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COMPLETELY 1-COMPLEMENTED SUBSPACES OF THE SCHATTEN SPACES S^p

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Abstract. We describe the subspaces of S^p $(1 \leq p \neq 2 < \infty)$ which are the range of a completely contractive projection.

1. Introduction. The results of this paper are taken from [8]. The study of subspaces of L^p $(1 \leq p \neq 2 < \infty)$ which are the range of a contractive projection (1-*complemented subspaces* in short) begun in the sixties. In [5], R. Douglas proved that a 1-complemented subspace of $L^1(\Omega)$ is isometric to a certain $L^1(\Omega')$, with $\Omega' \subset \Omega$. T. Ando showed (in [1]) that the previous result remains valid for $1 \leq p \neq 2 < \infty$. We would like to treat this complementation question for noncommutative L^p-spaces but in the category of operator spaces.

Recall the construction of a noncommutative L^p-space associated with a semifinite von Neumann algebra. Let $1 \leq p < \infty$ and M a von Neumann algebra equipped with a normal faithful semifinite trace τ. Consider the set

$$\{x \in M \mid \|x\|_p := \tau(|x|^p)^{1/p} < \infty\},$$

$\|\cdot\|_p$ is a norm and we denote by $L^p(M,\tau)$ its completion; this is the noncommutative L^p-space associated with (M,τ). The first example of this construction is the Schatten space $S^p(H)$ which is the noncommutative L^p-space associated with $B(H)$ and the usual trace. We denote $S^p = S^p(\ell^2)$.

Using interpolation, G. Pisier has equipped the Banach space $L^p(M,\tau)$ with an operator space structure (see [13]); in this category the morphisms are the *completely bounded* ones (see e.g. [6], [12] for operator spaces theory). A noncommutative L^1-space is just the predual of a von Neumann algebra. In [9], P.W. Ng and N. Ozawa proved the noncommutative analog of Douglas result, more precisely: a completely 1-complemented subspace

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of the predual of a von Neumann algebra is completely isometric to the predual of a certain W^*-TRO (we recall that a subspace X of $B(H)$ is a W^*-TRO if it is w^*-closed and $xy^*z \in X$, for any $x, y, z \in X$). When $1 < p \neq 2 < \infty$, we know little about completely 1-complemented subspaces of a given noncommutative L^p-space. Our purpose is to describe the completely 1-complemented subspaces of $S^p(H)$. Here we will suppose H separable but the main results are valid for H non-separable (see [8]).

Concretely, we can define the completely contractive maps of $S^p(H)$ as follows. For $x \in M_n(S^p(H))$, we define

$$\|x\|_{S_n^p[S^p(H)]} := \|x\|_{S^p(\ell_n^2 \otimes^2 H)}.$$

A linear map $T : S^p(H) \to S^p(K)$ is said to be *completely contractive* (resp. *completely isometric*) if and only if for any $n \in \mathbb{N}$,

$$id_n \otimes T : S_n^p[S^p(H)] \to S_n^p[S^p(K)]$$

is contractive (resp. isometric). Moreover, T is *2-contractive* if and only if $id_2 \otimes T$ is contractive.

Actually, we will need various degrees of complementation. A subspace $X \subset S^p(H)$ is called 2-1-*complemented* (resp. *completely* 1-*complemented*) if and only if there is a 2-contractive (resp. completely contractive) projection of $S^p(H)$ on X.

Now the question is: can we describe the completely 1-complemented subspaces of the Schatten space S^p?

2. Arazy-Friedman's theorem. Our work is based on [4], where J. Arazy and Y. Friedman have described 1-complemented subspaces of $S^p(H)$, their description uses classical Cartan factors.

Cartan factors are a special class of JC^*-triples. Recall that a subspace $X \subset B(H)$ is a *JC^*-triple* if

$$xx^*x \in X, \quad \text{for any } x \in X.$$

A map $T : X \to B(H)$ which preserves the triple-product above is called *a triple-representation*.

Up to triple-isomorphism, there are four types of classical Cartan factors:

- $U(I_{n,m}) = B(\ell_n^2, \ell_m^2)$.
- $U(II_n) = \{x \in B(\ell_n^2) \ : \ t(x) = -x\}$, here t denotes the transpose map.
- $U(III_n) = \{x \in B(\ell_n^2) \ : \ t(x) = x\}$.
- $U(IV_n) = \mathrm{vect}\{1, \omega_1, \ldots, \omega_{n-1}\}$, where the ω_i denote the fermions (see [7] for details on the CAR-algebra and fermionic analysis).

