text{-cycles;} \\ \text{(ii) contains no } \beta\text{-branches} \end{array} \right\}.$$

In other words, in $\mathcal{L} \in \Lambda_m^*(\{\beta, \gamma\})$ every β-edge appears only as a linear segment (β-segment).

With the above argument Theorem 4.3 is simplified as follows.

THEOREM 5.1. *The m-th moment of the generalized random graph $\mathcal{G}_I(n,p,p')$ is characterized by the moment:*

$$M_m(n,p,p') = \sum_{\mathcal{L}\in\Lambda_m^*(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)}. \tag{5.1}$$

EXAMPLE 5.2. The first four moments are given as follows:

$$\begin{aligned}
&M_1(n,p,p') = 0,\\
&M_2(n,p,p') = 2\beta_2 + (n-3)\gamma_2,\\
&M_3(n,p,p') = 6\beta_1^2\gamma_1 + 6(n-4)\beta_1\gamma_1^2 + (n-4)(n-5)\gamma_1^3,\\
&M_4(n,p,p') = 2\beta_4 + (n-3)\gamma_4 + 4\beta_2^2 + 8(n-3)\beta_2\gamma_2 + 2(n-3)(n-4)\gamma_2^2\\
&\qquad + 8\beta_1^3\gamma_1 + 8(2n-9)\beta_1^2\gamma_1^2 + 8(n^2-9n+21)\beta_1\gamma_1^3\\
&\qquad + (n^3-14n^2+67n-110)\gamma_1^4,
\end{aligned}$$

where β_κ and γ_κ are given in (4.5). The computation is easy and tedious.

5.2. *Model I in the sparse limit.* By Theorem 4.2 the mean degree of $\mathcal{G}_I(n,p,p')$ is given by

$$\bar{d}(\mathcal{G}_I(n,p,p')) = p + (n-2)p',$$

which suggests studying the sparse limit:

$$n\to\infty, \quad p'\to 0, \quad np'\to\lambda \text{ (constant)}. \tag{5.2}$$

In this limit the mean degree tends to a finite constant $p+\lambda$ and

$$\lim \beta_\kappa = \frac{p}{2^\kappa}, \qquad \lim n\gamma_\kappa = \frac{2\lambda}{2^\kappa}. \tag{5.3}$$

Throughout this section the symbol "lim" means the sparse limit taken as in (5.2).

Our main interest is to obtain the limit of the m-th moment of the mean spectral distribution of $\mathcal{G}_I(n,p,p')$. More explicitly, in view of the formula in Theorem 5.1, we will compute the limit:

$$\lim M_m(n,p,p') = \lim \sum_{\mathcal{L}\in\Lambda_m^*(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)}. \tag{5.4}$$

Let us consider Example 5.2 on trial. In fact, we obtain

$$\begin{aligned}
&\lim M_1(n,p,p') = 0,\\
&\lim M_2(n,p,p') = \frac{p}{2} + \frac{\lambda}{2},\\
&\lim M_3(n,p,p') = 0,\\
&\lim M_4(n,p,p') = \frac{p}{8} + \frac{\lambda}{8} + \frac{p^2}{4} + p\lambda + \frac{\lambda^2}{2}.
\end{aligned} \tag{5.5}$$

Since $\gamma_\kappa = O(n^{-1})$ by (5.3), each term of $M_3(n,p,p')$ vanishes in the sparse limit. For $M_4(n,p,p')$ the first five terms in the first line contribute to the limit and the rest does not for the same reason. We will explicate the general structure behind by means of graph-geometric observation.

To begin with, we will estimate $t(\mathcal{L};n)$ as $n \to \infty$ by using the idea of *β-edge contraction.* With each $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa) \in \Lambda_m^*(\{\beta,\gamma\})$ we will associate a connected graph $\tilde{\mathcal{L}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ as follows. First we introduce a notation. Two vertices $v, v' \in \mathcal{V}$ are called *β-equivalent*, denoted by $v \overset{\beta}{\sim} v'$, if $v = v'$ or if there exists a path from v to v' consisting of β-edges. The set of β-equivalence classes is denoted by $\tilde{\mathcal{V}} = \mathcal{V}/\overset{\beta}{\sim}$. For $v \in \mathcal{V}$ let $\tilde{v}$ denote the β-equivalence class containing v. Then $\tilde{\mathcal{E}}$ consists of edges of the form $\{\tilde{v}, \tilde{v}'\}$, where $\{v, v'\}$ runs over $\mathcal{E}$. It may happen that $\tilde{v} = \tilde{v}'$, which does not yield an edge of $\tilde{\mathcal{L}}$ (we do not allow a loop). By construction,

