FIXED POINT THEORY AND ITS APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 77
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2007

NONLINEAR CONTRACTIVE CONDITIONS: A COMPARISON AND RELATED PROBLEMS

JACEK JACHYMSKI

Institute of Mathematics, Technical University of Łódź
Wólczańska 215, 93-005 Łódź, Poland
E-mail: jachym@p.lodz.pl

IZABELA JÓŹWIK

Institute of Mathematics and Centre of Mathematics and Physics
Technical University of Łódź
al. Politechniki 11, 90-924 Łódź, Poland
E-mail: ijozwik@p.lodz.pl

Abstract. We establish five theorems giving lists of nonlinear contractive conditions which turn out to be mutually equivalent. We derive them from some general lemmas concerning subsets of the plane which may be applied both in the single- or set-valued case as well as for a family of mappings. A separation theorem for concave functions is proved as an auxiliary result. Also, we discuss briefly the following problems for several classes of contractions: stability of procedure of successive approximations, existence of approximate fixed points, continuous dependence of fixed points on parameters, existence of invariant sets for iterated function systems. Moreover, James Dugundji's contribution to the metric fixed point theory is presented. Using his notion of contractions, we also establish an extension of a domain invariance theorem for contractive fields.

1. Introduction. Let (X, d) be a complete metric space and T be a selfmap of X. We say that x_* in X is a *contractive fixed point* (abbr. CFP) of T if $x_* = Tx_*$ and the Picard iterates $T^n x$ converge to x_* as $n \to \infty$ for all $x \in X$. There are numerous results in the literature giving sufficient conditions for the existence of a CFP. It seems, however, that the Banach Principle is still the most important here for its simplicity and an amazing efficiency in applications. Nevertheless, in this paper we wish to give a detailed analysis of

2000 *Mathematics Subject Classification*: Primary: 47H10, 54H25; Secondary: 26A15, 26A18, 26A48, 26A51, 54C10.

Key words and phrases: Nonlinear contractive conditions, φ-contractive map, contractive fixed point, approximate fixed point, domain invariance theorem, separation theorem, semi-continuous function, monotonic function, concave function, subadditive function.

The paper is in final form and no version of it will be published elsewhere.

several contractive conditions; in particular, our main purpose is to illuminate connections between them. Part of this study has already been done in our articles [Ja97P] and [Ja99], and we shall partially use this material in Sections 3 and 4. In general, the rest of the results given in Sections 2, 5 and 6 seem to be new.

As far as we know, the first significant generalization of Banach's Principle was obtained by Rakotch [Ra62] in 1962 (the problem was suggested by H. Hanani). We say that T is a *Rakotch contraction* if there is a decreasing function $\alpha : \mathbb{R}_+ \to [0,1]$ such that $\alpha(t) < 1$ for all $t > 0$, and

$$d(Tx,Ty) \le \alpha(d(x,y))d(x,y) \text{ for all } x,y \in X. \tag{1}$$

(Here $\mathbb{R}_+$ denotes the set of all non-negative reals; in the sequel we use 'decreasing' for 'non-increasing' and 'increasing' for 'non-decreasing'.) Then each Rakotch contraction has a CFP.

Subsequently, in 1968 Browder [Br68] introduced a more general definition. Given a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(t) < t \text{ for all } t > 0, \tag{2}$$

we say that T is *φ-contractive* if

$$d(Tx,Ty) \le \varphi(d(x,y)) \text{ for all } x,y \in X. \tag{3}$$

Then we call T a *Browder contraction* if φ is increasing and right continuous. Such a T has a CFP. (Actually, that was shown by Browder [Br68] under the additional assumption that (X,d) be bounded.) This result, in turn, was extended by Boyd and Wong [BW69] in 1969 who observed that it was enough to assume only the right upper semi-continuity of φ, i.e.,

$$\limsup_{s\to t^+} \varphi(s) \le \varphi(t) \text{ for all } t \in \mathbb{R}_+.$$

(Moreover, the boundedness of (X,d) is superfluous.) Then T is said to be a *Boyd–Wong contraction.*

Another natural generalization of Banach's Principle was given in [KV72] in 1972. We say that T is a *Krasnosel'skiĭ contraction* if given $a,b \in \mathbb{R}_+$ with $0 < a < b$, there is an $L(a,b) \in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty) \le L(a,b)d(x,y) \text{ if } a \le d(x,y) \le b. \tag{4}$$

However, we have shown in [Ja97P] that this definition is equivalent to the one of Browder. In particular, this corrects a remark in [GD03, p. 16].