As S^p is not injective (in the category of operator spaces), we cannot identify its completely 1-complemented subspaces up to complete isometry. Thus we define an equivalence relation which preserves the complementation property. Let $X \subset S^p(H)$ and $Y \subset S^p(K)$, then we say that X and Y are *equivalent* if there exist partial isometries $u \in B(K, H)$, $v \in B(H, K)$ such that

$$X = uYv \quad \text{and} \quad u^*uyvv^* = y, \quad \forall y \in Y.$$

This equivalence relation is written $\backsimeq$. It can easily be proved that if $X \subset S^p(H)$ and $Y \subset S^p(K)$ are equivalent, then X is respectively 1-complemented, completely 1-complemented, 2-1-complemented in $S^p(H)$ if and only if Y is respectively 1-complemented, completely 1-complemented, 2-1-complemented in $S^p(K)$.

A reduction of the problem is to decompose the complemented subspace into an orthogonal sum of its "indecomposable" subspaces. Two subspaces $X, Y \subset S^p(H)$ are called *orthogonal* if

$$x^*y = xy^* = 0, \quad \forall x \in X,\ y \in Y.$$

A subspace $X \subset S^p$ is called *indecomposable* if it cannot be written as the orthogonal sum of two of its non-trivial subspaces. Moreover X is complemented if and only if its indecomposable subspaces are complemented.

The main result of [4] could be summarized as follows.

THEOREM 2.1 (Arazy-Friedman). *Let $1 \leq p \neq 2 < \infty$ and $X \subset S^p(H)$ be an indecomposable 1-complemented subspace. Then there are two Hilbert spaces K, L, a classical Cartan factor U, an orthogonal finite family $\{a_j\}_{j\in J} \subset S^p(K)$ and a family of faithful triple-representations $T_j : U \to B(L)$ such that*

$$X \backsimeq \Big\{ \sum_{j\in J} a_j \otimes T_j(y),\ y \in U_p \Big\} \subset S^p(K \otimes^2 L), \tag{2.1}$$

with $U_p = U \cap S^p$.

In that case, the subspace X is said to be of the type of U.

3. Main results. Clearly $S^p_{n,m}$ (where n, m are countable cardinals) is completely 1-complemented in S^p. Using the description of isometries of $S^p_{n,m}$ in [2], we can show a little bit more:

PROPOSITION 3.1. *Let $1 \leq p \neq 2 < \infty$ and $i : S^p_{n,m} \to S^p(H)$ be a complete isometry. Then the range of i is completely 1-complemented.*

The purpose is to prove the converse of the previous proposition, i.e. every completely 1-complemented indecomposable subspace of S^p "looks like" an $S^p_{n,m}$.

From Arazy-Friedman's theorem, the strategy is the following: as $S^p(H)$ is smooth and reflexive, the contractive projection P on a complemented subspace X is unique. For the spaces of type $I_{n,m}$ $(n, m \geq 2)$, II_n, III_n and IV_n, we exhibit this projection. The complete contractivity of P imposes conditions on the a_j's in (2.1). For the type $I_{1,n}$, the strategy is somewhat different; we examine the contractive projection on a completely 1-complemented subspace of X. Nevertheless, we only use the 2-contractivity of P and we obtain:

THEOREM 3.2. *Let $1 \leq p \neq 2 < \infty$ and X an indecomposable subspace of $S^p(H)$. Suppose that X is the range of a 2-contractive projection. Then there exist n, m, a Hilbert space K and an operator $a \in S^p(K)$ of norm 1 such that*

$$X \backsimeq a \otimes S^p_{n,m}.$$

Corollary 3.3. *Let $1 \leq p \neq 2 < \infty$ and X a subspace of $S^p(H)$. The following are equivalent:*

(i) X is completely 1-complemented.
(ii) X is the range of a 2-contractive projection.
(iii) There exist two sequences (n_k), (m_k), a Hilbert space K and orthogonal operators $a_k \in S^p(K)$ of norm 1 such that

$$X \simeq \oplus_k^p a_k \otimes S^p_{n_k,m_k}.$$

(iv) There exist two sequences (n_k), (m_k) such that

$$X = \oplus_k^p S^p_{n_k,m_k} \quad \text{completely isometrically.}$$

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