NONCOMMUTATIVE HARMONIC ANALYSIS
WITH APPLICATIONS TO PROBABILITY
BANACH CENTER PUBLICATIONS, VOLUME 78
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2007

OPERATORS OF THE q-OSCILLATOR

FRANCISZEK HUGON SZAFRANIEC

Instytut Matematyki, Uniwersytet Jagielloński
Reymonta 4, 30-059 Kraków, Poland
E-mail: umszafra@cyf-kr.edu.pl

Abstract. We scrutinize the possibility of extending the result of [19] to the case of q-deformed oscillator for q real; for this we exploit the whole range of the deformation parameter as much as possible. We split the case into two depending on whether a solution of the commutation relation is bounded or not. Our *leitmotif* is *subnormality.*

The deformation parameter q is reshaped and this is what makes our approach effective. The newly arrived parameter, the operator C, has two remarkable properties: it separates in the commutation relation the annihilation and creation operators from the deformation as well as it q-commutes with those two. This is why introducing the operator C may have far-reaching consequences.

q-deformations of the quantum harmonic oscillator (the abbreviation the *q-oscillator* stands here for it) has been arresting attention of many [1] resulting among other things in quantum groups. Besides realizing the ever lasting temptation to generalize matters, it brings forth new attractive findings. This paper exhibits the *spatial* side of the story.

The q-oscillator algebra, which is the *milieu* of our considerations, is generated by three objects a_+, a_- and 1 (the latter being a unit in the algebra) satisfying the commutation relations

$$a_-a_+ - qa_+a_- = 1; \tag{1}$$

it goes back to the seventies with [1] as a specimen. The other versions which appear in the literature are equivalent, and this is described completely in [8] where a list of further references can be found.

2000 *Mathematics Subject Classification*: Primary 47B20, 81S05.

Key words and phrases: unbounded subnormal operator, q-oscillator.

Supported at its final stage by the MNiSzW grant N201 026 32/1350.

The paper is in final form and no version of it will be published elsewhere.

[1] q-deformations are vastly disseminated in Mathematical Physics and we would like to acknowledge here with pleasure [8] for bringing them closer to Mathematics.

Looking for $*$-representations of (1) usually means assuming that $a_- = a_+^*$, with the asterisk denoting the Hilbert space adjoint. Thus what we start with is a *given* Hilbert space and the commutation relation

$$S^*S - qSS^* = I \tag{$\mathcal{O}_{q,\mathrm{op}}$}$$

in it. Of course, q must perforce be <u>real</u> then; this is what we assume in this paper.

An easy-going consequence is

SAMPLE THEOREM. *If S is a weighted shift with respect to the basis $\{e_n\}_{n=0}^{\infty}$ and*

$$S^*Sf - qSS^*f = f, \quad f \in \operatorname{lin}\{e_n\}_{n=0}^{\infty},$$

then $Se_n = \sqrt{1 + q + \cdots + q^n}\, e_{n+1}$, $n \geqslant 0$.

'If S is a weighted shift'—this is usually tacitly assumed when dealing with the relation ($\mathcal{O}_{q,\mathrm{op}}$), like in [5]. It is sometimes made a bit more explicit by stating that a vacuum vector (or a ground state, depending on denomination in Mathematical Physics an author belongs to) of S exists. The point here (as it was in [19] for $q = 1$) is to discuss the case. It turns out that, like in [19], *subnormality* plays an important role in the matter (and this, the case $q = 1$ at least, is parallel to Rellich-Dixmier [12, 7] characterization of solutions to the CCR). Luckily, the above coincides with our belief that subnormality is the missing counterpart of complex variable in the quantization scheme.

Preliminary essentials

A short guide to subnormality. Recall that a densely defined operator A is said to be *hyponormal* if $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $\|A^*f\| \leqslant \|Af\|$, $f \in \mathcal{D}(A)$. A hyponormal operator N is said to be *formally normal* if $\|Nf\| = \|N^*f\|$, $f \in \mathcal{D}(N)$. Specifying more, a formally normal operator N is called *normal* if $\mathcal{D}(N) = \mathcal{D}(N^*)$. Finally, a densely defined operator S is called (*formally*) *subnormal* if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ isometrically and a (formally) normal operator N in $\mathcal{K}$ such that $S \subset N$.

The following diagram relates these notions.

$$\begin{tikzcd}
\text{normal} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \text{formally normal} \arrow[dd, Rightarrow] \arrow[dr, Rightarrow] & \\
 & & \text{hyponormal} \\
\text{subnormal} \arrow[r, Rightarrow] & \text{formally subnormal} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

Though the definitions of formal normality and normality look much alike, with a small difference concerning the domains involved, the operators they define may behave in a totally incomparable manner. However, needless to say, these two notions do not differ at all in the case of bounded operators.

If A and B are densely defined operators in $\mathcal{H}$ and $\mathcal{K}$ resp. such that $\mathcal{H} \subset \mathcal{K}$ and $A \subset B$ then

$$\mathcal{D}(A) \subset \mathcal{D}(B) \cap \mathcal{H}, \quad \mathcal{D}(B^*) \cap \mathcal{H} \subset P\mathcal{D}(B^*) \subset \mathcal{D}(A^*) \tag{2}$$

where P stands for the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$; moreover,

$$A^*Px = PB^*x, \quad x \in \mathcal{D}(B^*). \tag{3}$$

If B closable, then so is A and both A^* as well as B^* are densely defined. The extension B of A is said to be *tight* if $\mathcal{D}(\bar{A}) = \mathcal{D}(\bar{B}) \cap \mathcal{H}$ and $*$-*tight* if $\mathcal{D}(B^*) \cap \mathcal{H} = \mathcal{D}(A^*)$. If $\mathcal{D}(B) \subset \mathcal{D}(B^*)$ (and this happens for formally normal operators as we already know), the two chains in (2) glue together as[2]

$$\mathcal{D}(A) \subset \mathcal{D}(B) \cap \mathcal{H} \subset \mathcal{D}(B^*) \cap \mathcal{H} \subset P\mathcal{D}(B^*) \subset \mathcal{D}(A^*). \tag{4}$$

As we have already said a densely defined operator having a normal extension is just subnormal. However, normal extensions may not be uniquely determined in the unbounded case as their minimality becomes a rather fragile matter, see [17]; even though the inclusions (4) hold for any of them. Moreover, even if all of them turn into equalities none of the normal extensions may be minimal of cyclic type (this is what ensures uniqueness); this will become effective when we pass to the case of $q > 1$. So far we have got an obvious fact.

PROPOSITION 1. *A subnormal operator S has a normal extension which is both tight and $*$-tight if and only if*

$$\mathcal{D}(\bar{S}) = \mathcal{D}(S^*). \tag{5}$$

If this happens then any normal extension is both tight and $*$-tight.