Further two conditions were proposed by Geraghty in 1973 [Ge73] and 1974 [Ge74]. T is said to be a *Geraghty* (I) *contraction* if it satisfies (1) with a function $\alpha : \mathbb{R}_+ \to [0,1]$ having the property that given a sequence $(t_n)_{n\in\mathbb{N}}$,

$$\alpha(t_n) \to 1 \text{ implies } t_n \to 0. \tag{5}$$

This definition was slightly modified in [Ge74]—the above property of α was replaced there by the following: Given a sequence $(t_n)_{n\in\mathbb{N}}$,

$$\text{if } (t_n)_{n\in\mathbb{N}} \text{ is decreasing and } \alpha(t_n) \to 1, \text{ then } t_n \to 0. \tag{6}$$

Then we call T a *Geraghty* (II) *contraction*. It turns out that this class of mappings coincides with the Boyd–Wong class as shown by Hegedüs and Szilágyi [HS80]. On the other hand, in Section 2 we shall show that T is a Geraghty (I) contraction if and only if it is a Rakotch contraction. Also, we shall consider in Section 3 the following variant of the above conditions: T is a *Geraghty* (III) *contraction* if it satisfies (1) with a function $\alpha : \mathbb{R}_+ \to [0,1)$ such that given a sequence $(t_n)_{n\in\mathbb{N}}$,

$$\text{if } (t_n)_{n\in\mathbb{N}} \text{ is bounded and } \alpha(t_n) \to 1, \text{ then } t_n \to 0. \tag{7}$$

We shall show this definition is equivalent to the one of Browder.

Another variant of Browder's condition was given by Matkowski [Ma75] in 1975 who replaced the continuity assumption on φ by the condition:

$$\lim_{n\to\infty} \varphi^n(t) = 0 \text{ for all } t > 0. \tag{8}$$

Then T is said to be a *Matkowski contraction*. Though this class of mappings is essentially wider than Browder's class (see Proposition 2 and Example 2), in some sense both these conditions are equivalent. More precisely, in [Ja99] we have shown that if T is a Matkowski contraction, then the second iterate T^2 satisfies Browder's condition. Hence, since by Browder's theorem, T^2 has a CFP, so does T.

In 1976 James Dugundji [Du76] established a very general coincidence theorem from which he derived a fixed point theorem for the following class of mappings. We say T is a *Dugundji contraction* if the function $(x, y) \mapsto d(x, y) - d(Tx, Ty)$ is *positive definite* mod $\Delta(X)$, the diagonal in $X \times X$, i.e., given an $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$,

$$d(x, y) - d(Tx, Ty) < \delta \text{ implies } d(x, y) < \varepsilon. \tag{9}$$

As shown in [Du76], Dugundji's theorem yields Browder's result under the assumption that (X, d) be bounded. However, in Section 5 we present that without the boundedness assumption, the class of Browder's contractions is essentially wider than the one of Dugundji. Nevertheless, we also show that each Dugundji contraction enjoys a nice property: It has a fixed point which is both contractive and approximate. Recall that a mapping T has an *approximate fixed point* (abbr. AFP) x_* if $x_* = Tx_*$ and given a sequence $(x_n)_{n\in\mathbb{N}}$,

$$d(x_n, Tx_n) \to 0 \text{ implies } x_n \to x_*. \tag{10}$$

(Let us note that this definition of an AFP is different from that given in [MR03].) On the other hand, it is not clear if each Browder contraction has such a property. Also, in Section 5 we give an extension of a domain invariance theorem for Dugundji contractive fields (see Theorem 10). It seems to be interesting here that the domain of the mapping need not be open, and the normed linear space need not be complete whereas these additional assumptions were used by Dugundji and Granas [DG78] in the proof of their domain invariance theorem for Browder contractions (also see [GD03, p. 11] for the case of Banach contractions). We close Section 5 with a theorem on continuous dependence of fixed points on parameters for a family of Dugundji contractions (see Theorem 11).

Yet another contractive definition was given by Dugundji and Granas [DG78] in 1978. A mapping T is said to be a *Dugundji–Granas contraction* if

$$d(Tx,Ty) \le d(x,y) - \Theta(x,y) \text{ for all } x,y \in X, \tag{11}$$

where the function $\Theta : X \times X \to \mathbb{R}_+$ is *compactly positive* on X, i.e., given $a,b \in \mathbb{R}_+$ such that $0 < a < b$,

$$\inf\{\Theta(x,y) : a \le d(x,y) \le b\} > 0.$$

As observed in [DG78], the above definition is equivalent to that of Krasnosel'skiĭ [KV72]. A domain invariance theorem for such mappings was also given in [KV72].