$$|\tilde{\mathcal{E}}| \le |\mathcal{E}_\gamma| = |\{e \in \mathcal{E}\,;\, \nu(e) = \gamma\}| \tag{5.6}$$

is obvious. Next let us consider $(\mathcal{V}, \mathcal{E}_\beta)$. Since in $\mathcal{L}$ every β-edge appears as a linear segment, $(\mathcal{V}, \mathcal{E}_\beta)$ is a disjoint union of trees, or more precisely, of linear segments and isolated vertices. The number of connected components is equal to $|\tilde{\mathcal{V}}|$. Let $b(\mathcal{L})$ be the number of isolated vertices of $(\mathcal{V}, \mathcal{E}_\beta)$.

Lemma 5.3. *Let $\mathcal{L} \in \Lambda_m^*(\{\beta,\gamma\})$ and $(\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ the β-edge contraction. Then,*

$$2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})}\,(n-2m^2)^{|\tilde{\mathcal{V}}|-1} \le |t(\mathcal{L};n)| \le 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})}\,n^{|\tilde{\mathcal{V}}|-1}. \tag{5.7}$$

Proof. Let $\{T_0, T_1, \dots, T_{r-1}\}$ be the set of connected components of the graph $(\mathcal{V}, \mathcal{E}_\beta)$, where T_0 is the one which contains o. Apparently, $r = |\tilde{\mathcal{V}}|$. Recall that these connected components are linear segments or isolated vertices. For each $1 \le s \le r-1$ choose $\xi_s \in T_s$ arbitrarily. We shall define an A-admissible embedding $\varphi : \mathcal{V} \to V \equiv \{0, 1, 2, \dots, n-1\}$. Note the obvious inequality:

$$|T_s| \le |\mathcal{V}| \le m, \qquad 0 \le s \le r-1,$$

where the second inequality follows from condition (L1). Therefore, the connected component T_0 lies necessarily in the interval:

$$\varphi(T_0) \subset I_0 \equiv \{-(m-1), \dots, -1, 0, 1, \dots, m-1\} \subset V,$$

since $\varphi(o) = 0$ is a constraint by definition. Note that there are two ways of embedding if $|T_0| \ge 2$, but there is no such freedom if $|T_0| = 1$. Choose arbitrarily $x_1 \in V \backslash I_0$ and define $\varphi(\xi_1) = x_1$. Then $\varphi(T_1) \cap \varphi(T_0) = \emptyset$. Again there are two ways of embedding T_1 if $|T_1| \ge 2$, but only one otherwise. Set

$$I_1 = \{x_1 - (m-1), \dots, x_1, x_1+1, \dots, x_1 + (m-1)\}.$$

Choose arbitrarily $x_2 \in V \backslash (I_0 \cup I_1)$ and define $\varphi(\xi_2) = x_2$. Then, $\varphi(T_2), \varphi(T_1), \varphi(T_0)$ are mutually disjoint. There are one or two ways of embedding of T_2 according as T_2 consists how many vertices. Continuing this procedure up to $s = r-1$, we may define an A-admissible embedding φ of $\mathcal{L}$. In this way we are able to construct at least

$$2^{r-r_0}(n-(2m-1))(n-2(2m-1))\dots(n-(r-1)(2m-1)) \tag{5.8}$$

different φ, where r_0 is the number of T_s's consisting of single vertex, i.e., $r_0 = b(\mathcal{L})$.