Because equality (5) is undoubtedly decisive for a solution of the commutation relation of (any of) the oscillators to be a weighted shift, subnormality is properly settled into this context.

q-notions. For x an integer and q real, $[x]_q \stackrel{\text{def}}{=} (1-q^x)(1-q)^{-1}$ if $q \neq 1$ and $[x]_1 \stackrel{\text{def}}{=} x$. If x is a non-negative integer, $[x]_q = 1+q+\dots+q^{x-1}$ and this is usually referred to as a *basic* or q-number. A little step further, the q-factorial is like the conventional, $[0]_q! \stackrel{\text{def}}{=} 1$ and $[n]_q! \stackrel{\text{def}}{=} [0]_q \cdots [n-1]_q[n]_q$ and so is the q-binomial $\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]_q \stackrel{\text{def}}{=} \frac{[m]_q!}{[m-n]_q![n]_q!}$. Thus, if $-1 \leqslant q$ and $x \in \mathbb{N}$ the basic number $[x]_q$ is non-negative.

For arbitrary complex numbers a and q one can always define $(a;q)_k$ as follows:

$$(a;q)_0 \stackrel{\text{def}}{=} 1, \quad (a;q)_k \stackrel{\text{def}}{=} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}), \quad k = 1,2,3,\dots$$

Then for $n > 0$ one has $[n]_q! = (q;q)_n(1-q)^{-n}$. Moreover, there are (at least) two possible definitions of q-exponential functions

$$e_q(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} z^k, \quad z \in \omega_q,$$

$$E_q(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q;q)_k} z^k, \quad z \in \omega_{q^{-1}}, \quad q \neq 0,$$

where

$$\omega_q \stackrel{\text{def}}{=} \begin{cases} \{z;\ |z| < 1\} & \text{if } |q| < 1, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

These two functions are related via

$$e_q(z) = E_{q^{-1}}(-z), \quad z \in \omega_q, \quad q \neq 0.$$

[2] Description of domains of weighted shifts and their adjoints can be found in [15].

The q-oscillator

Spatial interpretation of $(\mathcal{O}_{q,\mathrm{op}})$**.** The relation $(\mathcal{O}_{q,\mathrm{op}})$ has nothing but a symbolic meaning unless someone says something more about it; this is because some of the solutions may be unbounded. For this reason we distinguish two, extreme in a sense, ways of looking at the relation $(\mathcal{O}_{q,\mathrm{op}})$:

The first meaning of $(\mathcal{O}_{q,\mathrm{op}})$ is

$$\begin{gathered} S \text{ closable, } \mathcal{D} \text{ is dense in } \mathcal{H} \text{ and} \\ \mathcal{D} \subset \mathcal{D}(S^*\bar{S}) \cap \mathcal{D}(\bar{S}S^*),\ S^*Sf - qSS^*f = f,\ f \in \mathcal{D}. \end{gathered} \tag{$\mathcal{O}_{q,\mathcal{D}}$}$$

The other is

$$\langle Sf, Sg\rangle - q\langle S^*f, S^*g\rangle = \langle f, g\rangle,\ f, g \in \mathcal{D}(S) \cap \mathcal{D}(S^*) \tag{$\mathcal{O}_{q,\mathrm{w}}$}$$

and, because this is equivalent to

$$\|Sf\|^2 - q\|S^*f\|^2 = \|f\|^2, \quad f \in \mathcal{D}(S) \cap \mathcal{D}(S^*)$$

it implies for S to be closable, $(\mathcal{O}_{q,\mathrm{w}})$ in turn is equivalent to

$$\langle \bar{S}f, \bar{S}g\rangle - q\langle S^*f, S^*g\rangle = \langle f, g\rangle, \quad f \in \mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*).$$

The occurring interdependence, which follows, let us play a variation on the theme of $(\mathcal{O}_{q,\mathrm{op}})$.

1° $(\mathcal{O}_{q,\mathcal{D}})$ *with* $\mathcal{D}$ *being a core of* $S \Rightarrow (\mathcal{O}_{q,\mathrm{w}})$ *and* $\mathcal{D}(\bar{S}) \subset \mathcal{D}(S^*)$.

Indeed, for $f \in \mathcal{D}(\bar{S})$ there is a sequence $(f_n)_n \subset \mathcal{D}$ such that $f_n \to f$ and $Sf_n \to \bar{S}f$. Because S^* is closed we get from $(\mathcal{O}_{q,\mathcal{D}})$ that $S^*f_n \to S^*f$ and consequently $f \in \mathcal{D}(S^*)$ as well as $(\mathcal{O}_{q,\mathrm{w}})$.

2° $(\mathcal{O}_{q,\mathcal{D}})$ *with* $\mathcal{D}$ *being a core of* $S^* \Rightarrow (\mathcal{O}_{q,\mathrm{w}})$ *and* $\mathcal{D}(S^*) \subset \mathcal{D}(\bar{S})$.

This uses the same argument as that for 1°.

3° $(\mathcal{O}_{q,\mathrm{w}}) \Rightarrow (\mathcal{O}_{q,\mathcal{D}})$ *with* $\mathcal{D} = \mathcal{D}(S^*\bar{S}) \cap \mathcal{D}(\bar{S}S^*)$.

This is because $\mathcal{D}(S^*\bar{S}) \cap \mathcal{D}(\bar{S}S^*) \subset \mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$.

4° $(\mathcal{O}_{q,\mathrm{w}})$ *and* $\mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$ *a core of* S *and* $S^* \Rightarrow \mathcal{D}(S^*\bar{S}) = \mathcal{D}(\bar{S}S^*)$.

Take $f \in \mathcal{D}(S^*\bar{S})$. This means $f \in \mathcal{D}(\bar{S})$ and $\bar{S}f \in \mathcal{D}(S^*)$. Because of this, picking $(f_n)_n \in \mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$, we get from $(\mathcal{O}_{q,\mathrm{w}})$ in the limit

$$\langle S^*\bar{S}f, g\rangle - q\langle S^*f, S^*g\rangle = \langle f, g\rangle \tag{6}$$

for $g \in \mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$ and, because $g \in \mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$ is a core of S^*, we get (6) to hold for $g \in \mathcal{D}(S^*)$. Finally, $S^*f \in \mathcal{D}(\bar{S})$. The reverse inequality needs the same kind of argument.

The above results in

5° $(\mathcal{O}_{q,\mathrm{w}})$ *and* $\mathcal{D}(\bar{S}) = \mathcal{D}(S^*) \Rightarrow \bar{S}$ *satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *on* $\mathcal{D} = \mathcal{D}(S^*\bar{S}) = \mathcal{D}(\bar{S}S^*)$.

Remark 2. Notice that when $q \neq -1$ and S satisfying $(\mathcal{O}_{q,\mathcal{D}})$ with $\mathcal{D} = \mathcal{D}(S^*\bar{S}) = \mathcal{D}(\bar{S}S^*)$ for $\mathcal{D}$ to be a core of S^* is necessary and sufficient $\mathcal{R}(S^*S)$ to be dense in $\mathcal{H}$.

The following is a kind of general observation and puts hyponormality (or boundedness) in the context of $(\mathcal{O}_{q,\mathcal{D}})$.

