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ON THE STRUCTURE OF POSITIVE MAPS BETWEEN MATRIX ALGEBRAS

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Abstract. The structure of the set of positive unital maps between $M_2(\mathbb{C})$ and $M_n(\mathbb{C})$ ($n \geq 3$) is investigated. We proceed with the study of the "quantized" Choi matrix thus extending the methods of our previous paper [MM2]. In particular, we examine the quantized version of Størmer's extremality condition. Maps fulfilling this condition are characterized. To illustrate our approach, a careful analysis of Tang's maps is given.

1. Introduction. We will be concerned with linear positive maps $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$. We begin with setting up the notation and the relevant terminology (cf. [MM3]). We say that ϕ is *positive* if $\phi(A)$ is a positive element in $M_n(\mathbb{C})$ for every positive matrix from $M_m(\mathbb{C})$. If $k \in \mathbb{N}$, then ϕ is said to be *k-positive* (respectively *k-copositive*) whenever $[\phi(A_{ij})]_{i,j=1}^k$ (respectively $[\phi(A_{ji})]_{i,j=1}^k$) is positive in $M_k(M_n(\mathbb{C}))$ for every positive element $[A_{ij}]_{i,j=1}^k$ of $M_k(M_m(\mathbb{C}))$. If ϕ is k-positive (respectively k-copositive) for every $k \in \mathbb{N}$ then we say that ϕ is *completely positive* (respectively *completely copositive*). Finally, we say that the map ϕ is *decomposable* if it has the form $\phi = \phi_1 + \phi_2$ where ϕ_1 is completely positive while ϕ_2 is completely copositive.

By $\mathcal{P}(m,n)$ we denote the set of all positive maps acting between $M_m(\mathbb{C})$ and $M_n(\mathbb{C})$ and by $\mathcal{P}_1(m,n)$ the subset of $\mathcal{P}(m,n)$ composed of all positive unital maps (i.e. such that $\phi(\mathbb{I}) = \mathbb{I}$). Recall that $\mathcal{P}(m,n)$ has the structure of a convex cone while $\mathcal{P}_1(m,n)$ is its convex subset.

In the sequel we will use the notion of a face of a convex cone.

Definition 1. Let C be a convex cone. We say that a convex subcone $F \subset C$ is a *face of C* if for every $c_1, c_2 \in C$ the condition $c_1 + c_2 \in F$ implies $c_1, c_2 \in F$.

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A face F is said to be a *maximal face* if F is a proper subcone of C and for every face G such that $F \subseteq G$ we have $G = F$ or $G = C$.

The following theorem of Kye gives a nice characterization of maximal faces in the cone $\mathcal{P}(m,n)$.

THEOREM 2 ([Kye]). *A convex subset $F \subset \mathcal{P}(m,n)$ is a maximal face of $\mathcal{P}(m,n)$ if and only if there are vectors $\xi \in \mathbb{C}^m$ and $\eta \in \mathbb{C}^n$ such that $F = F_{\xi,\eta}$ where*

$$F_{\xi,\eta} = \{\phi \in \mathcal{P}(m,n) : \phi(P_\xi)\eta = 0\} \tag{1}$$

and P_ξ denotes the one-dimensional orthogonal projection in $M_m(\mathbb{C})$ onto the subspace generated by the vector ξ.

The aim of this paper is to go one step further in clarification of the structure of positive maps between $M_2(\mathbb{C})$ and $M_n(\mathbb{C})$. It is worth pointing out that many open problems in quantum computing demand the better knowledge of this structure. Consequently, our results shed new light on the structure of positive maps as well as on the nature of entanglement (cf. [MM1], and for relation to quantum correlations see [Maj]).

We recall (see [S], [W]) that all elements of $\mathcal{P}(2,2)$, $\mathcal{P}(2,3)$ and $\mathcal{P}(3,2)$ are decomposable. Contrary, $\mathcal{P}(n,m)$ with $m,n \geq 3$ contains nondecomposable maps. In [MM2] we proved that if ϕ is extremal element of $\mathcal{P}_1(2,2)$ then its decomposition is unique. Moreover, we provided a full description of this decomposition. In the case $m > 2$ or $n > 2$ the problem of finding the decomposition is still unsolved. In this paper we consider the next step for partial solution of this problem, namely for the case $m = 2$ and $n \geq 3$. Our approach will be based on the method of the so called Choi matrix.

To give a brief exposition of this method, we recall (see [Choi1], [MM1] for details) that if $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear map and $\{E_{ij}\}_{i,j=1}^m$ is a system of matrix units in $M_m(\mathbb{C})$, then the matrix

$$\mathbf{H}_\phi = [\phi(E_{ij})]_{i,j=1}^m \in M_m(M_n(\mathbb{C})), \tag{2}$$

is called the Choi matrix of ϕ with respect to the system $\{E_{ij}\}$. Complete positivity of ϕ is equivalent to positivity of $\mathbf{H}_\phi$ while positivity of ϕ is equivalent to block-positivity of $\mathbf{H}_\phi$ (see [Choi1], [MM1]). A matrix $[A_{ij}]_{i,j=1}^m \in M_m(M_n(\mathbb{C}))$ (where $A_{ij} \in M_n(\mathbb{C})$) is called *block-positive* if $\sum_{i,j=1}^m \overline{\lambda_i}\lambda_j \langle \xi, A_{ij}\xi\rangle \geq 0$ for any $\xi \in \mathbb{C}^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$.

It was shown in Lemma 2.3 in [MM2] that the general form of the Choi matrix of a positive map ϕ belonging to some maximal face of $\mathcal{P}(2,2)$ is the following:

$$\mathbf{H}_\phi = \left[\begin{array}{cc|cc} a & c & 0 & y \\ \overline{c} & b & \overline{z} & t \\ \hline 0 & z & 0 & 0 \\ \overline{y} & \overline{t} & 0 & u \end{array}\right]. \tag{3}$$

Here $a, b, u \geq 0$, $c, y, z, t \in \mathbb{C}$ and the following inequalities are satisfied:

(I) $|c|^2 \leq ab$,
(II) $|t|^2 \leq bu$,
(III) $|y| + |z| \leq (au)^{1/2}$.