Rather roughly (5.8) is estimated from below as

$$\geq 2^{r-r_0}(n-2m^2)^{r-1} = 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})}(n-2m^2)^{|\tilde{\mathcal{V}}|-1},$$

which proves the first half of (5.7). The second half is similarly and more easily verified. ∎

PROPOSITION 5.4. *Let $G=(V,E)$ be a connected graph. Then $|V| \leq |E|+1$. Moreover, equality holds if and only if G is a tree.*

Proof. Obvious. ∎

LEMMA 5.5. *Let $\mathcal{L} \in \Lambda_m^*(\{\beta,\gamma\})$. Then*

$$\lim t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)} = 0 \tag{5.9}$$

holds if (i) $\tilde{\mathcal{L}}$ contains a cycle, or if (ii) $\tilde{\mathcal{L}}$ is a tree such that $|\tilde{\mathcal{E}}| < |\mathcal{E}_\gamma|$.

Proof. Let $\mathcal{L} \in \Lambda_m^*(\{\beta,\gamma\})$. We see from Lemma 5.3 that

$$\begin{aligned} t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)} &\leq 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})}\, n^{|\tilde{\mathcal{V}}|-1} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)} \\ &= 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})}\, n^{|\tilde{\mathcal{V}}|-1-|\mathcal{E}_\gamma|} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} n\gamma_{\kappa(e)}\,. \end{aligned} \tag{5.10}$$

By (5.3) the last product converges in the sparse limit. Hence for (5.9) it is sufficient to show that $|\tilde{\mathcal{V}}|-1-|\mathcal{E}_\gamma| < 0$. In case (i), $|\tilde{\mathcal{V}}| \leq |\tilde{\mathcal{E}}|$ holds (see Proposition 5.4) so, combining with (5.6) we obtain $|\tilde{\mathcal{V}}| \leq |\mathcal{E}_\gamma|$. In case (ii) we have $|\tilde{\mathcal{V}}|-1 = |\tilde{\mathcal{E}}| < |\mathcal{E}_\gamma|$. ∎

It follows from Lemma 5.5 combining with (5.6) that for computation of (5.4) we need only to consider $\mathcal{L} \in \Lambda_m^*(\{\beta,\gamma\})$ such that $\tilde{\mathcal{L}}$ is a tree and $|\tilde{\mathcal{E}}| = |\mathcal{E}_\gamma|$. Set

$$\Lambda_m^{**}(\{\beta,\gamma\}) = \left\{ \mathcal{L} \in \Lambda_m^*(\{\beta,\gamma\})\,;\, \tilde{\mathcal{L}} \text{ is a tree and } |\tilde{\mathcal{E}}| = |\mathcal{E}_\gamma| \right\}.$$

Thus, our limit (5.4) is slightly simplified.

LEMMA 5.6. *Let $M_m(n,p,p')$ be the m-th moment of the mean spectral distribution of $\mathcal{G}_I(n,p,p')$. Then, in the sparse limit (5.2) we have*

$$\begin{aligned} &\lim M_m(n,p,p') \\ &\quad = \lim \sum_{\mathcal{L}\in\Lambda_m^{**}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)}\,. \end{aligned} \tag{5.11}$$

We now show the following important result.

LEMMA 5.7. *For $\mathcal{L}=(\mathcal{V},\mathcal{E},o,\nu,\kappa) \in \Lambda_m^*(\{\beta,\gamma\})$ the following two conditions are equivalent:*

(i) $\mathcal{L} \in \Lambda_m^{**}(\{\beta,\gamma\})$, *i.e., $\tilde{\mathcal{L}}$ is a tree and $|\tilde{\mathcal{E}}| = |\mathcal{E}_\gamma|$;*
(ii) $\mathcal{L}$ *is a tree.*

In particular,

$$\Lambda_m^{**}(\{\beta,\gamma\}) = \left\{ \mathcal{L} \in \Lambda_m(\{\beta,\gamma\})\,;\ \begin{array}{l} \mathcal{L} \text{ is a tree;} \\ \mathcal{L} \text{ contains no } \beta\text{-branches} \end{array} \right\}.$$

Proof. We prove with the help of the characterization of trees (Proposition 5.4) that (i) implies (ii). Let $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa) \in \Lambda_m^{**}(\{\beta,\gamma\})$. We maintain the same notation as in the proof of Lemma 5.3. Let $\{T_0, T_1, \dots, T_{r-1}\}$ be the set of connected components of the graph $(\mathcal{V}, \mathcal{E}_\beta)$. Since T_s is a tree,

$$|\mathcal{E}_\beta| = \sum_{s=0}^{r-1} |E(T_s)| = \sum_{s=0}^{r-1} (|V(T_s)| - 1) = |\mathcal{V}| - r. \tag{5.12}$$