PROPOSITION 3. (*a*) *For* $0 \leqslant q < 1$ *and for* S *satisfying* $(\mathcal{O}_{q,\mathcal{D}})$, $S|_{\mathcal{D}}$ *is hyponormal if and only if* S *is bounded and* $\|S\| \leqslant (1-q)^{-1/2}$. (*b*) *For* $q < 0$ *and for* S *satisfying* $(\mathcal{O}_{q,\mathcal{D}})$, $S^*|_{\mathcal{D}}$ *is hyponormal if and only if* S *is bounded and* $\|S\| \leqslant (1-q)^{-1/2}$.

Proof. Write $(\mathcal{O}_{q,\mathcal{D}})$ as

$$(1-q)\|Sf\|^2 = q(\|S^*f\|^2 - \|Sf\|^2) + \|f\|^2, \quad f \in \mathcal{D},$$

and look at this. ∎

The selfcommutator. Assuming $\mathcal{D} \subset \mathcal{D}(SS^*) \cap \mathcal{D}(S^*S)$ we introduce the following operator:

$$C \stackrel{\text{def}}{=} I + (q-1)SS^*, \quad \mathcal{D}(C) \stackrel{\text{def}}{=} \mathcal{D}. \tag{7}$$

This operator turns out to be an important invention in the matter. In particular there are two immediate consequences of this definition. The first says if S satisfies $(\mathcal{O}_{q,\mathcal{D}})$ with $\mathcal{D}$ invariant for both S and S^* then $\mathcal{D}$ is invariant for C as well and

$$CSf = qSCf, \quad qCS^*f = S^*Cf, \quad f \in \mathcal{D}. \tag{8}$$

The other is that $(\mathcal{O}_{q,\mathcal{D}})$ takes now the form

$$S^*Sf - SS^*f = Cf, \quad f \in \mathcal{D}, \tag{9}$$

which means that C is just the selfcommutator of S on $\mathcal{D}$.

We would like to know the instances when C is a positive operator.

PROPOSITION 4. (a) *For* $q \geqslant 1$, $C > 0$ *always.* (b) *For* $q < 1$, $C \geqslant 0$ *if and only if* S *is bounded and* $\|S\| \leqslant (1-q)^{-1/2}$. (c) *For* S *satisfying* $(\mathcal{O}_{q,\mathcal{D}})$, $C \geqslant 0$ *if and only if* S *is hyponormal.*

Proof. While (a) is apparently trivial, (b) comes immediately from

$$\langle Cf, f\rangle = \|f\|^2 + (q-1)\|S^*f\|^2, \quad f \in \mathcal{D}.$$

For (c) write (using $(\mathcal{O}_{q,\mathcal{D}})$) with $f \in \mathcal{D}$

$$\langle Cf, f\rangle = \|f\|^2 + (q-1)\|S^*f\|^2 = \|f\|^2 + q\|S^*f\|^2 - \|S^*f\|^2 = \|Sf\|^2 - \|S^*f\|^2. \ ∎$$

EXAMPLE 5. On the other hand, with any unitary U the operator

$$S \stackrel{\text{def}}{=} (1-q)^{-1/2}U \tag{10}$$

satisfies $(\mathcal{O}_{q,\mathcal{D}})$ if $q < 1$. The operator S is apparently bounded and normal. Consequently (the Spectral Theorem) it may have a bunch of nontrivial reducing subspaces (even not necessarily one dimensional) or may be irreducible and this observation ought to be dedicated to all those who start too fast generating algebras from formal commutation relations.

PROPOSITION 6. *For* $q < 1$ *the only formally normal operators satisfying* $(\mathcal{O}_{q,\mathcal{D}})$ *are those of the form* (10). *For* $q \geqslant 1$ *there is no formally normal solution of* $(\mathcal{O}_{q,\mathcal{D}})$.

Proof. Straightforward. ∎

EXAMPLE 7. An *ad hoc* illustration can be given as follows. Take a separable Hilbert space with a basis $(e_n)_{n=-\infty}^{\infty}$ and look for a bilateral (or rather *two-sided*) weighted shift T defined as $Te_n = \tau_n e_{n+1}$, $n \in \mathbb{Z}$. Then, because $T^*e_n = \bar{\tau}_{n-1}e_{n-1}$, $n \in \mathbb{Z}$, for any

$\alpha \in \mathbb{C}$ and $N \in \mathbb{Z}$ we get $|\tau_n|^2 = \alpha q^{n+N} + (1-q^{n+N})(1-q)^{-1} = \alpha q^{n+N} + [n+N]_q$ for all n if $q \neq 1$ and $|\tau_n|^2 = \alpha + n$ if $q = 1$; this is for all $n \in \mathbb{Z}$. The only possibility for the right hand sides to be non-negative (and in fact positive) [3] is $\alpha \geqslant (1-q)^{-1}$ for $0 \leqslant q < 1$ and $\alpha = (1-q)^{-1}$ for $q < 0$; the latter corresponds to Example 10. Thus *the only* bilateral weighted shifts satisfying $(\mathcal{O}_{q,\mathcal{D}})$, with $\mathcal{D} = \operatorname{lin}\{e_n;\ n \in \mathbb{Z}\}$, are those $Te_n = \tau_n e_{n+1}$, $n \in \mathbb{Z}$ which have the weights

$$\tau_n \stackrel{\text{def}}{=} \begin{cases} \sqrt{(1-q)^{-1}}, & & q \leqslant 0 \\ \sqrt{\alpha q^{n+N} + [n+N]_q}, & \alpha > (1-q)^{-1},\ N \in \mathbb{Z}, & 0 \leqslant q < 1 \\ \text{none}, & & 1 \leqslant q \end{cases}$$

However, T *violates hyponormality* (pick up $f = e_0$ as a sample) if $0 < q < 1$. Also C defined by (7) is *neither positive nor negative* ($\langle Ce_0, e_0\rangle = a > 0$ while $\langle Ce_{-1}, e_{-1}\rangle < 0$). Let us mention that T is q^{-1}-hyponormal in the sense of [13]. Anyway, T is apparently *unbounded* if $q > 0$. The case of $q \leqslant 0$ is precisely that of Example 10.

EXAMPLE 8. Repeating the reasoning of Example 7 we get that *the only* unilateral weighted shifts satisfying $(\mathcal{O}_{q,\mathcal{D}})$ are those T, defined as $Te_n = \tau_n e_{n+1}$ for $n \in \mathbb{N}$, which have the weights

$$\tau_n = \sqrt{[n+1]_q}\,, \quad -1 \leqslant q.$$

This is so because the virtual, in this case, 'τ_{-1}' is 0 ($T^*e_0 = 0$). If $-1 \leqslant q < 0$ they are bounded and *not hyponormal*, if $0 \leqslant q < 1$ they are again bounded and *hyponormal*, and if $1 \leqslant q$ they are unbounded and *hyponormal*; the latter two are even subnormal (cf. Theorem 19 and 21 resp.).