In Section 6 we study the class of φ-contractive maps with an increasing and continuous φ satisfying the following limit condition

$$\lim_{t\to\infty}(t-\varphi(t)) = \infty \tag{12}$$

which was introduced by Matkowski [Ma77] in 1977; independently, it also appeared in Walter's [Wa81] paper in 1981. Such contractive maps are said to be *Matkowski–Walter contractions*. (Incidentally, both authors considered more general conditions than (3).) This class of φ-contractions is of some importance in the theory of iterated function systems (see Theorem 13) and the asymptotic fixed point theory.

We close our shortened history of studies on nonlinear contractive conditions with recalling the following definition given by Burton [Bu96] in 1996. We say that T is a *large contraction* if given an $a > 0$, there is an $L(a) \in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty) \le L(a)d(x,y) \text{ if } d(x,y) \ge a. \tag{13}$$

Somewhat unexpectedly, we are coming back here to the beginning point since Burton's condition turns out to be equivalent to the one of Rakotch (see Theorem 1).

Finally, we would like to emphasize that our study is far from being comprehensive since there is a huge number of papers dealing with contractive type conditions. Many useful references concerning this topic may be found, e.g., in the books [KS01, Chapter 1] and [RPP02].

Throughout this paper we use the convention that $\inf\emptyset = \infty$ and—since we are working on the half-line $\mathbb{R}_+$—$\sup\emptyset = 0$.

2. Rakotch contractions. We begin with the following auxiliary result concerning a subset of the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$. (The idea of such an approach goes back to the paper by Hegedüs and Szilágyi [HS80].) Given a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, set

$$E_\varphi := \{(t,u) \in \mathbb{R}_+ \times \mathbb{R}_+ : u \le \varphi(t)\}.$$

LEMMA 1. *Let D be a subset of $\mathbb{R}_+ \times \mathbb{R}_+$ such that for any $u \in \mathbb{R}_+$,*

$$(0,u) \in D \textit{ implies } u = 0. \tag{14}$$

The following statements are equivalent:

(i) *there is a decreasing function $\alpha : \mathbb{R}_+ \to [0,1]$ such that $\alpha(t) < 1$ for all $t > 0$, and $D \subseteq E_\varphi$, where $\varphi(t) := t\alpha(t)$ $(t \in \mathbb{R}_+)$;*

(ii) *given an* $a > 0$, *there is an* $L \in [0, 1)$ *such that* $(t, u) \in D$ *and* $t \geq a$ *imply* $u \leq Lt$, *i.e.*,

$$\sup\left\{\frac{u}{t} : (t, u) \in D \textit{ and } t \geq a\right\} < 1;$$

(iii) *there is a function* $\alpha : \mathbb{R}_+ \to [0, 1)$ *such that* (5) *holds and* $D \subseteq E_\varphi$, *where* $\varphi(t) := t\alpha(t)$ $(t \in \mathbb{R}_+)$;

(iv) *there is a strictly increasing and concave function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *and such that* $D \subseteq E_\varphi$;

(v) *there is a subadditive function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *and such that* $D \subseteq E_\varphi$;

(vi) *there is a subadditive function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *which is continuous at* 0 *and such that* $\varphi(t_n) < t_n$ *for some sequence* $(t_n)_{n\in\mathbb{N}}$ *convergent to* 0, *and* $D \subseteq E_\varphi$;

(vii) *there is an increasing and right continuous function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *such that* $\limsup_{t\to\infty} \varphi(t)/t < 1$ *and* $D \subseteq E_\varphi$;

(viii) *there is an upper semi-continuous function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *such that* $\limsup_{t\to\infty} \varphi(t)/t < 1$ *and* $D \subseteq E_\varphi$.

The proof of Lemma 1 will be preceded by the following separation theorem for concave functions which extends Matkowski's result [Ma93, Lemma 1]. Moreover, our proof seems to be simpler.