On the other hand, $r = |\tilde{\mathcal{V}}|$ is apparent and $|\tilde{\mathcal{V}}| = |\tilde{\mathcal{E}}| + 1$ by assumption that $\tilde{\mathcal{L}}$ is a tree. Hence

$$r = |\tilde{\mathcal{E}}| + 1 = |\mathcal{E}_\gamma| + 1. \tag{5.13}$$

Combining (5.12) and (5.13), we arrive at

$$|\mathcal{V}| = |\mathcal{E}_\beta| + |\mathcal{E}_\gamma| + 1 = |\mathcal{E}| + 1,$$

which shows that $(\mathcal{V}, \mathcal{E})$ is a tree. This is what we wanted to show. The rest of the assertion is straightforward by definition. ■

THEOREM 5.8. *Let M_m be the m-th moment of the spectral distribution of the generalized random graph $\mathcal{G}_I(n,p,p')$ in the sparse limit* (5.2). *Then, for an odd m we have*

$$M_m = 0$$

and, for an even m we have

$$M_m = \sum_{\mathcal{L} \in \Lambda_m^{**}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\mathcal{E}_\gamma|+1-b(\mathcal{L})-m}\, p^{|\mathcal{E}_\beta|} (2\lambda)^{|\mathcal{E}_\gamma|}\,. \tag{5.14}$$

Moreover, $u(\mathcal{L}) = 0$ unless the κ-label of $\mathcal{L} \in \Lambda_m^{**}(\{\beta,\gamma\})$ is even-valued.

Proof. We observe the right hand side of (5.11). First suppose that m is odd. Since $\mathcal{L} \in \Lambda_m^{**}(\{\beta,\gamma\})$ is a tree by Lemma 5.7, obviously there is no unicursal walk of odd steps on $\mathcal{L}$. Namely, $u(\mathcal{L}) = 0$ for all $\mathcal{L} \in \Lambda_m^{**}(\{\beta,\gamma\})$, and $M_m = 0$ follows.

Next suppose that m is even. For $\mathcal{L} \in \Lambda_m^{**}(\{\beta,\gamma\})$ we compute

$$\lim\, t(\mathcal{L}; n) \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e) = \gamma}} \gamma_{\kappa(e)} = \lim t(\mathcal{L}; n) n^{-|\mathcal{E}_\gamma|} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e) = \gamma}} n\gamma_{\kappa(e)}.$$

Applying Lemma 5.3 and

$$|\tilde{\mathcal{V}}| - 1 - |\mathcal{E}_\gamma| = |\tilde{\mathcal{V}}| - 1 - |\tilde{\mathcal{E}}| = 0, \tag{5.15}$$

which follows from Lemma 5.7, we obtain

$$\lim t(\mathcal{L}; n) \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e) = \gamma}} \gamma_{\kappa(e)} = 2^{|\tilde{\mathcal{V}}| - b(\mathcal{L})} \lim \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e) = \gamma}} n\gamma_{\kappa(e)}\,. \tag{5.16}$$

Consequently,

$$
\begin{aligned}
(5.17)\qquad M_m &= \lim M_m(n,p,p') \\
&= \sum_{\mathcal{L}\in\Lambda_m^{**}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})} \lim \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} n\gamma_{\kappa(e)}\,.
\end{aligned}
$$

Finally, applying (5.3) and (5.15), we arrive at

$$
\begin{aligned}
M_m &= \sum_{\mathcal{L}\in\Lambda_m^{**}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \frac{p}{2^{\kappa(e)}} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \frac{2\lambda}{2^{\kappa(e)}} \\
&= \sum_{\mathcal{L}\in\Lambda_m^{**}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\mathcal{E}_\gamma|+1-b(\mathcal{L})-m}\, p^{|\mathcal{E}_\beta|} (2\lambda)^{|\mathcal{E}_\gamma|}\,.
\end{aligned}
$$

This proves (5.14). ∎

With the help of Theorem 5.8 one may derive (5.5) easily.