REMARK 9. According to Lemma 2.3 of [10] for $0 < q < 1$ the only cases which may happen are the orthogonal sums of the operators considered in Examples 7, 8 and given by formula (10). For $q > 1$, due to the same Lemma, the orthogonal sum of that from Example 8 can be taken into account.

An auxiliary lemma of [14]. We state here a result, [14] Lemma 2.4, which justifies the examples above. We adapt the notation of [14] to ours as well as improve a bit the syntax of the conclusion therein.

LEMMA 10. *Let $0 < p < 1$ and $\varepsilon \in \{-1, +1\}$. Assume T is a closed densely defined operator in $\mathcal{H}$. Then*

$$T^*Tf - p^2TT^*f = \varepsilon(1-p^2)f, \quad f \in \mathcal{D}(T^*T) = \mathcal{D}(TT^*) \tag{11}$$

if and only if T is unitarily equivalent to an orthogonal sum of operators of the following type:

- *in the case of $\varepsilon = 1$*

  (I) $T_{\mathrm{I}}\colon f_n \to (1-p^{2(n+1)})^{1/2} f_{n+1}$ *in* $\mathcal{H} = \bigoplus_{n=0}^{+\infty} \mathcal{H}_n$ *with each* $\mathcal{H}_n \stackrel{\text{def}}{=} \mathcal{H}_0$*;*

[3] We avoid weights which are not non-negative, for instance complex, as they lead to a unitarily equivalent version only.

(II) $T_{\mathrm{II}}\colon f_n \to (1+q^{2(n+1)}A^2)^{1/2} f_{n+1}$ *in* $\mathcal{H} = \bigoplus_{n=-\infty}^{+\infty} \mathcal{H}_n$ *with each* $\mathcal{H}_n \overset{\mathrm{def}}{=} \mathcal{H}_0$ *and* A *being a selfadjoint operator in* $\mathcal{H}_0$ *with* $\mathrm{sp}(A) \subset [p,1]$ *and either* p *or* 1 *not being an eigenvalue of* A;

(III) T_{III} *a unitary operator;*

- *in the case of* $\varepsilon = -1$

(IV) $T_{\mathrm{IV}}\colon f_n \to (p^{2n}-1)^{1/2} f_{n-1}$ *in* $\mathcal{H} = \bigoplus_{n=0}^{+\infty} \mathcal{H}_n$ *with each* $\mathcal{H}_n \overset{\mathrm{def}}{=} \mathcal{H}_0$ *and always* $f_{-1} \overset{\mathrm{def}}{=} 0$.

A couple of remarks seem to be absolutely imperative.

Remark 11. The conclusion of Lemma 10 is a bit too condensed. Let us provide some hints to reading it. First of all the f_n's appearing in (I), (II) and (IV) should be understood as follows: take $f \in \mathcal{H}_0$ and define f_n as a (one sided or two sided, depending on circumstances) sequence having all the coordinates zero except the one labelled n which is equal to f. Then, with the definition

$$\mathcal{D}(\mathcal{E}) \overset{\mathrm{def}}{=} \mathrm{lin}\{f_n;\ f \in \mathcal{E} \subset \mathcal{H}_0,\ n \in \mathbb{Z} \text{ or } n \in \mathbb{N} \text{ depending on the case}\},$$

one has to guess that $\mathcal{D}(T_{\mathrm{I}}) = \mathcal{D}(T_{\mathrm{IV}}) = \mathcal{D}(\mathcal{H}_0)$ and $\mathcal{D}(T_{\mathrm{II}}) = \mathcal{D}(\mathcal{D}(A))$. Passing to closures in (I), (II) and (IV) we check that $\overline{T}_{\mathrm{I}}$ as well as $\overline{T}_{\mathrm{IV}}$ are everywhere defined bounded operators (use $0 < p < 1$) while $\overline{T}_{\mathrm{II}}$ is always unbounded (though satisfying $\mathcal{D}(T_{\mathrm{II}}^{*}\overline{T}_{\mathrm{II}}) = \mathcal{D}(\overline{T}_{\mathrm{II}}T_{\mathrm{II}}^{*})$ [4]).

Remark 12. To relate (11) to $(\mathcal{O}_{q,\mathcal{D}})$ set $\varepsilon = 1$, $p = \sqrt{q}$ and $T = \sqrt{1-p^2}S$ when $0<q<1$, and $\varepsilon = -1$, $p^{-1} = \sqrt{q}$ and $T = p^{-1}\sqrt{p^2-1}S^*$ when $q > 1$.

Positive definiteness from $(\mathcal{O}_{q,\mathcal{D}})$. The following formalism will be needed.

Proposition 13. *If* S *satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *with* $\mathcal{D}$ *invariant for both* S *and* S^*, *then*

$$S^{*i}S^j f = \sum_{k=0}^{\infty} [k]_q! \begin{bmatrix} i \\ k \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q S^{j-k} C^k S^{*(i-k)} f, \quad f \in \mathcal{D},\ i,j = 0,1,\ldots, \tag{12}$$

If, moreover, $C \geqslant 0$ *then*

$$\sum_{i,j=0}^{p} \langle S^i f_j, S^j f_i \rangle = \sum_{k=0}^{\infty} [k]_q! \left\| \sum_{i=0}^{p} \begin{bmatrix} i \\ k \end{bmatrix}_q C^{k/2} S^{*(i-k)} f_i \right\|^2, \quad f_0,\ldots,f_p \in \mathcal{D}. \tag{13}$$

All this under convention $S^l = (S^*)^l = 0$ *for* $l < 0$ *and* $\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]_q = 0$ *for* $j > i$.

Proof. Formula (12) is in [6, formula (35)]. Formula (13) is an immediate consequence of (12). ■

As a direct consequence of Fact A and (13) we get

Corollary 14. *Suppose* S *satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *where* $\mathcal{D}$ *is invariant for* S *and* S^*, *and* $\mathcal{D}$ *is a core of* S. *If* $C \geqslant 0$, *then*

$$\sum_{i,j=0}^{p} \langle S^i f_j, S^j f_i \rangle \geqslant 0, \quad f_0,\ldots,f_p \in \mathcal{D}. \tag{$\mathcal{PD}$}$$

[4] In this situation we have implications 4° and 5° on p. 296.

A useful lemma

LEMMA 15. *Let $q > 0$. Consider the following conditions:*

(a) *S satisfies $(\mathcal{O}_{q,\mathrm{w}})$ and $\mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$;*

(b) *$\mathcal{N}(S^*) \neq \{0\}$ and for $n = 0, 1, \ldots$*

$$f \in \mathcal{N}(S^*) \Rightarrow \bar{S}^n f \in \mathcal{D}(\bar{S}),\ \ \bar{S}^{(n-1)} f \in \mathcal{D}(S^*)\ \ \&\ \ S^* \bar{S}^{n-1} f = (n-1) \bar{S}^{n-2} f; \tag{14}$$

(c) *there is $f \neq 0$ such that $\bar{S}^n f \in \mathcal{D}(\bar{S})$, $n = 0, 1, \ldots$ and $\bar{S}^m f \perp \bar{S}^n$ for $m \neq n$.*

Then (a)$\Rightarrow$(b)$\Rightarrow$(c).