LEMMA 2. *Assume that functions* $\varphi, \lambda : \mathbb{R}_+ \to \mathbb{R}_+$ *are such that* $\varphi(t) < \lambda(t)$ *for all* $t > 0$, *and* λ *is concave and strictly increasing. If given an* $s > 0$,

$$\sup\left\{\frac{\varphi(t)}{\lambda(t)} : t \geq s\right\} < 1,$$

then there is a concave and strictly increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\varphi(t) < \psi(t) < \lambda(t) \textit{ for all } t > 0.$$

Proof. Set

$$\Phi := \{\eta : \mathbb{R}_+ \to \mathbb{R}_+ \mid \eta \text{ is concave, increasing and } \varphi(t) \leq \eta(t) \text{ for } t > 0\}.$$

Clearly, $\Phi \neq \emptyset$ since $\lambda \in \Phi$. Define

$$\pi(0) := 0 \text{ and } \pi(t) := \inf\{\eta(t) : \eta \in \Phi\} \text{ for } t > 0.$$

Then $\varphi(t) \leq \pi(t)$ for $t > 0$. It is easily seen that since every $\eta \in \Phi$ is increasing, so is π. Moreover, since the infimum of any family of concave functions which are equibounded from below is concave, we infer $\pi|_{(0,\infty)}$ is concave. Hence, also π is concave since π is non-negative and $\pi(0) = 0$. We show $\pi(t) < \lambda(t)$ for $t > 0$. Let $s > 0$. By hypothesis, there is an $\alpha \in (0, 1)$ such that

$$\varphi(t) \leq \alpha\lambda(t) \text{ for } t \geq s$$

which implies $\varphi(t) \leq \alpha\lambda(t) + (1 - \alpha)\lambda(s)$ for $t > 0$. Indeed, this holds for $t \geq s$ since $\lambda(s) > 0$; if $0 < t < s$, then by hypothesis,

$$\varphi(t) < \lambda(t) = \alpha\lambda(t) + (1 - \alpha)\lambda(t) < \alpha\lambda(t) + (1 - \alpha)\lambda(s).$$

Set

$$\mu_s(t) := \alpha\lambda(t) + (1 - \alpha)\lambda(s) \text{ for } t \in \mathbb{R}_+.$$

It is clear that $\mu_s \in \Phi$, so $\pi(t) \le \mu_s(t)$ for $t > 0$. In particular, if $t > s$, then by the strict monotonicity of λ,

$$\pi(t) \le \alpha\lambda(t) + (1-\alpha)\lambda(s) < \alpha\lambda(t) + (1-\alpha)\lambda(t) = \lambda(t).$$

Since s was any positive number, we infer $\pi(t) < \lambda(t)$ for all $t > 0$. Now it suffices to set $\psi(t) := (\lambda(t) + \pi(t))/2$ for $t \in \mathbb{R}_+$. ■

Remark 1. Lemma 2 was proved in [Ma93] in case when the function λ is linear.

Proof of Lemma 1. (i)⇒(ii). Let $a > 0$. Set $L := \alpha(a)$. By (i), $L \in [0, 1)$. Let $(t, u) \in D$ and $t \ge a$. By (i),

$$u \le t\alpha(t) \le t\alpha(a) = Lt,$$

so (ii) holds.

(ii)⇒(iii). Set

$$\alpha(0) := 0 \text{ and } \alpha(t) := \sup\left\{\frac{u}{t} : (t,u) \in D\right\} \text{ for } t > 0.$$

By (ii), $\alpha : \mathbb{R}_+ \to [0, 1)$. Assume that $(t_n)_{n\in\mathbb{N}}$ is such that $\alpha(t_n) \to 1$. Suppose, on the contrary, that $t_n \not\to 0$. Without loss of generality we may assume $t_n \ge \delta$ for some $\delta > 0$ and all $n \in \mathbb{N}$. By (ii), there is an $L(\delta) \in [0, 1)$ such that $(t_n, u) \in D$ implies $u \le L(\delta)t_n$. Hence we infer

$$\alpha(t_n) \le L(\delta) < 1$$

which yields a contradiction since $\alpha(t_n) \to 1$. Thus $t_n \to 0$. Moreover, by the definition of α, $D \cap (0,\infty)^2 \subseteq E_\varphi$. Hence and by (14), $D \subseteq E_\varphi$.

(iii)⇒(iv). Let φ be as in (iii), so that $\varphi(t)/t = \alpha(t)$ for $t > 0$. Hence and by (iii), given an $s > 0$,

$$\sup\{\varphi(t)/t : t \ge s\} < 1.$$

(Otherwise, there is a $(t_n)_{n\in\mathbb{N}}$ such that $t_n \ge s$ and $\alpha(t_n) \to 1$, a contradiction.) By Lemma 2 (with $\lambda(t) := t$ for $t \in \mathbb{R}_+$), there is a strictly increasing and concave function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) < \psi(t) < t$ for $t > 0$. Since $D \subseteq E_\varphi$ and $E_\varphi \subseteq E_\psi$, we get $D \subseteq E_\psi$, so ψ is a desirable function.

