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THE LÉVY–KHINTCHINE FORMULA AND NICA–SPEICHER PROPERTY FOR DEFORMATIONS OF THE FREE CONVOLUTION

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Abstract. We study deformations of the free convolution arising via invertible transformations of probability measures on the real line $T : \mu \mapsto T\mu$. We define new associative convolutions of measures by

$$\mu \boxplus_T \nu = T^{-1}(T\,\mu \boxplus T\,\nu).$$

We discuss infinite divisibility with respect to these convolutions, and we establish a Lévy–Khintchine formula. We conclude the paper by proving that for any such deformation of free probability all probability measures μ have the Nica–Speicher property, that is, one can find their convolution power $\mu^{\boxplus_T s}$ for all $s \geq 1$. This behaviour is similar to the free case, as in the original paper of Nica and Speicher [NS].

1. Introduction. For every invertible transformation of probability measures on the real line $T : \mu \mapsto T\mu$, we are able to define a new associative convolution of measures by the requirement

$$\mu \boxplus_T \nu = T^{-1}(T\,\mu \boxplus T\,\nu).$$

In this paper we establish a Lévy–Khintchine formula for the deformed free convolution. The method of proof is similar to the case of t-deformation, see [Wo1] and [Wo2]. We also show that any invertible deformation of the free convolution has the Nica–Speicher

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property, that is, for any number $s \geq 1$ there exists a probability measure μ_s such that $\mu_s = \mu^{\boxplus_T s}$.

In Section 2 we recall the notion of the free convolution of probability measures on the real line together with the moment–cumulant formula for the free convolution.

In Section 3 we define deformations of the free convolution.

In Section 4 we prove an analogue of the free Lévy–Khintchine formula for the deformed free convolution. Namely, we show that a probability measure μ on $\mathbb{R}$ is $\boxplus_T$-infinitely divisible if and only if there exists an $\alpha \in \mathbb{R}$ and a finite positive measure ρ such that for all $z \in \mathbb{C}^+$

$$\varphi_\mu^{\boxplus_T}(z) = \alpha + \int_{-\infty}^{\infty} \frac{1 + z\,x}{z - x}\, d\rho(x),$$

where $\varphi_\mu^{\boxplus_T}(z)$ is the T-deformed counterpart of the $\varphi_\mu^{\boxplus}(z)$-transform of Voiculescu. Both transforms linearize the respective convolutions.

In Section 5 we prove that for any deformation of free probability that we consider all probability measures μ have the Nica–Speicher property.

2. Free convolution of measures. A noncommutative probability space is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital complex $\star$-algebra and φ a linear positive functional such that $\varphi(1) = 1$. A noncommutative random variable is simply an element $X \in \mathcal{A}$. In the sequel we will be interested by self-adjoint random variables $X = X^\star$. By the distribution of X we understand the moments $\varphi(X^n)$, $n = 0, 1, \ldots$. Since the sequence of moments is positive definite, there exists a measure $\mu \in \mathrm{Prob}(\mathbb{R})$ such that $\varphi(X^n) = \int x^n d\mu(x)$, moreover, if $\mathcal{A}$ is a $C^\star$-algebra, the measure μ has compact support, hence is uniquely determined by its moments. A family of subalgebras $\mathcal{A}_i \subset \mathcal{A}$ is called free if

$$\varphi(a_1\, a_2 \cdots a_n) = \varphi(a_1)\, \varphi(a_2) \cdots \varphi(a_n) = 0$$
$$\text{whenever} \qquad \varphi(a_j) = 0, \quad a_j \in \mathcal{A}_{i_j}, \quad j = 1, 2, \ldots, n \quad \text{and} \quad i_1 \neq i_2 \neq \cdots \neq i_n. \tag{1}$$

Two random variables are called free if they belong to two distinct free subalgebras. For measures μ and ν with compact support, their free convolution $\mu \boxplus \nu$ is defined as the distribution of $X + Y \in \mathcal{A}$ where $X, Y \in \mathcal{A}$ are free and have distributions μ and ν respectively. To this concept there corresponds the notion of the free product of noncommutative probability spaces. Given $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ we define $\mathcal{A} = \mathcal{A}_1 \star \mathcal{A}_2$ as the free product with amalgamation of units, that is, the $\star$-algebra generated by the unit and words of the form $a_1^{i_1} b_1^{j_1} \ldots a_n^{i_n} b_n^{j_n}$ where $a_k \in \mathcal{A}_1$, $b_k \in \mathcal{A}_2$, $k, i_k, j_k \in \mathbb{N}$, $i_k, j_k > 0$, $i_1, j_n \geq 0$. The state $\varphi = \varphi_1 \star \varphi_2$ is defined so as to satisfy the relation (1). Then we have $\varphi|_{\mathcal{A}_i} = \varphi_i$, the algebras $\mathcal{A}_i$ naturally embedded into $\mathcal{A}$ are free and if $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$ then $m_{\mu_X \boxplus \mu_Y}(n) = \varphi((X + Y)^n)$.