Proof. (a)$\Rightarrow$(b). The polar decomposition for S^* is $S^* = V|S^*|$ where V is a partial isometry with the initial space $\mathcal{R}(|S^*|)$ and the final space $\mathcal{R}(SS^*)$. Suppose $\mathcal{N}(S^*) = \{0\}$. Then, because $\mathcal{N}(V) = \mathcal{R}(|S^*|)^{\perp} = \mathcal{N}(|S^*|) = \mathcal{N}(\bar{S}S^*) = \mathcal{N}(S^*)$, V is unitary. Since $\bar{S} = |S^*|V^*$, from 5° we get $V|S^*|^2 V^* = q|S^*|^2 + I$. Consequently, for the spectra we have $\mathrm{sp}(|S^*|) \subset q\,\mathrm{sp}(|S^*|) + 1 \subset [0, +\infty)$, which is absurd. Thus $\mathcal{N}(S^*) \neq \{0\}$.

We show (14) by induction. Of course, $\mathcal{N}(S^*) \subset \mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$, which establishes (14) for $n = 0$. Suppose $\mathcal{N}(S^*) \subset \mathcal{D}(\bar{S}^n)$ and $S^*\bar{S}^{n-1} f = (n-1)\bar{S}^{n-2} f$. Then, for $g \in \mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$,

$$\langle S^* \bar{S}^{n-1} f, S^* g\rangle = (n-1)\langle \bar{S}^{n-2} f, \bar{S}^* g\rangle. \tag{15}$$

Because already $\bar{S}^{(n-2)} f \in \mathcal{D}(\bar{S}) = \mathcal{D}(S^{**})$, we have

$$|\langle S^* \bar{S}^{n-1} f, S^* g\rangle| \le C\|g\|. \tag{16}$$

Because $\bar{S}^{(n-1)} \in \mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$, we can use $(\mathcal{O}_{q,\mathrm{w}})$ to get

$$\langle \bar{S}^n f, \bar{S} g\rangle = \langle \bar{S}\bar{S}^{(n-1)} f, \bar{S} g\rangle = \langle S^* \bar{S}^{(n-1)}, S^*\rangle + \langle \bar{S}^{(n-1)f}, g\rangle.$$

This, by (16), implies $\bar{S}^n f \in \mathcal{D}(S^*) = \mathcal{D}(\bar{S})$ and, consequently, by (15), gives us $S^*\bar{S}^n f = n\bar{S}^{n-1} f$, which completes the induction argument. Now a straightforward application of (14) gives $\bar{S}^n(\mathcal{N}(S^*)) \subset \mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$ for $n = 0, 1, \ldots$.

(b)$\Rightarrow$(c). Take any $f \in \mathcal{N}(S^*)$ and using (14) and (12) write

$$\langle S^m f, S^n f\rangle = \langle S^{n*} S^m f, f\rangle = \sum_{k=0}^{\min\{m,n\}} [k]_q! \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k \end{bmatrix}_q \langle S^{(n-k)} C^k S^{*(m-k)} f, f\rangle = 0, \quad m > n. \ \blacksquare$$

A matrix formation. Suppose $q > 0$ and S is a weighted shift with respect to $(e_k)_{k=0}^{\infty}$ with the weights $(\sqrt{[k+1]_q}\,)_{k=0}^{\infty}$. With

$$S_0 \overset{\text{def}}{=} S, \quad S_n \overset{\text{def}}{=} q^{n/2} S, \quad D_n \overset{\text{def}}{=} \sqrt{[n]_q}\ \mathrm{diag}(q^{k/2})_{k=0}^{\infty}, \quad n = 1, 2, \ldots \tag{17}$$

the matrix

$$\begin{pmatrix} S_0 & D_1 & 0 & 0 & \\ 0 & S_1 & D_2 & 0 & \ddots \\ 0 & 0 & S_2 & D_3 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{18}$$

defines an operator $\boldsymbol{N}$ in $\bigoplus_{n=0}^{\infty} \mathcal{H}_n$, $\mathcal{H}_n = \mathcal{H}$, with domain composed of all those $\bigoplus_{n=0}^{\infty} f_n$

for which $f_n = 0$ but for a finite number of n's. This matrix for the familiar creation operator was set out in [21].

First we need to determine $\mathcal{D}(\boldsymbol{N}^*)$ and relate it to $\mathcal{D}(\boldsymbol{N})$. If $0 < q < 1$ then each D_n is bounded. In that case Remark 9 in [20] gives us

$$\mathcal{D}(\boldsymbol{N}^*) = \bigoplus_{n=0}^{\infty} \mathcal{D}(S_n^*). \tag{19}$$

If $q > 1$ then each $S_n D_n^{-1}$ is bounded. According to Proposition 4.5 in [11] and Corollary 8 in [20] we can deduce (19) as well. In either case, what we get is the adjoint of $\boldsymbol{N}$ can be taken as a matrix of adjoints (which is rather an exceptional case). Because the same argument concerning the adjoint of a matrix operator applies now to $\boldsymbol{N}^*$ we can assert that the closure operation for the operator $\boldsymbol{N}$ goes entrywise as well. Now, due to the fact that the apparent norm equality for $\boldsymbol{N}$ and $\boldsymbol{N}^*$ holds on $\mathcal{D}(\boldsymbol{N})$, we get essential normality of $\boldsymbol{N}$. Consequently,

$$S \text{ is subnormal and } \bar{N} \text{ is its tight and } *\text{-tight normal extension.} \tag{20}$$

Subnormality in the q-oscillator

The case of S bounded. The next result says a little more about boundedness of solutions of $(\mathcal{O}_{q,\mathcal{D}})$.

PROPOSITION 16. *Suppose S is bounded and satisfies $(\mathcal{O}_{q,\mathcal{D}})$. (a) If $q < 0$ then $\|S\| \geqslant (1-q)^{-1/2}$. (b) If $0 \leqslant q < 1$ then $\|S\| \leqslant (1-q)^{-1/2}$. (c) If $q \geqslant 1$ then no such an S exists.*

Proof. For (a) look at $\|Sf\|^2 = \|f\|^2 + q\|S^*f\|^2 \geqslant \|f\|^2 + q\|S\|^2\|f\|^2$, and for (b) at $\|Sf\|^2 = \|f\|^2 + q\|S^*f\|^2 \leqslant \|f\|^2 + q\|S\|^2\|f\|^2$. For (c) write $\|Sf\|^2 = \|f\|^2 + q\|S^*f\|^2 \geqslant q\|S\|^2\|f\|^2$ which gives $1 \geqslant q$. The case of $q = 1$ is excluded by the well known result of Winter. ■

The case of $q < 0$. Here we get at once

COROLLARY 17. *For $q < 0$ the only bounded operator S with norm $\|S\| = (1-q)^{-1/2}$ satisfying $(\mathcal{O}_{q,\mathcal{D}})$ is that given by (10).*

Proof. By Proposition 16 (a) and Proposition 3 (b) $S^*|_{\mathcal{D}}$ is hyponormal. On the other hand, by Proposition 4 (b) and (c) $S|_{\mathcal{D}}$ is hyponormal too. Proposition 6 makes the conclusion. ■

Pauli matrices, which are neither hyponormal nor cohyponormal[5], provide an example of operators satisfying $(\mathcal{O}_{-1,\mathrm{op}})$ with norm $1 > 2^{-1/2} = (1-q)^{-1/2}$. Are there bounded operators satisfying $(\mathcal{O}_{q,\mathrm{op}})$ with norm not equal to $(1-q)^{-1/2}$ for arbitrary $q < 0$, different from -1 say?