It will turn out that in the case $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$, $n \geq 2$, the Choi matrix has the form which is similar to (3) but some of the coefficients have to be matrices (see [MM3]). The main result of our paper is an analysis of Tang's maps in the Choi matrix setting and proving some partial results about the structure of positive maps in the case $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$.

2. $\mathcal{P}(2, n+1)$ maps and Tang's maps. In this section we summarize without proofs the relevant material on the Choi matrix method for $\mathcal{P}(2, n+1)$ (see [MM3]) and we indicate how this technique may be used to the analysis of nondecomposable maps. Let $\{e_1, e_2\}$ and $\{f_1, f_2, \dots, f_{n+1}\}$ denote the standard orthonormal bases of the spaces $\mathbb{C}^2$ and $\mathbb{C}^{n+1}$ respectively, and let $\{E_{ij}\}_{i,j=1}^2$ and $\{F_{kl}\}_{k,l=1}^{n+1}$ be systems of matrix units in $M_2(\mathbb{C})$ and $M_{n+1}(\mathbb{C})$ associated with these bases. We assume that $\phi \in F_{\xi,\eta}$ for some $\xi \in \mathbb{C}^2$ and $\eta \in \mathbb{C}^{n+1}$. By taking the map $A \mapsto V^*\phi(WAW^*)V$ for suitable $W \in U(2)$ and $V \in U(n+1)$ we can assume without loss of generality that $\xi = e_2$ and $\eta = f_1$. Then the Choi matrix of ϕ has the form

$$\mathbf{H} = \left[\begin{array}{cccc|cccc} a & c_1 & \dots & c_n & x & y_1 & \dots & y_n \\ \overline{c_1} & b_{11} & \dots & b_{1n} & \overline{z_1} & t_{11} & \dots & t_{1n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \overline{c_n} & b_{n1} & \dots & b_{nn} & \overline{z_n} & t_{n1} & \dots & t_{nn} \\ \hline \overline{x} & z_1 & \dots & z_n & 0 & 0 & \dots & 0 \\ \overline{y_1} & \overline{t_{11}} & \dots & \overline{t_{n1}} & 0 & u_{11} & \dots & u_{1n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \overline{y_n} & \overline{t_{1n}} & \dots & \overline{t_{nn}} & 0 & u_{n1} & \dots & u_{nn} \end{array}\right] \tag{4}$$

We introduce the following notations:

$$C = \left[\begin{array}{ccc} c_1 & \dots & c_n \end{array}\right], \quad Y = \left[\begin{array}{ccc} y_1 & \dots & y_n \end{array}\right], \quad Z = \left[\begin{array}{ccc} z_1 & \dots & z_n \end{array}\right],$$

$$B = \left[\begin{array}{ccc} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{array}\right], \quad T = \left[\begin{array}{ccc} t_{11} & \dots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \dots & t_{nn} \end{array}\right], \quad U = \left[\begin{array}{ccc} u_{11} & \dots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \dots & u_{nn} \end{array}\right].$$

The matrix (4) can be rewritten in the following form:

$$\mathbf{H} = \left[\begin{array}{cc|cc} a & C & x & Y \\ C^* & B & Z^* & T \\ \hline \overline{x} & Z & 0 & 0 \\ Y^* & T^* & 0 & U \end{array}\right]. \tag{5}$$

The symbol 0 in the right-bottom block has three different meanings. It denotes 0, $\left[\begin{array}{ccc} 0 & \dots & 0 \end{array}\right]$ or $\left[\begin{array}{c} 0 \\ \vdots \\ 0 \end{array}\right]$ respectively. We have the following

PROPOSITION 3 ([MM3]). *Let $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ be a positive map with the Choi matrix of the form (5). Then the following relations hold:*

1. $a \geq 0$ and B, U are positive matrices,
2. if $a = 0$ then $C = 0$, and if $a > 0$ then $C^*C \leq aB$,
3. $x = 0$,
4. the matrix $\left[\begin{array}{c|c} B & T \\ \hline T^* & U \end{array}\right] \in M_2(M_n(\mathbb{C}))$ is block-positive. ■

In the example below, we will be concerned with the two-parameter family of nondecomposable maps (cf. [Tang]). Here the important point to note is the fact that $\mathcal{P}(2,4)$ and $\mathcal{P}(3,3)$ are the lowest dimensional cases having nondecomposable maps. Therefore the detailed analysis of such maps should yield necessary information for explanations of the occurrence of nondecomposability.

Example 4. Let $\phi_0 : M_2(\mathbb{C}) \to M_4(\mathbb{C})$ be the linear map defined by

$$\phi_0\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} (1-\varepsilon)a + \mu^2 d & -b & \mu c & -\mu d \\ -c & a + 2d & -2b & 0 \\ \mu b & -2c & 2a + 2d & -2b \\ -\mu d & 0 & -2c & a + d \end{bmatrix}, \tag{6}$$

where $0 < \mu < 1$ and $0 < \varepsilon \leq \frac{1}{6}\mu^2$. It is proved in [Tang] that ϕ_0 is nondecomposable. One can see that ϕ_0 has the following Choi matrix:

$$H_{\phi_0} = \left[\begin{array}{cccc|cccc} 1-\varepsilon & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & \mu & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \mu & 0 & \mu^2 & 0 & 0 & -\mu \\ -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & -\mu & 0 & 0 & 1 \end{array}\right]. \tag{7}$$

Observe that

$$\phi_0(\mathbb{I}) = \begin{bmatrix} 1-\varepsilon+\mu^2 & 0 & 0 & -\mu \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ -\mu & 0 & 0 & 2 \end{bmatrix}.$$

Let $\rho = \sqrt{1-\varepsilon+\mu^2}$ and

$$\delta = \begin{vmatrix} 1-\varepsilon+\mu^2 & -\mu \\ -\mu & 2 \end{vmatrix}^{1/2} = \sqrt{2 - 2\varepsilon + \mu^2}.$$