(iv)⇒(v). Let φ be as in (iv). Then, by [HP57, Theorem 7.2.5], φ is also subadditive, so (v) holds.

(v)⇒(vi). It is easily seen that for any $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, φ is subadditive if its restriction $\varphi|_{(0,\infty)}$ is subadditive. Hence, if φ is as in (v), then we may assume without loss of generality that $\varphi(0) = 0$. This implies the continuity of φ at 0 since $\varphi(t) < t$ for $t > 0$. Moreover, by (14), $D \subseteq E_\varphi$ with such a modified φ. Thus (vi) holds.

(vi)⇒(vii). Let φ be as in (vi). By [Ma93, Corollary 1], φ satisfies (2). Hence and by [Ma93, Remark 1], we have for any $s > 0$,

$$\sup\left\{\frac{\varphi(t)}{t} : t \ge s\right\} < 1. \tag{15}$$

By Lemma 2, there is an increasing and concave function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) < \psi(t) < t$ for $t > 0$. By concavity, ψ is continuous on $(0,\infty)$. By monotonicity, $\psi(0) \le \psi(t) < t$ for $t > 0$, and hence ψ is also continuous at 0. Since ψ is non-negative and concave, [HP57, Theorem 7.2.5] implies ψ is subadditive. Now [HP57, Theorem 7.6.2] yields the existence of the limit $\lim_{t\to\infty}\psi(t)/t$ and, moreover,

$$\lim_{t\to\infty}\frac{\psi(t)}{t}=\inf\left\{\frac{\psi(t)}{t}:t>0\right\}\le\psi(1)<1.$$

Finally, since $D\subseteq E_\varphi\subseteq E_\psi$, we infer ψ is a desirable function for (vii).

(vii)⇒(viii) is obvious since each increasing function is left upper semi-continuous.

(viii)⇒(i). Let φ be as in (viii). We show given an $s>0$, (15) holds. Suppose, on the contrary, this supremum is equal to 1. There is a $(t_n)_{n\in\mathbb{N}}$ such that $t_n\ge s$ and $\varphi(t_n)/t_n\to 1$. Then $(t_n)_{n\in\mathbb{N}}$ is bounded; otherwise, we would get $\limsup_{t\to\infty}\varphi(t)/t=1$, a contradiction. Thus, without loss of generality, we may assume $(t_n)_{n\in\mathbb{N}}$ converges to some $t_0\ge s$. By the upper semi-continuity,

$$1=\lim_{n\to\infty}\frac{\varphi(t_n)}{t_n}\le\limsup_{t\to t_0}\frac{\varphi(t)}{t}\le\frac{\varphi(t_0)}{t_0}<1$$

which yields a contradiction. Now Lemma 2 ensures the existence of a concave function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ such that $\varphi(t)<\psi(t)<t$ for $t>0$. Set

$$\alpha(0):=1 \text{ and } \alpha(t):=\psi(t)/t \text{ for } t>0.$$

Since ψ is concave, we infer α is decreasing (cf. [HP57, proof of Theorem 7.2.5]). Moreover, $t\alpha(t)=\psi(t)$ for $t\in\mathbb{R}_+$, and $D\subseteq E_\psi$, so (i) holds. ∎

THEOREM 1. *Let (X,d) be a metric space and T be a selfmap of X. The following statements are equivalent*:

(i) *T is a Rakotch contraction* (*cf.* (1));

(ii) *T is Burton's large contraction* (*cf.* (13));

(iii) *T is a Geraghty* (I) *contraction* (*cf.* (5));

(iv) *T is φ-contractive, where φ is strictly increasing and concave*;

(v) *T is φ-contractive, where φ is subadditive*;

(vi) *T is φ-contractive* (*here* (2) *is not required*), *where φ is subadditive and continuous at* 0, *and such that $\varphi(t_n)<t_n$ for some sequence $(t_n)_{n\in\mathbb{N}}$ convergent to* 0;

(vii) *T is φ-contractive, where φ is increasing and right continuous, and such that* $\limsup_{t\to\infty}\varphi(t)/t<1$;

(viii) *T is φ-contractive, where φ is upper semi-continuous and* $\limsup_{t\to\infty}\varphi(t)/t<1$.

Proof. Set $D:=\{(d(x,y),d(Tx,Ty)):x,y\in X\}$. It is clear that D satisfies (14), so it suffices to apply Lemma 1. ∎

REMARK 2. The implication (ii)⇒(i) was essentially used by Reich and Zaslavski [RZ00] though they were unaware of the existence of the paper [Bu96].