---

[5] An operator A is said to be *cohyponormal* if A^* is hyponormal; for unbounded A this may not be the same as $A^*|_{\mathcal{D}(A)}$ being hyponormal.

The case of $0 \leqslant q < 1$. We list two results which hold in this case.

PROPOSITION 18. *Suppose* S *satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *with* $\mathcal{D}$ *dense in* $\mathcal{H}$. *If* $0 \leqslant q < 1$, *then the following facts are equivalent:*

(i) S *is bounded and* $\|S\| \leqslant (1-q)^{-1/2}$;
(ii) S *is bounded;*
(iii) S *is subnormal;*
(iv) S *is hyponormal.*

Proof. Because of conclusion (a) of Proposition 4 the only remaining implication to argue for is (ii)⇒(iii). But, in virtue of (13), this follows from the Halmos-Bram characterization [4] of subnormality of bounded operators. ■

THEOREM 19. *If* $0 \leqslant q < 1$, *then the following facts are equivalent:*

(i) *there is an orthonormal basis* $(e_n)_{n=0}^{\infty}$ *in* $\mathcal{H}$ *such that* $Se_n = \sqrt{[n+1]_q}\, e_{n+1}$, $n = 0, 1, \ldots$;
(ii) S *is irreducible*[6], *satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *with some* $\mathcal{D}$ *dense in* $\mathcal{H}$, *is bounded and* $\|S\| = (1-q)^{-1/2}$;
(iii) S *is irreducible, satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *with some* $\mathcal{D}$ *dense in* $\mathcal{H}$, *is bounded and* $\|S\| \leqslant (1-q)^{-1/2}$;
(iv) S *is irreducible, satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *with some* $\mathcal{D}$ *dense in* $\mathcal{H}$ *and is bounded;*
(v) S *is irreducible, satisfies with some* $\mathcal{D}$ *dense in* $\mathcal{H}$ $(\mathcal{O}_{q,\mathcal{D}})$ *and is subnormal;*
(vi) S *is irreducible, satisfies* $(\mathcal{O}_{q,\mathcal{D}})$ *with some* $\mathcal{D}$ *dense in* $\mathcal{H}$ *and is hyponormal.*

Proof. Proposition 18 establishes the equivalence of (ii) up to (vi).

Because $\sup\{\sqrt{[n+1]_q};\ n \geqslant 0\} = (1-q)^{-1}$ and because S is a weighted shift $\|S\| = \sup\{\sqrt{[n+1]_q};\ n \geqslant 0\}$, we get (i)⇒(ii).

Assume (iv). Because $\mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$, condition (c) of Lemma 15 let us calculate the weights of $\bar{S}$ starting with $e_0 \in \mathcal{N}(N^*)$. Because S is irreducible the sequence $(e_n)_{n=0}^{\infty}$ is complete. This establishes (i). ■

REMARK 20. From Theorem 19 and Example 5 we get that there are two, of different nature, solutions of $(\mathcal{O}_{q,\mathcal{D}})$. Are there any others?

The case of $q > 1$. No bounded solution exists, cf. Proposition 16(c).

Let us record what is known already in the bounded case in the following tableau.

[6] Let us recall relevant definitions: a subspace $\mathcal{D} \subset \mathcal{D}(A)$ is *invariant* for A if $A\mathcal{D} \subset \mathcal{D}$; $A|_{\mathcal{D}}$ stands for the restriction of A to $\mathcal{D}$. On the other hand, a *closed* subspace $\mathcal{L}$ is *invariant* for A if $A(\mathcal{L} \cap \mathcal{D}(A)) \subset \mathcal{D}(A)$; then the restriction $A{\restriction}_{\mathcal{L}} \stackrel{\text{def}}{=} A|_{\mathcal{L}\cap\mathcal{D}(A)}$. A step further, a closed subspace $\mathcal{L}$ *reduces* an operator A if both $\mathcal{L}$ and $\mathcal{L}^{\perp}$ are invariant for A as well as $P\mathcal{D}(A) \subset \mathcal{D}(A)$, where P is the orthogonal projection of $\tilde{\mathcal{H}}$ onto $\mathcal{L}$; all this is the same as to require $PA \subset AP$. Then the restriction $A{\restriction}_{\mathcal{L}}$ is called the *part* of A in $\mathcal{L}$. A is *irreducible* if it has no nontrivial reducing subspace. Compared to the more familiar case of bounded operators some nuances become requisite here. Therefore, if $\mathcal{L}$ reduces A, then $\overline{(A{\restriction}_{\mathcal{L}})} = \bar{A}{\restriction}_{\mathcal{L}}$ and $(A{\restriction}_{\mathcal{L}})^* = A^*{\restriction}_{\mathcal{L}}$.

		$q<0$	$0 \leqslant q < 1$	$1 \leqslant q$
normal	general	SOME Exa. 10	SOME Exa. 10	
	unilat. shift		SOME Th. 19	
subnormal	bilat. shift	NONE Exa. 7	NONE Exa. 7	
	others	SOME Exa. 5	SOME Exa. 5	NONE Prop. 16(a)
	unilat. shifts		SOME Th. 19	
hyponormal	bilat. shift	NONE Exa. 7	NONE Exa. 7	
	other	SOME Exa. 5	SOME Exa. 5	

The case of S unbounded

The case of $q < 0$. There is no hope to find subnormal solutions of $(\mathcal{O}_{q,\mathrm{op}})$ among weighted shifts, neither one- nor two-sided.

The only one-sided weighted shifts satisfying $(\mathcal{O}_{q,\mathrm{op}})$ are for $-1 < q < 0$ and they are given as in (i) of Theorem 19. They are apparently not hyponormal (their weights are not increasing).

The only two-sided weighted shifts which satisfy $(\mathcal{O}_{q,\mathrm{op}})$ are those of Example 7. They are normal bilateral weighted shifts. So if there are subnormal operators satisfying $(\mathcal{O}_{q,\mathrm{op}})$ they cannot be weighted shifts or bounded operators of norm less than or equal to $(1-q)^{-1/2}$, cf. Corollary 17.