Then $\phi_0(\mathbb{I})^{-1/2}$ is of the form

$$\phi_0(\mathbb{I})^{-1/2} = \begin{bmatrix} \dfrac{\beta}{\delta} & 0 & 0 & -\dfrac{\gamma}{\delta} \\ 0 & \dfrac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & \dfrac{1}{2} & 0 \\ -\dfrac{\gamma}{\delta} & 0 & 0 & \dfrac{\alpha}{\delta} \end{bmatrix}$$

where and $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$ are such that

$$\begin{aligned} \alpha^2 + \gamma^2 &= \rho^2, \\ \beta^2 + \gamma^2 &= 2, \\ (\alpha + \beta)\gamma &= -\mu. \end{aligned} \tag{8}$$

Let us define $\phi_1 : M_2(\mathbb{C}) \to M_4(\mathbb{C})$ by

$$\phi_1(A) = \phi_0(\mathbb{I})^{-1/2}\phi_0(A)\phi_0(\mathbb{I})^{-1/2}, \quad A \in M_2(\mathbb{C}).$$

Then

$$\phi_1(E_{11}) = \begin{bmatrix} \frac{(1-\varepsilon)\beta^2 + \gamma^2}{\delta^2} & 0 & 0 & -\frac{[(1-\varepsilon)\beta + \alpha]\gamma}{\delta^2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{[(1-\varepsilon)\beta + \alpha]\gamma}{\delta^2} & 0 & 0 & \frac{(1-\varepsilon)\gamma^2 + \alpha^2}{\delta^2} \end{bmatrix},$$

$$\phi_1(E_{22}) = \begin{bmatrix} \frac{(\mu\beta + \gamma)^2}{\delta^2} & 0 & 0 & -\frac{(\mu\beta + \gamma)(\mu\gamma + \alpha)}{\delta^2} \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ -\frac{(\mu\beta + \gamma)(\mu\gamma + \alpha)}{\delta^2} & 0 & 0 & \frac{(\mu\gamma + \alpha)^2}{\delta^2} \end{bmatrix},$$

$$\phi_1(E_{12}) = \begin{bmatrix} 0 & -\frac{\beta}{\delta\sqrt{3}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ \frac{\mu\beta + 2\gamma}{2\delta} & 0 & 0 & -\frac{\mu\gamma + 2\alpha}{2\delta} \\ 0 & \frac{\gamma}{\delta\sqrt{3}} & 0 & 0 \end{bmatrix}.$$

One can deduce from (8) that

$$(\mu\gamma + \alpha)^2 + (\mu\beta + \gamma)^2 = \rho^2. \tag{9}$$

Let

$$W = \begin{bmatrix} \frac{\mu\gamma + \alpha}{\sqrt{1 - \varepsilon + \mu^2}} & 0 & 0 & \frac{\mu\beta + \gamma}{\sqrt{1 - \varepsilon + \mu^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\mu\beta + \gamma}{\sqrt{1 - \varepsilon + \mu^2}} & 0 & 0 & -\frac{\mu\gamma + \alpha}{\sqrt{1 - \varepsilon + \mu^2}} \end{bmatrix}.$$

It follows from (9) that W is a unitary matrix. Define $\phi : M_2(\mathbb{C}) \to M_4(\mathbb{C})$ by $\phi(A) =$

$W^*\phi_1(A)W$. Then the Choi matrix of ϕ is of the form

$$\mathbf{H}_\phi = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}\rho} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & -\frac{\mu}{2\rho} & 0 & 0 & \frac{\delta}{2\rho} \\ 0 & 0 & 0 & \frac{1-\varepsilon}{\delta^2} & 0 & -\frac{\mu}{\sqrt{3}\delta\rho} & 0 & 0 \\ \hline 0 & 0 & -\frac{\mu}{2\rho} & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{3}\rho} & 0 & 0 & -\frac{\mu}{\sqrt{3}\delta\rho} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{\delta}{2\rho} & 0 & 0 & 0 & 0 & \frac{\rho^2}{\delta^2} \end{array}\right]. \tag{10}$$

One can see that $\psi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0$, so $\psi \in F_{\eta,\xi}$ (cf. Theorem 2), where $\eta = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\xi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Observe that blocks which form the Choi matrix (10) as in (5) are of the form

$$a = 1, \qquad C = 0, \qquad Y = \begin{bmatrix} -\frac{1}{\sqrt{3}\delta} & 0 & 0 \end{bmatrix}, \qquad Z = \begin{bmatrix} 0 & -\frac{\mu}{2\rho} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1-\varepsilon}{\delta^2} \end{bmatrix}, \quad U = \begin{bmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{\rho^2}{\delta^2} \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{\delta}{2\rho} \\ -\frac{\mu}{\sqrt{3}\delta\rho} & 0 & 0 \end{bmatrix}.$$

It is worth observing that the vectors C, Y, Z are orthogonal, the matrices B, U are diagonal, while T is "complementary" to the diagonal matrices B and U. This observation is useful in understanding the peculiarity of nondecomposable maps

In the sequel we will need the following technicalities. For $X = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \in M_{1,n}(\mathbb{C})$ we define $\|X\| = (\sum_{i=1}^n |x_i|^2)^{1/2}$. By $|X|$ we denote the square $(n \times n)$-matrix $(X^*X)^{1/2}$. We identify elements of $M_{n,1}(\mathbb{C})$ with vectors from $\mathbb{C}^n$ and for any $X \in M_{1,n}(\mathbb{C})$ define a unit vector $\xi_X \in \mathbb{C}^n$ by $\xi_X = \|X\|^{-1}X^*$.

PROPOSITION 5. *Let* $X, X_1, X_2 \in M_{1,n}(\mathbb{C})$. *Then*

(1) $|X| = \|X\| P_{\xi_X}$, *where* P_ξ *denotes the orthogonal projection onto the one-dimensional subspace in* $\mathbb{C}^n$ *generated by a vector* $\xi \in \mathbb{C}^n$*;*
(2) $|X_1||X_2| = \langle \xi_{X_1}, \xi_{X_2} \rangle X_1^* X_2$.