Since the measure $\mu \boxplus \nu$ depends only on the measures μ and ν, it is essential to be able to describe it only in terms of μ and ν. This is done with the use of the R-transforms $R_\mu^{\boxplus}(z)$, $R_\nu^{\boxplus}(z)$, the analogues of the logarithm of the Fourier transform in classical probability. If we define

$$R_\mu^{\boxplus}(z) = G_\mu^{-1}(z) - \frac{1}{z}, \tag{2}$$

where $G_\mu^{-1}(z)$ is the right inverse of the Cauchy transform of the measure μ with respect to composition of functions, we have

$$R^{\boxplus}_{\mu\boxplus\nu}(z) = R^{\boxplus}_{\mu}(z) + R^{\boxplus}_{\nu}(z). \tag{3}$$

Recall that the Cauchy transform of a probability measure μ for $z \in \mathbb{C}^+ = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ is defined as follows:

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{d\mu(x)}{z-x}.$$

$G_\mu^{-1}(z)$ and $R^{\boxplus}_{\nu}(z)$ are well defined in some neighbourhood of zero. We can thus write equation (2) in an alternative form

$$G_\mu(z) = \frac{1}{z - R^{\boxplus}_{\mu}(G_\mu(z))}. \tag{4}$$

Moreover, since $R^{\boxplus}_{\mu}(z)$ is analytic, it can be written as the series $\sum_{k=0}^{\infty} R^{\boxplus}_{\mu}(k+1)\, z^k$. The coefficients $R^{\boxplus}_{\mu}(k)$ can be calculated by the results of Speicher [Sp] from the combinatorial moment–cumulant formula

$$m_\mu(n) = \sum_{\substack{\pi\in \mathrm{NC}(n)\\ \pi=(\pi_1,\ldots,\pi_k)}} \prod_{i=1}^{k} R^{\boxplus}_{\mu}(|\pi_i|), \tag{5}$$

where $\mathrm{NC}(n)$ is the set of noncrossing partitions of the ordered set $\{1,2,\ldots,n\}$, $\pi_i,\ i = 1,\ldots,k$ are blocks of the partition π and $|\pi_i|$ is the cardinality of the block. Equivalently, one can start with reciprocals instead of Cauchy transforms

$$F_\mu(z) = \frac{1}{G_\mu(z)}$$

and define

$$\varphi^{\boxplus}_{\mu}(z) = F_\mu^{-1}(z) - z$$

getting a similar linearity relation

$$\varphi^{\boxplus}_{\mu\boxplus\nu}(z) = \varphi^{\boxplus}_{\mu}(z) + \varphi^{\boxplus}_{\nu}(z).$$

Moreover, this approach extends to measures with unbounded support and with infinite moments, one only has to find an appropriate domain for z. This has been done by Maassen [Ma] for the case of measures with finite variance and by Bercovici and Voiculescu without this assumption in [BV2]. Bercovici and Voiculescu prove that for any probability measure $\mu \in \mathrm{Prob}(\mathbb{R})$ and any $\alpha > 0$ there exists $\beta > 0$ such that the function $\varphi_\mu(z)$ is analytic in a domain of the form

$$\{z : |z| > \beta, \operatorname{Im}(z) > 0, \operatorname{Re}(z) < \alpha \operatorname{Im}(z)\}$$

and that such an analytic function determines a corresponding probability measure. Since the sum of two such functions is again analytic in such a truncated angle for β big enough, the corresponding measure is determined.

3. Deformations of free convolution. For every invertible transformation $T : \mu_i \mapsto T\mu_i$, we are able to define a new deformation of the free convolution of measures on the real line by

$$\mu \boxplus_T \nu = T^{-1}(T\mu \boxplus T\nu)$$

and the convolution $\boxplus_T$ is associative, since $\boxplus$ is.

For such a deformation of free convolution we can define cumulants, deformed R-transforms and deformed φ-functions:

Proposition 1. *For any positive integer n and $z \in \mathbb{C}^+$ define*

$$R_\mu^{\boxplus_T}(n) := R_{T\mu}^{\boxplus}(n),$$
$$R_\mu^{\boxplus_T}(z) := R_{T\mu}^{\boxplus}(z),$$
$$\varphi_\mu^{\boxplus_T}(z) := \varphi_{T\mu}^{\boxplus}(z).$$

Then

$$R_{\mu\boxplus_T\nu}^{\boxplus_T}(z) = R_\mu^{\boxplus_T}(z) + R_\nu^{\boxplus_T}(z),$$
$$\varphi_{\mu\boxplus_T\nu}^{\boxplus_T}(z) = \varphi_\mu^{\boxplus_T}(z) + \varphi_\nu^{\boxplus_T}(z).$$

4. Infinite divisibility for deformed free convolution

Definition 1. We say that a probability measure μ is *infinitely divisible with respect to a convolution* $\star$ if for every $N \in \mathbb{N}$ there exists a measure μ_N such that $\mu = \mu_N^{\star N}$.