Below we inform the reader about other possible applications of Lemma 1.

REMARK 3. If T is a set-valued mapping with values in $\mathrm{CB}(X)$, the family of all nonempty closed and bounded subsets of X endowed with the Hausdorff metric, then applying Lemma 1 with D defined by

$$D:=\{(d(x,y),H(Tx,Ty)):x,y\in X\}$$

we obtain a counterpart of Theorem 1 for set-valued mappings. (In particular, such a class of contractive set-valued mappings was studied by Reich [Re72] and Reich–Zaslavski [RZ02].) As another application of Lemma 1, we may set (for a single-valued T)

$$D := \{(M(x,y), d(Tx,Ty)) : x, y \in X\},$$

where $M(x,y) := \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$ to get the result on the equivalence of other contractive conditions studied in the literature.

REMARK 4. It follows from Theorem 1 ((ii)⇔(vii)) that a theorem of Chen [Ch00, Theorem 3.2] on pointwise convergence of a sequence $(T_n \circ \ldots \circ T_1)_{n\in\mathbb{N}}$ and a result of Jachymski [Ja97A, Theorem 5] are identical. Moreover, applying Lemma 1 ((i)⇔(ii)) with D defined as

$$D := \{(d(x,y), d(T_n x, T_n y)) : x, y \in X,\ n \in \mathbb{N}\},$$

we infer that the assumptions of another result of Chen [Ch00, Theorem 3.3] mean all maps T_n are Rakotch contractions with a function α independent of n.

In the rest of this section we wish to quote two results dealing with Rakotch contractions. The first one concerns the problem of stability of iteration procedure. Recall that the procedure of successive approximations for a mapping T having a CFP is *stable* (see Ostrowski [Os67]) if for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive reals convergent to 0, every sequence $(y_n)_{n=0}^{\infty}$ such that

$$d(y_n, Ty_{n-1}) \le \varepsilon_n \text{ for } n \in \mathbb{N}$$

converges to the fixed point of T. (The reals ε_n can be interpreted as the 'round-off errors' of $T^n y_0$.) Ostrowski [Os67] proved that for any Banach contraction, the procedure of successive approximations is stable. This result was extended in [Ja97A] in the following way.

THEOREM 2. *Let (X,d) be a complete metric space and T be a φ-contractive selfmap of X such that* $\limsup_{t\to\infty} \varphi(t)/t < 1$, *and φ is increasing and right continuous. Then the procedure of successive approximations for T is stable.*

Now it follows from Theorem 1 ((i)⇔(vii)) that Theorem 2 deals in fact with the class of Rakotch contractions. It remains an open question whether this result can be extended to whichever class considered in the next sections. Recently, a partial answer to this question was given by Butnariu, Reich and Zaslavski [BRZ06, Corollary 1.1].

We also wish to mention a result related to generic aspects of metric fixed point theory in which the class of Rakotch contractions plays an important role. Let X be a Banach space, and K be a non-empty closed bounded convex subset of X. Denote by $\mathcal{A}$ the set of all non-expansive selfmaps of K, i.e.,

$$\|Ax - Ay\| \le \|x - y\| \text{ for all } A \in \mathcal{A} \text{ and } x, y \in K.$$

Endow $\mathcal{A}$ with the metric h defined by

$$h(A,B) := \sup\{\|Ax - Bx\| : x \in K\} \text{ for } A, B \in \mathcal{A}.$$

Then $(\mathcal{A}, h)$ is a complete metric space. In 1976 De Blasi and Myjak [DM76] proved that most of mappings in $\mathcal{A}$—in the sense of Baire's category—have a CFP. Recently, their result was improved by Reich and Zaslavski [RZ00] (see also [KS01, p. 561]) who obtained the following

THEOREM 3 (Reich–Zaslavski). *There exists a set* $\mathfrak{F}$ *which is a countable intersection of open everywhere dense sets in* $\mathcal{A}$ *such that each* $A \in \mathfrak{F}$ *is a Rakotch contraction.*

In particular, a typical non-expansive mapping is a Rakotch contraction, and hence it has a CFP. On the other hand, in a Hilbert space the set of all Banach contractions is only of the first category as shown by De Blasi and Myjak [DM76]. Finally, let us note that Theorem 3 has been strengthened in [RZ01C] by using the concept of sigma-porosity.

3. Browder contractions. We again begin with a result concerning a subset of the quadrant $\mathbb{R}_+ \times \mathbb{R}_+$.