5.3. *Model II.* We consider $\mathcal{G}_{II}(n,p,p') = \mathcal{G}(n,R;p,p')$, where a relation R is given by

$$R = \{(i, i\pm 1)\,;\, i\in V\}.$$

In this case too, $\mathcal{G}_{II}(n,p,p')$ satisfies the symmetry condition. Since $(R\cup R^t)\backslash(R\cap R^t) = \emptyset$, there is no matrix element A_{ij} which obeys the distribution β. Hence, on computing the m-th moment for the mean spectral distribution of $\mathcal{G}_{II}(n,p,p')$ we need only to consider $\Lambda_m(\{\alpha,\gamma\})$, see Theorem 4.3.

Since $R\cap R^t = \{(i,i\pm 1)\,;\, i\in V\}$, an A-admissible embedding φ of $\mathcal{L}\in\Lambda_m(\{\alpha,\gamma\})$ maps an α-edge to $\{i, i\pm 1\}$. Thus, the argument on $\mathcal{G}_I(n,p,p')$ is applicable just by replacing β by α. Corresponding to Theorem 5.1, we have

Theorem 5.9. *The m-th moment of the generalized random graph $\mathcal{G}_{II}(n,p,p')$ is characterized by the moment:*

$$
(5.18)\qquad M_m(n,p,p') = \sum_{\mathcal{L}\in\Lambda_m^{*}(\{\alpha,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\alpha}} \alpha_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)}.
$$

It follows from Theorem 4.2 that the mean degree of $\mathcal{G}_{II}(n,p,p')$ is given by

$$\bar{d}(\mathcal{G}_I(n,p,p')) = 2p + (n-3)p'.$$

We are again interested in the sparse limit as in (5.2). In this limit the mean degree tends to a finite constant $2p+\lambda$. Moreover, we note from (4.5) that

$$
(5.19)\qquad \lim \alpha_\kappa = p^2 + \frac{2p(1-p)}{2^\kappa}\,, \qquad \lim n\gamma_\kappa = \frac{2\lambda}{2^\kappa}.
$$

In fact, α_κ is independent of n and p'.

We are now convinced that the argument in the previous subsection is valid for $\mathcal{G}_{II}(n,p,p')$ by replacing β by α. In fact, it is sufficient to modify (5.17) to obtain

$$\begin{aligned}(5.20)\qquad M_m &= \lim M_m(n,p,p') \\ &= \sum_{\mathcal{L}\in\Lambda_m^{**}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})} \lim \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\alpha}} \alpha_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} n\gamma_{\kappa(e)}\,.\end{aligned}$$

Then, using (5.19), we come to the final claim.

THEOREM 5.10. *Let M_m be the m-th moment of the spectral distribution of the generalized random graph $\mathcal{G}_I(n,p,p')$ in the sparse limit* (5.2)*. Then, for an odd m we have*

$$M_m = 0$$

and, for an even m we have

$$M_m = \sum_{\mathcal{L}\in\Lambda_m^{**}(\{\alpha,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\mathcal{E}_\gamma|+1-b(\mathcal{L})-m}\, (2\lambda)^{|\mathcal{E}_\gamma|} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\alpha}} ((2^{\kappa(e)}-2)p^2+2p).$$

Moreover, $u(\mathcal{L})=0$ unless the κ-label of $\mathcal{L}\in\Lambda_m^{**}(\{\alpha,\gamma\})$ is even-valued.

6. Appendix. The Erdős–Rényi random graph

6.1. *Mean spectral distribution.* For an integer $n\ge 1$ and a constant number $0<p<1$ let $\mathcal{G}(n,p)$ denote the probability space consisting of graphs G with vertex set $V=\{0,1,2,\dots,n-1\}$ with probability $P(\{G\})$ defined by

$$P(\{G\}) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|},$$

where $E(G)$ stands for the set of edges of G. We call $\mathcal{G}(n,p)$ the *Erdős–Rényi random graph*. This random graph is generated in such a way that for a pair of vertices we decide by a coin toss whether to draw an edge or not. The mean degree of $\mathcal{G}(n,p)$ is given by

$$\bar{d}(\mathcal{G}(n,p)) = \frac{1}{n}\sum_{i\in V}\mathbf{E}(\deg_G(i)) = (n-1)p.$$

Note that $\mathcal{G}(n,p)$ is not recovered by specializing parameters of a generalized Erdős–Rényi random graph $\mathcal{G}(n,R;p,p')$ introduced in Section 4.