Proof. (1) Let $\eta \in \mathbb{C}^n$. Since η is considered also as an element of $M_{n,1}(\mathbb{C})$ the multiplication of matrices $X\eta$ makes sense. As a result we obtain a 1×1-matrix which can be interpreted as a number. With this identification we have the equality

$$X\eta = \langle X^*, \eta \rangle$$

where X^* on the right hand side is considered as a vector from $\mathbb{C}^n$, and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{C}^n$. Now we can calculate

$$\langle \eta, X^* X \eta \rangle = \langle X\eta, X\eta \rangle = \|X\eta\|^2 = |\langle X^*, \eta \rangle|^2 = \|X\|^2 |\langle \xi_X, \eta \rangle|^2.$$

(2) If $X_1 = 0$ or $X_2 = 0$ then the equality is obvious. In the case both X_1 and X_2 are nonzero the equality follows from the following computations

$$\begin{aligned} |X_1||X_2| &= \|X_1\|^{-1}\|X_2\|^{-1}|X_1|^2|X_2|^2 = \|X_1\|^{-1}\|X_2\|^{-1} X_1^* X_1 X_2^* X_2 \\ &= \|X_1\|^{-1}\|X_2\|^{-1} X_1^*(X_1 X_2^*) X_2 = \|X_1\|^{-1}\|X_2\|^{-1} \langle X_1^*, X_2^* \rangle X_1^* X_2 \\ &= \langle \xi_{X_1}, \xi_{X_2} \rangle X_1^* X_2. \ \blacksquare \end{aligned}$$

To proceed with the study of Tang's maps we recall some general properties of maps in $\mathcal{P}(2, n+1)$ (cf. [MM3]). We start with

PROPOSITION 6 ([MM3]). *A map ϕ with the Choi matrix of the form*

$$\mathbf{H} = \left[\begin{array}{cc|cc} a & C & 0 & Y \\ C^* & B & Z^* & T \\ \hline 0 & Z & 0 & 0 \\ Y^* & T^* & 0 & U \end{array} \right]. \tag{11}$$

is positive if and only if the inequality

$$\left| \langle Y^*, \Gamma^\tau \rangle + \overline{\langle Z^*, \Gamma^\tau \rangle} + \operatorname{Tr}(\Lambda^\tau T) \right|^2 \leq [\alpha a + \operatorname{Tr}(\Lambda^\tau B) + 2\Re\{\langle C^*, \Gamma^\tau \rangle\}] \operatorname{Tr}(\Lambda^\tau U) \tag{12}$$

holds for every $\alpha \in \mathbb{C}$, matrices $\Gamma = \begin{bmatrix} \gamma_1 & \dots & \gamma_n \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \lambda_{11} & \dots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \dots & \lambda_{nn} \end{bmatrix}$,
$\gamma_i \in \mathbb{C}$, $\lambda_{ij} \in \mathbb{C}$ for $i, j = 1, 2, \dots, n$, such that

1. *$\alpha \geq 0$ and $\Lambda \geq 0$,*
2. *$\Gamma^* \Gamma \leq \alpha \Lambda$.*

The superscript τ denotes the transposition of matrices.

THEOREM 7 ([MM3]). *If the assumptions of Proposition 3 are fulfilled, then*

$$|Y| + |Z| \leq a^{1/2} U^{1/2}. \tag{13}$$

REMARK 8. One can easily check that the nondecomposable maps described in Example 4 fulfill the above inequality. It is easy to check that in this case the inequality is strict. This observation will be crucial for the next section.

As we mentioned, for $\mathcal{P}(2, n)$, $n > 3$, there are nondecomposable maps. The proposition below provides the characterization of completely positive and completely copositive components of $\mathcal{P}(2, n)$.

PROPOSITION 9 ([MM3]). *Let $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ be a linear map with the Choi matrix of the form (11). Then the map ϕ is completely positive (resp. completely copositive) if and only if the following conditions hold:*

(1) $Z = 0$ *(resp.* $Y = 0$*),*

(2) *the matrix* $\begin{bmatrix} a & C & Y \\ C^* & B & T \\ Y^* & T^* & U \end{bmatrix}$ *(resp.* $\begin{bmatrix} a & C & Z \\ C^* & B & T^* \\ Z^* & T & U \end{bmatrix}$*) is a positive element of the algebra* $M_{2n+1}(\mathbb{C})$.

In particular, the condition (2) implies:

(3) *if B is an invertible matrix, then* $T^*B^{-1}T \leq U$ *(resp.* $TB^{-1}T^* \leq U$*),*

(4) $C^*C \leq aB$,

(5) $Y^*Y \leq aU$ *(resp.* $Z^*Z \leq aU$*).*

This proposition yields information about possible splitting of a decomposable map into completely positive and completely copositive components. To go one step further let us make the following observation. Let $\phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be a decomposable map and $\phi = \phi_1 + \phi_2$ for some completely positive ϕ_1 and completely copositive ϕ_2. Then from the Kadison inequality we easily obtain

$$\phi(E_{ij})^*\phi(E_{ij}) \leq \|\phi(\mathbb{I})\| \, (\phi_1(E_{ii}) + \phi_2(E_{jj})) \tag{14}$$

for $i, j = 1, 2, \ldots, m$.

Assume now that $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ has the Choi matrix of the form (5). It follows from Proposition 9 that the Choi matrices of ϕ_1 and ϕ_2 are respectively

$$\mathbf{H_1} = \left[\begin{array}{cc|cc} a_1 & C_1 & 0 & Y \\ C_1^* & B_1 & 0 & T_1 \\ \hline 0 & 0 & 0 & 0 \\ Y^* & T_1^* & 0 & U_1 \end{array}\right], \quad \mathbf{H_2} = \left[\begin{array}{cc|cc} a_2 & C_2 & 0 & 0 \\ C_2^* & B_2 & Z^* & T_2 \\ \hline 0 & Z & 0 & 0 \\ 0 & T_2^* & 0 & U_2 \end{array}\right]. \tag{15}$$

Clearly, $\mathbf{H_1} + \mathbf{H_2} = \mathbf{H}$, where $\mathbf{H}$ is the Choi matrix corresponding to ϕ. The inequality (14) leads to additional relations between components of the Choi matrices

$$\begin{bmatrix} \|Z\|^2 & ZT \\ T^*Z^* & |Y|^2 + T^*T \end{bmatrix} \leq \|\phi(\mathbb{I})\| \begin{bmatrix} a_1 & C_1 \\ C_1^* & B_1 + U_2 \end{bmatrix}$$

and

$$\begin{bmatrix} \|Y\|^2 & YT^* \\ TY^* & |Z|^2 + TT^* \end{bmatrix} \leq \|\phi(\mathbb{I})\| \begin{bmatrix} a_2 & C_2 \\ C_2^* & B_2 + U_1 \end{bmatrix}.$$

It is worth pointing out that the above inequalities give a partial answer to Choi's question (cf. [Choi2]). Furthermore, turning to Tang's maps one can observe that the matrix corresponding to $\phi(E_{ij})^*\phi(E_{ij})$ is relatively large, which precludes the possibility of decomposition of these maps.