LEMMA 3. *Let* D *and* E_φ *be as in Lemma* 1. *The following statements are equivalent*:

(i) *there is an increasing and right continuous function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *such that* $D \subseteq E_\varphi$;

(ii) *given* $a, b > 0$ *with* $a < b$, *there is an* $L \in [0, 1)$ *such that* $(t, u) \in D$ *and* $a \le t \le b$ *imply* $u \le Lt$, *i.e.*,

$$\sup\left\{\frac{u}{t} : (t, u) \in D \text{ and } t \in [a, b]\right\} < 1;$$

(iii) *given* $a, b > 0$ *with* $a < b$, *there is a* $\delta > 0$ *such that* $(t, u) \in D$ *and* $a \le t \le b$ *imply* $u \le t - \delta$, *i.e.*,

$$\inf\{t - u : (t, u) \in D \text{ and } t \in [a, b]\} > 0;$$

(iv) *there is a strictly increasing and continuous function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *such that* $D \subseteq E_\varphi$;

(v) *there is a continuous function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *such that* $D \subseteq E_\varphi$;

(vi) *there is a continuous function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\psi(t) > 0$ *for* $t > 0$ *and* $D \subseteq E_\varphi$, *where* $\varphi(t) := t - \psi(t)$ $(t \in \mathbb{R}_+)$;

(vii) *there is an upper semi-continuous function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *such that* $D \subseteq E_\varphi$;

(viii) *there is a function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfying* (2) *such that*

$$\limsup_{s\to t} \varphi(s) < t \text{ for all } t > 0,$$

and $D \subseteq E_\varphi$;

(ix) *there is a strictly increasing function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* (8) *holds, and* $D \subseteq E_\varphi$;

(x) *there is a function* $\alpha : \mathbb{R}_+ \to [0, 1)$ *such that* (7) *holds for any sequence* $(t_n)_{n\in\mathbb{N}}$, *and* $D \subseteq E_\varphi$, *where* $\varphi(t) := t\alpha(t)$ $(t \in \mathbb{R}_+)$.

Proof. (i)⇒(vii) follows immediately from the fact that each increasing function is left upper semi-continuous.

(vii)⇒(viii). If φ is as in (vii), then given $t > 0$,

$$\limsup_{s\to t} \varphi(s) \le \varphi(t) < t,$$

so (viii) holds.

(viii)⇒(iv) follows directly from a separation lemma [Ja97P, Lemma 1].

(iv)⇒(i) is obvious.

(We have shown (i), (iv), (vii) and (viii) are equivalent.)

(iv)⇒(ix). If φ is as in (iv), then (8) holds (see [Br68]), so (ix) holds.

(ix)⇒(viii). If φ is as in (ix), then φ satisfies (2) (see [GD03, p. 15]), so by the monotonicity, given $t > 0$,

$$\lim_{s\to t^-} \varphi(s) \le \varphi(t) < t.$$

Moreover, $\lim_{s\to t^+} \varphi(s) < t$; otherwise, $\lim_{s\to t^+} \varphi(s) = t$ and then, by [Ja97P, Theorem 7], there is a $\delta > 0$ such that $\varphi|_{(t,t+\delta)}$ is constant which contradicts the strict monotonicity of φ. Thus $\lim_{s\to t} \varphi(s) < t$, so (viii) holds.

(iv) implies (v) *a fortiori*. To show (v)⇒(vi), it suffices to set $\psi(t) := t-\varphi(t)$ $(t \in \mathbb{R}_+)$. Similarly, to prove (vi)⇒(vii), it is enough to set $\varphi(t) := t - \psi(t)$ for $t \in \mathbb{R}_+$.

(We have shown (i), (iv), (v), (vi), (vii), (viii) and (ix) are equivalent. Thus it suffices to prove: (v)⇒(ii)⇒(iii)⇒(viii) and (v)⇒(x)⇒(viii).)

(v)⇒(ii). Let $0 < a < b$, $(t, u) \in D$ and $t \in [a, b]$. By (v),

$$\frac{u}{t} \le \frac{\varphi(t)}{t} \le \sup\left\{\frac{\varphi(s)}{s} : s \in [a, b]\right\} = \frac{\varphi(s_0)}{s_0}$$

for some $s_0 \in [a, b]$, because $s \mapsto \varphi(s)/s$ $(s \in [a, b])$ attains its maximum in view of the continuity of φ. It suffices to set $L := \varphi(s_0)/s_0$. Indeed, $u \le Lt$ and by (2), $L \in [0, 1)$.