This can be rewritten equivalently in terms of $R^\star$-transforms: a probability measure μ is infinitely divisible with respect to a convolution $\star$ if for every $N \in \mathbb{N}$ there exists a measure μ_N such that $R_\mu^\star(z) = N\, R_{\mu_N}^\star(z)$.

In the case of the deformed free convolution $\boxplus_T$ we prefer to use the $\varphi_\mu^{\boxplus_T}(z)$ transform, since it allows to treat measures with unbounded support. We have an analogue of the free Lévy–Khintchine formula:

Theorem 1. *A measure μ is $\boxplus_T$-infinitely divisible if and only if there exist $\alpha \in \mathbb{R}$ and a finite positive measure ρ such that for all $z \in \mathbb{C}^+$*

$$\varphi_\mu^{\boxplus_T}(z) = \alpha + \int_{-\infty}^{\infty} \frac{1 + z\,x}{z - x}\, d\rho(x).$$

Proof. In the case of our convolution a measure μ is $\boxplus_T$-infinitely divisible if for every $N \in \mathbb{N}$ there exists a measure μ_N such that on some domain

$$\varphi_\mu^{\boxplus_T}(z) = N \cdot \varphi_{\mu_N}^{\boxplus_T}(z).$$

By a double application of the definition of the $\varphi^{\boxplus_T}$-transform to the left and right hand side of the above equation we get

$$\varphi_{T\mu}^{\boxplus}(z) = \varphi_\mu^{\boxplus_T}(z) = N \cdot \varphi_{\mu_N}^{\boxplus_T}(z) = N \cdot \varphi_{T\mu_N}^{\boxplus}(z),$$

hence, the measure μ is $\boxplus_T$-infinitely divisible if and only if

$$\varphi_{T\mu}^{\boxplus}(z) = N \cdot \varphi_{T\mu_N}^{\boxplus}(z),$$

which is equivalent to $\boxplus$-infinite divisibility of $T\mu$.

By the free Lévy–Khintchine formula (see [Ma],[BV1]) we know that the measure $T\mu$ is freely infinitely divisible if and only if there exist $\tilde{\alpha} \in \mathbb{R}$ and a positive measure $\tilde{\rho}$ such that for all $z \in \mathbb{C}^+$

$$\varphi_{T\mu}^{\boxplus}(z) = \tilde{\alpha} + \int_{-\infty}^{\infty} \frac{1 + z\,x}{z - x}\, d\tilde{\rho}(x).$$

Hence

$$\varphi_{\mu}^{\boxplus_T}(z) = \varphi_{T\mu}^{\boxplus}(z) = \alpha + \int_{-\infty}^{\infty} \frac{1 + z\,x}{z - x}\, d\rho(x),$$

where $\alpha = \tilde{\alpha}$ and $\rho = \tilde{\rho}$. ■

5. Nica–Speicher theorem for the deformation of free probability. By a result of Nica and Speicher [NS], we know that for any probability measure μ, possibly not infinitely divisible, and any number $s \geq 1$ there exists a probability measure μ^s such that $\mu^s = \mu^{\boxplus s}$, which is understood as

$$s\,\varphi_{\mu}^{\boxplus}(z) = \varphi_{\mu^s}^{\boxplus}(z).$$

For the deformation of the free convolution we also have the following

THEOREM 2. *For an arbitrary probability measure* $\mu \in \mathrm{Prob}(\mathbb{R})$ *there exists a measure* $\mu^s \in \mathrm{Prob}(\mathbb{R})$ *such that*

$$\mu^s = \mu^{\boxplus_T s} \quad \textit{for} \quad s \geq 1.$$

Proof. By definition

$$\varphi_{\mu}^{\boxplus_T}(z) = \varphi_{T\,\mu}^{\boxplus}(z) = \varphi_{\nu}^{\boxplus}(z),$$

where $T\,\mu = \nu \in \mathrm{Prob}(\mathbb{R})$. By the Nica and Speicher theorem for all $s \geq 1$ there exist measures ν^s such that $\nu^s = \nu^{\boxplus s}$, which gives

$$s\,\varphi_{\nu}^{\boxplus}(z) = \varphi_{\nu^s}^{\boxplus}(z) = \varphi_{T^{-1}\nu^s}^{\boxplus_T}(z).$$

Hence

$$s\,\varphi_{\mu}^{\boxplus_T}(z) = \varphi_{T^{-1}\nu^s}^{\boxplus_T}(z).$$

We have therefore found a family of measures $T^{-1}\nu^s$ which has the required property

$$\left(T^{-1}\nu^s\right) = \mu^{\boxplus_T s} \quad \text{for} \quad s \geq 1.$$

It is not possible to prove this theorem for $0 < s < 1$, since that would imply that the free convolution version would also hold for $0 < s < 1$, which is known to be false. ■

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