3. On the structure of elements of $\mathcal{P}(2, n+1)$. Giving a full description of the situation in $\mathcal{P}(2,2)$ in [MM2] we proved that if $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is from a large class of extremal positive unital maps, then the constituent maps ϕ_1 and ϕ_2 are uniquely determined (cf. Theorem 2.7 in [MM2]). We recall that the Choi matrix of such an extremal map $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is of the form (cf. (3))

$$\mathbf{H}_\phi = \left[\begin{array}{cc|cc} 1 & 0 & 0 & y \\ 0 & 1-u & \overline{z} & t \\ \hline 0 & z & 0 & 0 \\ \overline{y} & \overline{t} & 0 & u \end{array}\right], \tag{16}$$

where, in particular, the following equality is satisfied (cf. (III) from Section 1):

$$|y| + |z| = u^{1/2}. \tag{17}$$

In this section, motivated by the results given in the previous section (we 'quantized' the relations (I)-(III) given at the end of Section 1), we consider maps $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$. If such a map is positive unital and $\phi \in F_{e_2,f_1}$ then its Choi matrix has the form

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & Y \\ 0 & B & Z^* & T \\ \hline 0 & Z & 0 & 0 \\ Y^* & T^* & 0 & U \end{array}\right], \tag{18}$$

where B and U are positive matrices such that $B + U = 1$ and the conditions listed in Propositions 3 and 6 are satisfied.

Our object is to examine consequences of the property

$$|Y| + |Z| = U^{1/2} \tag{19}$$

which for $n \geq 1$ is a natural analog of (17).

First, we recall the following technical

Lemma 10. *Let* $\mathbf{A} = \begin{bmatrix} P & S \\ S^* & Q \end{bmatrix} \in M_2(M_n(\mathbb{C}))$, *where* $P, Q, S \in M_n(\mathbb{C})$, *and* $P, Q \geq 0$. *The following are equivalent:*

(i) $\mathbf{A}$ *is block-positive;*
(ii) $pP + sS + \overline{s}S^* + qQ \geq 0$ *for all numbers* p, q, s *such that* $p, q \geq 0$ *and* $|s|^2 \leq pq$;
(iii) $|\langle \eta, S\eta \rangle|^2 \leq \langle \eta, P\eta \rangle \langle \eta, Q\eta \rangle$ *for every* $\eta \in \mathbb{C}^n$.

Proof. (i)$\Rightarrow$(ii). Let $\eta \in \mathbb{C}^n$. It follows from the definition of block-positivity (cf. [MM2]) that the matrix

$$\begin{bmatrix} \langle \eta, P\eta \rangle & \langle \eta, S\eta \rangle \\ \langle \eta, S^*\eta \rangle & \langle \eta, Q\eta \rangle \end{bmatrix}$$

is positive. Hence the matrix

$$\begin{bmatrix} \langle \eta, pP\eta \rangle & \langle \eta, sS\eta \rangle \\ \langle \eta, \overline{s}S^*\eta \rangle & \langle \eta, qQ\eta \rangle \end{bmatrix}$$

being a Hadamard product of two positive matrices is positive as well. Consequently,

$$\langle \eta, (pP + sS + \overline{s}S^* + qQ)\eta \rangle \geq 0.$$

Since η is arbitrary, (ii) is proved.

(ii)⇒(i). To prove that $\mathbf{A}$ is block-positive one should show that for any $\eta \in \mathbb{C}^n$ and $\mu_1, \mu_2 \in \mathbb{C}$ one has

$$|\mu_1|^2\langle\eta, P\eta\rangle + 2\Re\{\mu_1\overline{\mu_2}\langle\eta, S\eta\rangle\} + |\mu_2|^2\langle\eta, Q\eta\rangle \geq 0.$$

Observe that $p = |\mu_1|^2$, $q = |\mu_2|^2$, $s = \mu_1\overline{\mu_2}$ fulfill $p, q \geq 0$ and $|s|^2 = pq$. So,

$$|\mu_1|^2\langle\eta, P\eta\rangle + 2\Re\{\mu_1\overline{\mu_2}\langle\eta, S\eta\rangle\} + |\mu_2|^2\langle\eta, Q\eta\rangle = \langle\eta, (pP + sS + \overline{s}S^* + qQ)\eta\rangle \geq 0.$$

(i)⇔(iii). Let $\eta \in \mathbb{C}^n$. The positivity of the matrix $\begin{bmatrix} \langle\eta, P\eta\rangle & \langle\eta, S\eta\rangle \\ \langle\eta, S^*\eta\rangle & \langle\eta, Q\eta\rangle \end{bmatrix}$ is equivalent to non-negativity of its determinant $\langle\eta, P\eta\rangle\langle\eta, Q\eta\rangle - |\langle\eta, S\eta\rangle|^2$. ∎

Here we give another (cf. Proposition 6) characterisation of positive maps in the language of their Choi matrices

PROPOSITION 11. *Let $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ be a linear unital map with the Choi matrix of the form*

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & Y \\ 0 & B & Z^* & T \\ \hline 0 & Z & 0 & 0 \\ Y^* & T^* & 0 & U \end{array}\right] \tag{20}$$

where $B, U, T \in M_n(\mathbb{C})$, $Y, Z \in M_{1,n}(\mathbb{C})$, and $B, U \geq 0$. Then the map ϕ is positive if and only if

$$pB + sT + \overline{s}T^* + qU \geq 0$$

and

$$(\overline{s}Y^* + sZ^*)(sY + \overline{s}Z) \leq p^2B + p(sT + \overline{s}T^*) + pqU \tag{21}$$

for every $p, q, s \in \mathbb{C}$ such that $p, q \geq 0$ and $|s|^2 \leq pq$.