Let μ_G be the spectral distribution of $G\in\mathcal{G}(n,p)$ and $\mu_{n,p}=\mathbf{E}(\mu_G)$ its mean distribution. We are interested in asymptotics of $\mu_{n,p}$ in the sparse limit:

$$(6.1)\qquad n\to\infty,\quad p\to 0,\quad np\to\lambda \text{ (constant)}.$$

Note that the mean degree of $\mathcal{G}(n,p)$ tends to λ in the sparse limit.

Let $A=(A_{ij})$ be the adjacency matrix of $G\in\mathcal{G}(n,p)$. Obviously, A satisfies conditions (A0)–(A3) and for all $i\ne j$,

$$P(A_{ij}=1)=p,\qquad P(A_{ij}=0)=1-p,$$

namely, A_{ij} obeys a Bernoulli distribution with success probability p. Since $\mathcal{G}(n,p)$ is symmetric, the mean spectral distribution $\mu_{n,p}$ is characterized by its moment sequence:

$$(6.2)\qquad M_m(\mu_{n,p}) = \frac{1}{n}\mathbf{E}(\mathrm{Tr}\,A^m) = \frac{1}{n}\sum_{i\in V}\mathbf{E}((A^m)_{ii}) = \mathbf{E}((A^m)_{00}),\quad m=1,2,\dots.$$

PROPOSITION 6.1. *The m-th moment of the spectral distribution $\mu_{n,p}$ of the random graph $\mathcal{G}(n,p)$ is given by*

$$M_m(\mu_{n,p}) = \sum_{\mathcal{L}\in\Lambda_m} |\mathrm{Aut}\,(\mathcal{L})|^{-1}u(\mathcal{L})(n-1)(n-2)\cdots(n-(|V(\mathcal{L})|-1))\,p^{|E(\mathcal{L})|}, \tag{6.3}$$

where Λ_m is the complete set of representatives of labeled rooted graphs of size m with constant ν-label. Obviously, $M_1(\mu_{n,p})=0$.

Proof. By Theorem 3.3, (6.2) becomes

$$M_m(\mu_{n,p}) = \sum_{\mathcal{L}\in\Lambda_m(\mathcal{D}(A))} |\mathrm{Aut}\,(\mathcal{L})|^{-1}u(\mathcal{L})t(\mathcal{L};n)\prod_{e\in E(\mathcal{L})} M_{\kappa(e)}(\nu(e)). \tag{6.4}$$

Since $\mathcal{D}(A)$ consists of a single distribution, that is, the Bernoulli distribution with success probability p. Hence $M_\kappa(\nu(e)) = p$ for all $\kappa \geq 1$. Moreover, every injection $\mathcal{V}\backslash\{o\} \to \{1,2,\dots,n-1\}$ is A-admissible so that

$$t(\mathcal{L};n) = (n-1)(n-2)\cdots(n-(|V(\mathcal{L})|-1)).$$

Combining these arguments, we come to (6.3). ∎

6.2. *The sparse limit.* Using Proposition 6.1 we will calculate the sparse limit:

$$M_m = \lim M_m(\mu_{n,p}),$$

where the limit is taken as (6.1). In view of (6.3) we need only to consider

$$(n-1)(n-2)\cdots(n-(|V(\mathcal{L})|-1))\,p^{|E(\mathcal{L})|} \sim n^{|V(\mathcal{L})|-1-|E(\mathcal{L})|}(np)^{|E(\mathcal{L})|}. \tag{6.5}$$

If $\mathcal{L}$ is not a tree, i.e., contains a cycle, then we have $|V(\mathcal{L})| \leq |E(\mathcal{L})|$ and (6.5) vanishes in the sparse limit. If $\mathcal{L}$ is a tree, we have $|V(\mathcal{L})| = |E(\mathcal{L})|+1$. In this case, (6.5) implies that

$$\lim(n-1)(n-2)\cdots(n-(|V(\mathcal{L})|-1))\,p^{|E(\mathcal{L})|} = \lambda^{|E(\mathcal{L})|}.$$