The case of $0 \leqslant q < 1$. Lemma 10 does not leave any hope for subnormal solutions different than those in Theorem 19 but they must necessarily be bounded.

The case of $q \geqslant 1$. This is the right case for unbounded solutions to exist.

Theorem 21. *For a densely defined closable operator S in a complex Hilbert space $\mathcal{H}$ consider the following conditions:*

(i) *$\mathcal{H}$ is separable and has an orthonormal basis of the form $\{e_n\}_{n=0}^{\infty}$ contained in $\mathcal{D}(\bar{S})$ and such that*

$$\bar{S} e_n = \sqrt{[n+1]_q}\, e_{n+1}, \quad n = 0, 1, \ldots; \tag{21}$$

(ii) *S is irreducible, satisfies $(\mathcal{O}_{q,\mathcal{D}})$ with some $\mathcal{D}$ invariant for S and S^* and being a core of S, and S is a* **subnormal** *operator having a tight and $*$-tight normal extension;*

(iii) *S is irreducible, satisfies $(\mathcal{O}_{q,\mathcal{D}})$ with some $\mathcal{D}$ being a core of both S and S^*;*

(iv) *S is irreducible, satisfies $(\mathcal{O}_{q,\mathrm{w}})$ and $\mathcal{D}(\bar{S}) = \mathcal{D}(S^*)$;*

(v) *S is irreducible, satisfies $(\mathcal{O}_{q,\mathrm{w}})$ with $\mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$ dense in $\mathcal{H}$, $\mathcal{N}(S^*) \neq \{0\}$ and $\bar{S}^n(\mathcal{N}(S^*)) \subset \mathcal{D}(\bar{S}) \cap \mathcal{D}(S^*)$ for $n = 0, 1, \ldots$.*

Then (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i).

Proof. The implication (i)$\Rightarrow$(ii) follows from (20). Proposition 1 leads us from (ii) to (iii), and then Lemma 15 drives us up to (v). Now, like in the proof of Theorem 19, calculating the weights closes the chain of implications. ∎

Now we visualize this section's findings in the following table.

<table>
<tr><th></th><th></th><th>$q < 0$</th><th>$0 \leqslant q < 1$</th><th>$1 \leqslant q$</th></tr>
<tr><td>normal</td><td>general</td><td rowspan="7">NONE
Prop. 3(b)</td><td rowspan="7">NONE
Prop. 3(a)</td><td>NONE
Prop. 6</td></tr>
<tr><td rowspan="3">subnormal</td><td>unilat. shift</td><td></td></tr>
<tr><td>bilat. shift</td><td>NONE
Exa. 7</td></tr>
<tr><td>others</td><td></td></tr>
<tr><td rowspan="3">hyponormal</td><td>unilat. shifts</td><td></td></tr>
<tr><td>bilat. shift</td><td>NONE
Prop. 3(b)</td></tr>
<tr><td>others</td><td>MAY
Prop. 4(a)&(b)</td></tr>
</table>

The q-oscillator: models in RKHS

A general look. A reproducing kernel Hilbert space $\mathcal{H}$ and its kernel K which suits our considerations is of the form

$$K(z,w) \overset{\text{def}}{=} \sum_{n=0}^{+\infty} c_n z^n \overline{w}^n, \quad z, w \in D, \quad D = \mathbb{C} \text{ or } D = \{z;\ |z| < R \leqslant 1\}. \tag{22}$$

Notice $(\sqrt{c_n} Z^n)_{n=0}^{+\infty}$ is an orthonormal basis of $\mathcal{H}$.

The following fact is a byproduct of some general results on subnormality in [16]; we give here an *ad hoc* argument. Let us make a shorthand notation

$$\mathcal{H} \subset \mathcal{L}^2(\mathbb{C}, \mu) \text{ isometrically.} \tag{23}$$

PROPOSITION 22. *There is a measure μ such that* (23) *holds if and only if there is a Stieltjes moment sequence $(a_n)_{n=0}^{+\infty}$ such that*

$$a_{2n} = c_n^{-1}, \quad n = 0, 1, \ldots \tag{24}$$

If this happens then a measure μ can be chosen to be rotationally invariant[7]*, that is such that $\mu(\mathrm{e}^{\mathrm{i}\,t}\sigma) = \mu(\sigma)$ for all t's and σ's.*

[7] Or *radial* as some authors say.

Proof. Suppose (23) holds. Because $(\sqrt{c_n}Z^n)_{n=0}^{+\infty}$ is an orthonormal sequence in $\mathcal{L}^2(\mathbb{C},\mu)$, we have

$$c_n^{-1} = \int_{\mathbb{C}} |z|^{2n}\mu(\mathrm{d}z), \quad n = 0, 1, \dots$$

Let m_μ be the measure on $[0,+\infty)$ transported from μ via the mapping $\mathbb{C} \ni z \to |z| \in [0,+\infty)$. Then

$$a_n \stackrel{\text{def}}{=} \int_0^{+\infty} r^n m_\mu(\mathrm{d}r) = \int_{\mathbb{C}} |z|^n \mu(\mathrm{d}z), \quad n = 0, 1, \dots \tag{25}$$

satisfies (24) as well as the sequence $(a_n)_{n=0}^{+\infty}$ is a Stieltjes moment sequence.

If $(a_n)_{n=0}^{+\infty}$ is any Stieltjes moment sequence with a representing measure m and satisfying (24) then the rotationally invariant measure

$$\mu(\sigma) \stackrel{\text{def}}{=} (2\pi)^{-1} \int_0^{2\pi} \int_0^{+\infty} \chi_\sigma(r\,\mathrm{e}^{\mathrm{i}\,t}) m(\mathrm{d}r)\,\mathrm{d}t, \quad \sigma \text{ a Borel subset of } \mathbb{C}, \tag{26}$$

makes the imbedding (23) happen. ∎

THEOREM 23. *Under the circumstances of Proposition 22 there exists a non-rotationally invariant measure μ such that* (23) *holds if and only if there is a sequence $(a_n)_{n=0}^{+\infty}$ satisfying* (24) *which is not Stieltjes determinate.*

Proof. Suppose (23) with μ not rotationally invariant and define $(a_n)_{n=0}^{+\infty}$ as in (25). Thus there is and $s \in \mathbb{R}$ such that $\mu(\tau) \neq \mu(\mathrm{e}^{\mathrm{i}\,s}\,\tau)$ for some subset τ of $\mathbb{C}$; make τ maximal closed with respect to this property. Let ν be a measure on $\mathbb{C}$ transported from μ via the rotation $z \to \mathrm{e}^{-\mathrm{i}\,s}\,z$ and let m_ν be the measure on $[0,+\infty)$ constructed from ν in the way m_μ was from μ, cf. (25). Because, by a straightforward calculation, m_μ and m_ν differ on $\{|z|;\ z \in \tau\}$, we get indeterminacy of $(a_n)_{n=0}^{+\infty}$ at once.