Proof. It follows from the definition of the Choi matrix and from (20) that

$$\phi\left(\begin{bmatrix} p & s \\ v & q \end{bmatrix}\right) = \begin{bmatrix} p & sY + vZ \\ sZ^* + vY^* & pB + sT + vT^* + qU \end{bmatrix}.$$

So, the map ϕ is positive if and only if the matrix

$$\begin{bmatrix} p & sY + \overline{s}Z \\ sZ^* + \overline{s}Y^* & pB + sT + \overline{s}T^* + qU \end{bmatrix}. \tag{22}$$

is a positive element of $M_{n+1}(\mathbb{C})$ for numbers p, q, s such that $p, q \geq 0$ and $|s|^2 \leq pq$ (i.e. such that the matrix $\begin{bmatrix} p & s \\ \overline{s} & q \end{bmatrix}$ is positive in $M_2(\mathbb{C})$). The positivity of the matrix (22) is equivalent to both inequalities from the statement of the proposition. ∎

The following generalizes Lemma 8.10 from [S].

PROPOSITION 12. *Let ϕ be a positive unital map with the Choi matrix (20). Assume that B is invertible. Then the matrix*

$$\begin{bmatrix} 2B & T \\ T^* & U - |Y|^2 - |Z|^2 \end{bmatrix} \tag{23}$$

is block-positive.

Proof. Let $\eta \in \mathbb{C}^n$, $\eta \neq 0$, and $p, q, s \in \mathbb{C}$ be numbers such that $p, q \geq 0$ and $|s|^2 = pq$. Then from (21) we have

$$|s|^2\langle \eta, (|Y|^2 + |Z|^2)\eta\rangle + 2\Re\left\{s^2\langle \eta, Z^*Y\eta\rangle\right\} \leq p^2\langle \eta, B\eta\rangle + 2p\,\Re\left\{s\langle \eta, T\eta\rangle\right\} + pq\langle \eta, U\eta\rangle.$$

Replace s in this inequality by is and obtain

$$|s|^2\langle \eta, (|Y|^2 + |Z|^2)\eta\rangle - 2\Re\left\{s^2\langle \eta, Z^*Y\eta\rangle\right\} \leq p^2\langle \eta, B\eta\rangle + 2p\,\Re\left\{is\langle \eta, T\eta\rangle\right\} + pq\langle \eta, U\eta\rangle.$$

Adding the above two inequalities one gets

$$|s|^2\langle \eta, (|Y|^2 + |Z|^2)\eta\rangle \leq p^2\langle \eta, B\eta\rangle + p\,\Re\left\{(1+i)s\langle \eta, T\eta\rangle\right\} + pq\langle \eta, U\eta\rangle. \tag{24}$$

Let $pq = 1$, and s be such that $|s| = 1$ and $\Re\left\{(1+i)s\langle \eta, T\eta\rangle\right\} = -\sqrt{2}\,|\langle \eta, T\eta\rangle|$. Then the inequality (24) takes the form

$$\langle \eta, (|Y|^2 + |Z|^2)\eta\rangle \leq p^2\langle \eta, B\eta\rangle - \sqrt{2}\,p\,|\langle \eta, T\eta\rangle| + \langle \eta, U\eta\rangle. \tag{25}$$

Following the argument of Størmer in the proof of Lemma 8.10 in [S] we observe that the function $f(x) = \langle \eta, B\eta\rangle\, x^2 - \sqrt{2}\,|\langle \eta, T\eta\rangle| x + \langle \eta, U\eta\rangle$ has its minimum for $x = 2^{-1/2}\langle \eta, B\eta\rangle^{-1}|\langle \eta, T\eta\rangle|$. Hence, (25) leads to the inequality

$$\langle \eta, (|Y|^2 + |Z|^2)\eta\rangle \leq -2^{-1}\langle \eta, B\eta\rangle^{-1}|\langle \eta, T\eta\rangle|^2 + \langle \eta, U\eta\rangle$$

and finally

$$|\langle \eta, T\eta\rangle|^2 \leq 2\langle \eta, B\eta\rangle\langle \eta, (U - |Y|^2 - |Z|^2)\eta\rangle.$$

By Lemma 10 this implies block-positivity of the matrix (23). ∎

Our next results show that the property (19) in the case $n \geq 2$ has rather restrictive consequences.

PROPOSITION 13. *Let $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$, $n \geq 2$, be a positive linear map with the Choi matrix of the form (20). Assume $|Y| + |Z| = U^{1/2}$. Then Y and Z are linearly dependent.*

Proof. Assume on the contrary that Y and Z are linearly independent. We will show that ϕ can not be positive in this case. To this end let p, q, s be numbers such that $p > 0$, $q > 0$ and $|s|^2 \leq pq$ and define

$$D = p^2B + p(sT + \overline{s}T^*) + pqU - (\overline{s}Y^* + sZ^*)(sY + \overline{s}Z).$$

By Proposition 11 (cf. (21)) it is enough to find numbers p, q, s and a vector $\xi_0 \in \mathbb{C}^n$ such that $\langle \xi_0, D\xi_0\rangle < 0$.