We set

$$\Lambda_m^{**} = \{\mathcal{L}\in\Lambda_m\,;\,\mathcal{L}\text{ is a tree}\},$$

which is consistent with the notation in the previous sections. Summing up,

$$M_m = \lim M_m(\mu_{n,p}) = \sum_{\mathcal{L}\in\Lambda_m^{**}} |\mathrm{Aut}\,(\mathcal{L})|^{-1}u(\mathcal{L})\,\lambda^{|E(\mathcal{L})|}.$$

Since a tree admits no unicursal walk of odd steps, for an odd m we have $u(\mathcal{L})=0$ so the odd moments vanish.

THEOREM 6.2. *Let M_m be the sparse limit of the m-th moment of mean spectral distribution the Erdős–Rényi random graph $\mathcal{G}(n,p)$. Then for an odd m we have*

$$M_m = 0,$$

and for an even m,

$$M_m = \sum_{\mathcal{L}\in\Lambda_m^{**}} |\mathrm{Aut}\,(\mathcal{L})|^{-1}u(\mathcal{L})\,\lambda^{|E(\mathcal{L})|}. \tag{6.6}$$

The formulae (6.3) and (6.6) are essentially known in the literature with different notations, see e.g., Bauer–Golinelli [2].

6.3. *Partition statistics and approximations.* There is another expression for M_m in Theorem 6.2. In fact, we are interested only in the moments of even orders. Using Lemma 3.1 we start with

$$M_{2m} = \lim \sum_{[i]\in\mathcal{W}(V,2m)} p^{|E(G[i])|}. \tag{6.7}$$

Taking Theorem 6.2 into account, for the limit in the right hand side it is sufficient to take the sum over $[i] \in \mathcal{W}(V, 2m)$ whose underlying graph $G[i]$ is a tree.

With each $[i] \in \mathcal{W}(V, 2m)$ such that $G[i]$ is a tree, we associate a partition ϑ of $\{1, 2, \dots, 2m\}$. Write $[i]$ as

$$[i] : 0 \equiv i_0 \neq i_1 \neq i_2 \neq \cdots \neq i_{2m-1} \neq i_{2m} \equiv 0.$$

For $s, t \in \{1, 2, \dots, 2m\}$ we write $s \sim t$ if $\{i_{s-1}, i_s\} = \{i_{t-1}, i_t\}$. Then $s \sim t$ becomes an equivalence relation, which in turn yields a partition of $\{1, 2, \dots, 2m\}$, denoted by $\vartheta = \vartheta[i]$. Let $\mathcal{P}_{\mathrm{T}}(2m)$ denote the set of all partitions of $\{1, 2, \dots, 2m\}$ obtained in this way. Obviously, for $\vartheta = \vartheta[i]$ we have $|E(G[i])| = |\vartheta|$. Then (6.7) becomes

$$\begin{aligned} M_{2m} &= \lim \sum_{\vartheta\in\mathcal{P}_{\mathrm{T}}(2m)} \sum_{\substack{[i]\in\mathcal{W}(V,2m)\\ \vartheta[i]=\vartheta}} p^{|\vartheta|} \\ &= \lim \sum_{\vartheta\in\mathcal{P}_{\mathrm{T}}(2m)} (n-1)(n-2)\cdots(n-|\vartheta|)\, p^{|\vartheta|} \\ &= \sum_{\vartheta\in\mathcal{P}_{\mathrm{T}}(2m)} \lambda^{|\vartheta|}. \end{aligned}$$

Summing up,

THEOREM 6.3. *The sparse limit of the* $2m$*-th moment of mean spectral distribution of the Erdős–Rényi random graph* $\mathcal{G}(n, p)$ *is given by*

$$M_{2m} = \sum_{\vartheta\in\mathcal{P}_{\mathrm{T}}(2m)} \lambda^{|\vartheta|} \tag{6.8}$$

It is obvious by construction that each block of $\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)$ consists of even number of points. Let $\mathcal{P}_{\mathrm{NC}}(2m)$ be the set of non-crossing partitions of $\{1, 2, \dots, 2m\}$ and set

$$\mathcal{P}_{\mathrm{TNC}}(2m) = \{\vartheta \in \mathcal{P}_{\mathrm{NC}}(2m)\,;\ \text{each } v \in \vartheta \text{ consists of even number of points}\}.$$

It is then shown that $\mathcal{P}_{\mathrm{TNC}}(2m) \subset \mathcal{P}_{\mathrm{T}}(2m)$. However, $\mathcal{P}_{\mathrm{T}}(2m)$ contains some crossing partitions too. This would hinder us from getting an explicit expression of the limit distribution. An analytical approach, which yields also an implicit description of the limit distribution, is found in Dorogovtsev–Goltsev–Mendes–Samukhin [7].