The other way around, if m_1 and m_2 are two different measures on $[0,+\infty)$ representing the Stieltjes moment sequence $(a_n)_{n=0}^{+\infty}$ satisfying (24), then the measure μ on $\mathbb{C}$ defined by

$$\mu(\sigma) \stackrel{\text{def}}{=} (2\pi)^{-1}(s\int_0^a \mathrm{d}t \int_0^{+\infty} \chi_\sigma(r\,\mathrm{e}^{\mathrm{i}\,t}) m_1(\mathrm{d}r) + (1-s)\int_a^{2\pi} \mathrm{d}t \int_0^{+\infty} \chi_\sigma(r\,\mathrm{e}^{\mathrm{i}\,t})(s m_2(\mathrm{d}r),$$
$$\sigma \text{ a Borel subset of } \mathbb{C},\ 0 < s < 1,\ 0 < a < 2\pi$$

is not rotationally invariant while still (23) is maintained. ∎

Résumé. Define two linear operators M and D_q acting on functions

$$(Mf)(z) \stackrel{\text{def}}{=} zf(z), \quad (D_q f)(z) \stackrel{\text{def}}{=} \begin{cases} \frac{f(z)-f(qz)}{z-qz} & \text{if } q \neq 1, \\ f'(z) & \text{if } q = 1. \end{cases} \tag{27}$$

It turns out that for $a_+ = M$ and $a_- = D_q$ the commutation relation (1) is always satisfied. What Bargmann did in [3] was to find, for $q = 1$, a Hilbert space of entire functions such that M and D_1 are formally adjoint. This for arbitrary $q > 0$ leads to the reproducing kernel Hilbert space $\mathcal{H}_q$ of analytic functions with the kernel

$$K(z,w) \stackrel{\text{def}}{=} e_q((1-q)z\bar{w}), \quad z, w \in |1-q|^{-1/2}\omega_q,$$

where

$$\omega_q = \begin{cases} \{z;\ |z|<1\} & \text{if } 0<q<1, \\ \mathbb{C} & \text{if } q>1. \end{cases}$$

Under these circumstances we always have

$$\langle Z^m, Z^n \rangle_{\mathcal{H}_q} = \delta_{m,n}[m]_q!$$

and the operator $S = M$ acts as a weighted shift with the weights $(\sqrt{[n+1]_q})$ as in Sample Theorem on p. 294.

Our keynote, subnormality of M now means precisely that (23) with some μ is retained. Here we have three qualitatively different situations:

(a) for $0 < q < 1$ the multiplication operator M is bounded and subnormal, this implies uniqueness of μ;

(b) for $q = 1$ the multiplication operator is unbounded and subnormal, it has a normal extension of cyclic type in the sense of [17] and consequently μ is uniquely determined as well;

(c) for $q > 1$ the multiplication operator is unbounded and subnormal, it has no normal extension of cyclic type in the sense of [17] though it does have plenty of those of spectral type in the sense of [17], which are not unitarily equivalent [8]; an explicit example, based on [2], can be found in [18] (one has to replace q by q^{-1} there to get the commutation relation (1) satisfied), an explicit example of a non-radially invariant measure μ given in [9] also results from Theorem 23.

The author's afterword. The fundamentals of this paper have been presented on several occasions for the last couple of years, recently at the Będlewo 9th Workshop *Noncommutative Harmonic Analysis with Applications to Probability.* It was Marek Bożejko's contagious enthusiasm that catalysed converting at long last my loose notes into a cohesive exposition.

References

[1] M. Arik and D. D. Coon, *Hilbert spaces of analytic functions and generalized coherent states*, J. Math. Phys. 17 (1976), 524–527.

[2] R. Askey, *Ramanujan's extension of the gamma and beta functions*, Amer. Math. Monthly 87 (1980), 346–359.

[3] V. Bargmann, *On a Hilbert space of analytic functions and an associated integral transform*, Comm. Pure Appl. Math. 14 (1961), 187–214.

[4] J. Bram, *Subnormal operators*, Duke Math. J. 22 (1955), 75–94.

[5] M. Chaichian, H. Grosse, and P. Presnajder, *Unitary representations of the q-oscillator algebra*, J. Phys. A: Math. Gen. 27 (1994), 2045–2051.

[6] J. Cigler, *Operatormethoden für q-Identitäten*, Mh. Math. 88 (1979), 87–105.

[7] J. Dixmier, *Sur la relation* $\mathrm{i}(PQ-QP)=1$, Compositio Math. 13 (1958), 263–269.

[8] That is, there is no unitary map between the $\mathcal{L}^2$ spaces in question, which is the identity on $\mathcal{H}_q$.

[8] A. U. Klimyk and W. Schempp, *Classical and quantum Heisenberg groups, their representations and applications*, Acta Appl. Math. 45 (1996), 143–194.

[9] I. Królak, *Measures connected with Bargmann's representation of the q-commutation relation for* $q > 1$, in: Quantum Probability, Banach Center Publications 43, Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1998, 253–257.

[10] K-D. Kürsten and E. Wagner, *Invariant integration theory on non-compact quantum spaces: Quantum* $(n, 1)$*-matrix ball*, arXiv:QA/0305380v1.

[11] M. Möller and F. H. Szafraniec, *Adjoints and formal adjoints of matrices of unbounded operators*, Proc. Amer. Math. Soc., to appear.

[12] F. Rellich, *Der Eindeutigkeitssatz für die Lösungen der quantunmechanischen Vertauschungsrelationen*, Nachrichten Akad. Wiss. Göttingen, Math.-Phys. Klasse 1946, 107–115.

[13] S. Ôta, *Some classes of q-deformed operators*, J. Operator Th. 48 (2002), 151–186.

[14] K. Schmüdgen and E. Wagner, *Hilbert space representations of cross product algebras II*, Algebr. Represent. Theor. 9 (2006), 431–464.

[15] J. Stochel and F. H. Szafraniec, *A few assorted questions about unbounded subnormal operators*, Univ. Iagel. Acta Math. 28 (1991), 163–170.

[16] J. Stochel and F. H. Szafraniec, *On normal extensions of unbounded operators. II*, Acta Sci. Math. (Szeged) 53 (1989), 153–177.

[17] J. Stochel and F. H. Szafraniec, *On normal extensions of unbounded operators. III. Spectral properties*, Publ. RIMS Kyoto Univ. 25 (1989), 105–139.

[18] F. H. Szafraniec, *A RKHS of entire functions and its multiplication operator. An explicit example*, Operator Theory: Advances and Applications 43 (1990), 309–312.

[19] F. H. Szafraniec, *Subnormality in the quantum harmonic oscillator*, Commun. Math. Phys. 210 (2000), 323–334.

[20] F. H. Szafraniec, *On normal extensions of unbounded operators. IV. A matrix construction*, Oper. Th. Adv. Appl. 163 (2005), 337–350.

[21] F. H. Szafraniec, *How to recognize the creation operator*, Rep. Math. Phys. 59 (2007), 401–408.