It follows from the assumption and Proposition 5 that

$$\begin{aligned} D &= p^2B + p(sT + \overline{s}T^*) + pq(|Y| + |Z|)^2 \\ &\quad - |s|^2(|Y|^2 + |Z|^2) - \overline{s}^2Y^*Z - s^2Z^*Y \\ &= p^2B + (pq - |s|^2)(|Y|^2 + |Z|^2) + pq(|Y|\,|Z| + |Z|\,|Y|) \\ &\quad + p(sT + \overline{s}T^*) - \overline{s}^2Y^*Z - s^2Z^*Y \\ &= p^2B + (pq - |s|^2)(|Y|^2 + |Z|^2) + psT + p\overline{s}T^* \\ &\quad + (pq\langle \xi_Y, \xi_Z\rangle - \overline{s}^2)Y^*Z + (pq\langle \xi_Z, \xi_Y\rangle - s^2)Z^*Y. \end{aligned}$$

Let $\xi \in \mathbb{C}^n$. Then

$$\begin{aligned}\langle \xi, D\xi\rangle &= p^2\langle \xi, B\xi\rangle + (pq - |s|^2)\langle \xi, (|Y|^2 + |Z|^2)\xi\rangle + 2p\,\Re\{s\langle \xi, T\xi\rangle\}\\ &\quad + 2\Re\{(pq\langle \xi_Y, \xi_Z\rangle - \overline{s}^2)\langle \xi, Y^*Z\xi\rangle\}\\ &= p^2\langle \xi, B\xi\rangle + (pq - |s|^2)\langle \xi, (|Y|^2 + |Z|^2)\xi\rangle + 2p\,\Re\{s\langle \xi, T\xi\rangle\}\\ &\quad + 2\|Y\|\|Z\|\,\Re\{(pq\langle \xi_Y, \xi_Z\rangle - \overline{s}^2)\langle \xi, \xi_Y\rangle\langle \xi_Z, \xi\rangle\}.\end{aligned}$$

Let $\xi_0 = \xi_Y + \xi_Z$ and $s = (pq)^{1/2}e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then

$$\begin{aligned}\langle \xi_0, D\xi_0\rangle &= p^2\langle \xi_0, B\xi_0\rangle + 2p^{3/2}q^{1/2}\,\Re\left\{e^{i\theta}\langle \xi_0, T\xi_0\rangle\right\}\\ &\quad + 2pq\|Y\|\|Z\|\Re\{(\langle \xi_Z, \xi_Y\rangle - e^{-2i\theta})\left(1 + \langle \xi_Y, \xi_Z\rangle\right)^2\}.\end{aligned}$$

By the assumption ξ_Y and ξ_Z are linearly dependent. Moreover $\|\xi_Y\| = \|\xi_Z\| = 1$. This implies that $|\langle \xi_Z, \xi_Y\rangle| < 1$, so $(1 + \langle \xi_Z, \xi_Y\rangle)^2 \neq 0$. Now, choose θ such that

$$\Re\left\{e^{-2i\theta}(1 + \langle \xi_Z, \xi_Y\rangle)^2\right\} = |1 + \langle \xi_Z, \xi_Y\rangle|^2.$$

Then

$$\begin{aligned}\langle \xi_0, D\xi_0\rangle &= p^2\langle \xi_0, B\xi_0\rangle + 2p^{3/2}q^{1/2}\,\Re\left\{e^{i\theta}\langle \xi_0, T\xi_0\rangle\right\}\\ &\quad + 2pq\|Y\|\,\|Z\|\,\left[\Re\left\{\langle \xi_Y, \xi_Z\rangle(1 + \langle \xi_Z, \xi_Y\rangle)^2\right\} - |1 + \langle \xi_Z, \xi_Y\rangle|^2\right].\end{aligned}$$

Observe that

$$\Re\left\{\langle \xi_Y, \xi_Z\rangle(1 + \langle \xi_Z, \xi_Y\rangle)^2\right\} < |1 + \langle \xi_Z, \xi_Y\rangle|^2,$$

so it is possible to find p sufficiently small and q sufficiently large so that $\langle \xi_0, D\xi_0\rangle$ is negative. This ends the proof. ∎

Proposition 14. *Let $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ satisfy the assumptions of the previous Proposition. If $Z = 0$ and $\|Y\| < 1$ (resp. $Y = 0$ and $\|Z\| < 1$) then ϕ is completely positive (resp. completely copositive).*

Proof. It follows that $U = |Y|^2$. Moreover, the assumption $\|Y\| < 1$ implies that $B = 1 - |Y|^2$ is invertible. As we also have $U - |Y|^2 - |Z|^2 = 0$, by Proposition 12 the matrix $\begin{bmatrix} 2B & T \\ T^* & 0 \end{bmatrix}$ is block-positive. Hence $T = 0$. We conclude that the Choi matrix of ϕ has the form

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & Y \\ 0 & 1 - |Y|^2 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ Y^* & 0 & 0 & |Y|^2 \end{array}\right].$$

In order to finish the proof one should show (cf. Proposition 9) that the matrix

$$\begin{bmatrix} 1 & 0 & Y \\ 0 & 1 - |Y|^2 & 0 \\ Y^* & 0 & |Y|^2 \end{bmatrix}$$

is positive, but this can be done by straightforward computations.

The proof in the case $Y = 0$ follows in the same way. ∎

As a consequence of the above results we get the following description of maps satisfying the "quantized" properties (17).

THEOREM 15. *Let $\phi : M_2(\mathbb{C}) \to M_{n+1}(\mathbb{C})$ be a positive unital map with the Choi matrix of the form (20) where $|Y| + |Z| = U^{1/2}$. Then*

(1) *there are vectors $\xi \in \mathbb{C}^2$ and $\eta_0 \in \mathbb{C}^{n+1}$ such that*

$$\phi \in \bigcap_{\eta \perp \eta_0} F_{\xi,\eta}; \tag{26}$$

(2) *ϕ is unitarily equivalent to a map with the Choi matrix of the form*

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & y \\ 0 & 1 & 0 & 0 & 0 & W^* \\ 0 & 0 & 1-u & \overline{z} & V & t \\ \hline 0 & 0 & z & 0 & 0 & 0 \\ 0 & 0 & V^* & 0 & 0 & 0 \\ \overline{y} & W & \overline{t} & 0 & 0 & u \end{array}\right] \tag{27}$$

where in each block there are numbers on positions (11), (13), (31) *and* (33), *one-row matrices from $M_{1,n-1}(\mathbb{C})$ on positions* (12) *and* (32), *one-column matrices from $M_{n-1,1}(\mathbb{C})$ on positions* (21) *and* (23), *and square matrices from $M_{n-1}(\mathbb{C})$ on positions* (22). *Here $u = (|y| + |z|)^2$. Moreover, coefficients satisfy the inequality*

$$|\langle \rho, Y_1^* \rangle| + |\langle \rho, Z_1^* \rangle| \le u^{1/2} \tag{28}$$

for any unit vector $\rho \in \mathbb{C}^n$ where $Y_1, Z_1 \in M_{1,n}(\mathbb{C})$ are defined as

$$Y_1 = \left[\begin{array}{cc} \overline{y} & W \end{array}\right], \qquad Z_1 = \left[\begin{array}{cc} \overline{z} & V \end{array}\right].$$