To conclude, we show two approximations for the limit distribution whose m-th moment is M_m.

PROPOSITION 6.4. *Let* $\pi_{\lambda/2}$ *be the free Poisson distribution with parameter* $\lambda/2$ *and* $\pi^{\vee}_{\lambda/2}$ *its reflection, i.e.,* $\pi^{\vee}_{\lambda/2}(dx) = \pi_{\lambda/2}(-dx)$. *Then*

$$M_{2m-1}(\pi_{\lambda/2} \boxplus \pi^{\vee}_{\lambda/2}) = 0, \quad M_{2m}(\pi_{\lambda/2} \boxplus \pi^{\vee}_{\lambda/2}) = \sum_{\vartheta\in\mathcal{P}_{\mathrm{TNC}}(2m)} \lambda^{|\vartheta|}. \tag{6.9}$$

Proof. The free Poisson distribution $\pi_{\lambda/2}$ is characterized by the constant free cumulants $r_k(\pi_{\lambda/2}) = \lambda/2$. Then, $r_k(\pi^{\vee}_{\lambda/2}) = (-1)^k\lambda/2$ and

$$r_k(\pi_{\lambda/2} \boxplus \pi^{\vee}_{\lambda/2}) = r_k(\pi_{\lambda/2}) + r_k(\pi^{\vee}_{\lambda/2}) = \begin{cases} \lambda, & k \text{ is even,} \\ 0, & k \text{ is odd.} \end{cases}$$

Applying the free moment–cumulant formula:

$$M_k = \sum_{\vartheta \in \mathcal{P}_{\mathrm{NC}}(k)} \prod_{v \in \vartheta} r_{|v|}\,,$$

we have

$$M_{2m-1} = 0, \quad M_{2m} = \sum_{\vartheta \in \mathcal{P}_{\mathrm{TNC}}(2m)} \lambda^{|\vartheta|},$$

which completes the proof. (This proof is within the standard framework of the free probability theory, see e.g., Hiai–Petz [10].) ∎

Comparing (6.8) and (6.9), we can expect that the sparse limit of mean spectral distribution of the Erdős–Rényi random graph is a kind of deformation of the free Poisson distributions.

Next we look for the leading term of M_{2m} for a large λ. In fact,

$$\begin{aligned} M_{2m} &= \sum_{k=1}^{m} |\{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)\,;\, |\vartheta| = k\}|\, \lambda^k \\ &= |\{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)\,;\, |\vartheta| = m\}|\, \lambda^m + O(\lambda^{m-1}) \\ &= |\mathcal{P}_{\mathrm{NCP}}(2m)|\, \lambda^m + O(\lambda^{m-1}), \end{aligned}$$

where $\mathcal{P}_{\mathrm{NCP}}(2m)$ stands for the set of non-crossing pair partitions of $\{1, 2, \dots, 2m\}$. The number $|\mathcal{P}_{\mathrm{NCP}}(2m)|$ is well known as the Catalan number and is the $2m$-th moment of the Wigner semicircle law.

PROPOSITION 6.5. *For the m-th moment of mean spectral distribution the Erdős–Rényi random graph $\mathcal{G}(n,p)$ we have*

$$\lim_{\lambda\to\infty} \lim \lambda^{-m/2} M_m(\mu_{n,p}) = \frac{1}{2\pi}\int_{-2}^{+2} x^m \sqrt{4-x^2}\, dx, \qquad m = 1, 2, \dots,$$

where the second limit is the sparse limit as in (6.1).

The above result supports from the viewpoint of spectral analysis that the Erdős–Rényi random graph behaves like a tree in the sparse limit.

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