Proof. It follows from Proposition 13 that there is a unit vector $\eta_0 \in \mathbb{C}^n$ such that $Y^* = \overline{y}\eta_0$ and $Z^* = \overline{z}\eta_0$ for some $y, z \in \mathbb{C}$. Hence $|Y| = |y|P_{\eta_0}$, $|Z| = |z|P_{\eta_0}$, and $U = (|y| + |z|)^2 P_{\eta_0}$, where P_{η_0} is the orthogonal projector onto the one-dimensional subspace generated by the vector η_0. As

$$\phi(P_{e_2}) = \left[\begin{array}{cc} 0 & 0 \\ 0 & U \end{array}\right] \in M_{n+1}(\mathbb{C})$$

then $\phi(P_{e_2})\eta = 0$ for any η orthogonal to η_0. So, from Theorem 2 we obtained (26).

By choosing a suitable basis of $\mathbb{C}^{n+1}$ we may assume that $f_{n+1} = \eta_0$. Then the Choi matrix (20) takes the form

$$\left[\begin{array}{ccccc|ccccc} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & y \\ 0 & 1 & \cdots & 0 & 0 & 0 & t_{11} & \cdots & t_{1,n-1} & t_{1n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & t_{n-1,1} & \cdots & t_{n-1,n-1} & t_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1-u & \overline{z} & t_{n1} & \cdots & t_{n,n-1} & t_{nn} \\ \hline 0 & 0 & \cdots & 0 & z & 0 & 0 & \cdots & 0 & 0 \\ 0 & \overline{t_{11}} & \cdots & \overline{t_{n-1,1}} & \overline{t_{n1}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \overline{t_{1,n-1}} & \cdots & \overline{t_{n-1,n-1}} & \overline{t_{n,n-1}} & 0 & 0 & \cdots & 0 & 0 \\ \overline{y} & \overline{t_{1n}} & \cdots & \overline{t_{n-1,n}} & \overline{t_{nn}} & 0 & 0 & \cdots & 0 & u \end{array}\right].$$

Block-positivity of this matrix implies that the matrix

$$\left[\begin{array}{ccc|ccc} 1 & \cdots & 0 & t_{11} & \cdots & t_{1,n-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & t_{1,n-1} & \cdots & t_{n-1,n-1} \\ \hline \overline{t_{11}} & \cdots & \overline{t_{n-1,1}} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \overline{t_{1,n-1}} & \cdots & \overline{t_{n-1,n-1}} & 0 & \cdots & 0 \end{array}\right].$$

is also block-positive, so $t_{ij} = 0$ for $i, j = 1, 2, \ldots, n-1$. Thus we obtained that the Choi matrix has the form (27).

Now, for any $\rho \in \mathbb{C}^n$, where $\rho = \left[\begin{array}{ccc} \rho_1 & \ldots & \rho_n \end{array}\right]$, define the following matrix from $M_{n+1,2}(\mathbb{C})$:

$$V_\rho = \left[\begin{array}{cccc} \overline{\rho_1} & \ldots & \overline{\rho_n} & 0 \\ 0 & \ldots & 0 & 1 \end{array}\right].$$

One can easily check that $VV^* = 1$, so the map $\psi_\rho : M_{n+1}(\mathbb{C}) \to M_2(\mathbb{C}) : A \mapsto VAV^*$ is unital and completely positive. As a consequence, the map $\psi_\rho \circ \phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is positive and unital. Moreover, by a straightforward calculation one can check that the Choi matrix of this map has the form

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & \langle \rho, Y_1^* \rangle \\ 0 & 1-u & \overline{\langle \rho, Z_1^* \rangle} & t \\ \hline 0 & \langle \rho, Z_1^* \rangle & 0 & 0 \\ \overline{\langle \rho, Y_1^* \rangle} & \overline{t} & 0 & u \end{array}\right].$$

The inequality (28) follows from (III) in Section 1. ∎

4. Conclusions. In our previous paper [MM2] we proved that for any positive unital map $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ from some maximal face there exists a unique decomposition of ϕ onto completely positive and completely copositive parts. To prove this result we have used the techniques based on the so called Choi matrix (see (3)). It turned out that these techniques can be extended for an analysis of maps $\phi : M_2(\mathbb{C}) \to M_n(\mathbb{C})$ ($n \geq 3$). In particular, we have shown that the appropriate Choi matrix (see (5)) has very analogous form but some of the coefficients have to be matrices. In other words, there is some kind of "quantization" of the lowest dimensional case. In Propositions 3, 6 and 12 and Theorem 7 we have shown several necessary conditions for positivity of the map ϕ in terms of its Choi matrix while in Proposition 9 we did it for complete positivity. It is worth pointing out these conditions are generalizations of those given in [S] and [MM2]. Further we emphasize that Theorem 7 demonstrates rather strikingly that a generalization of the inequality (III) from Section 1 is valid. Furthermore, guided by the 2×2 case, the natural strengthening of the (in)equality (III) was examined. To this end in Proposition 13 we show that this quantized condition is very restrictive. This gives the possibility to prove Theorem 15 which fully characterizes maps from $M_2(\mathbb{C})$ into $M_n(\mathbb{C})$ satisfying the condition (28).

We end this paper by a remark that Theorem 15 gives a very useful tool for describing properties of extremal maps in $\mathcal{P}(2, n+1)$ and it seems that following this line of research can give a possibility to construct some new examples of nondecomposable maps. The details will be given in forthcoming publications.

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