(ii)⇒(iii). Let $0 < a < b$ and L be as in (ii). Set $\delta := a(1 - L)$. Clearly, $\delta > 0$ and given $(t, u) \in D$ with $t \in [a, b]$, we have

$$u \le Lt = t - (1 - L)t \le t - (1 - L)a = t - \delta.$$

Thus (iii) holds.

(iii)⇒(viii). By (iii), there is a sequence $(\delta_n)_{n\in\mathbb{N}}$ of positive reals such that if $(t, u) \in D$ and $t \in [1/n, n+1]$, then $u \le t-\delta_n$. Without loss of generality, we may assume $\delta_n \le 1/n$. Set

$$\varphi(0) := 0,\ \varphi(t) := t - \delta_1 \text{ for } t \in [1, 2],$$
$$\varphi(t) := t - \delta_n \text{ for } t \in [1/n, 1/(n - 1)) \cup (n, n + 1] \text{ and } n \ge 2.$$

It is easily seen that φ has all the properties we need, so (viii) holds. (Here we also use condition (14).)

(v)⇒(x). Let φ be as in (v). Set

$$\alpha(0) := 0 \text{ and } \alpha(t) := \frac{\varphi(t)}{t} \text{ for } t > 0.$$

Let $(t_n)_{n\in\mathbb{N}}$ be bounded and such that $\alpha(t_n) \to 1$. Suppose, on the contrary, that $t_n \not\to 0$. Without loss of generality, we may assume there are $a, b > 0$ such that $a \le t_n \le b$ for all $n \in \mathbb{N}$, and $(t_n)_{n\in\mathbb{N}}$ converges to some $t_0 \in [a, b]$. Then, by continuity of φ, we have

$$1 = \lim_{n\to\infty} \alpha(t_n) = \lim_{n\to\infty} \frac{\varphi(t_n)}{t_n} = \frac{\varphi(t_0)}{t_0} < 1$$

which yields a contradiction. Thus $t_n \to 0$, so (x) holds.

(x)⇒(viii). Let α be as in (x). Set $\varphi(t) := t\alpha(t)$ for $t \in \mathbb{R}_+$. Clearly, φ satisfies (2). Let $t > 0$. Suppose, on the contrary, $\limsup_{s\to t} \varphi(s) = t$. There is an $(s_n)_{n\in\mathbb{N}}$ such that $s_n \to t$ and $\varphi(s_n) \to t$. Hence $\alpha(s_n) = \varphi(s_n)/s_n \to 1$, so by hypothesis, $s_n \to 0$ which yields a contradiction. ∎

THEOREM 4. *Let (X, d) be a metric space and T be a selfmap of X. The following statements are equivalent*:

(i) *T is a Browder contraction* (*cf.* (3));

(ii) *T is a Krasnosel'skiĭ contraction* (*cf.* (4));

(iii) *T is a Dugundji–Granas contraction* (*cf.* (11));

(iv) *T is φ-contractive, where φ is strictly increasing and continuous*;

(v) *T is φ-contractive, where φ is continuous*;

(vi) *there is a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) > 0$ for $t > 0$, and for all $x, y \in X$,*

$$d(Tx, Ty) \le d(x, y) - \psi(d(x, y));$$

(vii) *T is φ-contractive, where φ is upper semi-continuous*;

(viii) *T is φ-contractive, where φ is such that* $\limsup_{s\to t} \varphi(s) < t$ *for* $t > 0$;

(ix) *T is φ-contractive, where φ is strictly increasing and such that* (8) *holds*;

(x) *T is a Geraghty* (III) *contraction* (*cf.* (7)).

Proof. Set $D := \{(d(x, y), d(Tx, Ty) : x, y \in X\}$ and apply Lemma 3. For example, to show that (ii) implies (iii), define $\Theta(x, y) := d(x, y) - d(Tx, Ty)$, and then use Lemma 3 ((ii)⇒(iii)). ∎

REMARK 5. Condition (vi) is taken from the book of Krasnosel'skiĭ et al. [KV72, p. 52]. (vii) is a stronger version of the Boyd–Wong [BW69] condition in which φ is *right* upper semi-continuous, however, these two versions are not equivalent (see Proposition 3). Similarly, (ix) is a variant of Matkowski's [Ma75] condition in which φ need not be *strictly* monotonic. (viii) was given by Tasković [Ta78].

Theorem 1 ((i)⇒(vii)) implies that every Rakotch contraction is a Browder contraction. The following result illuminates more relations between these classes. In particular, they do not coincide.

PROPOSITION 1. *Assume that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and right continuous, and satisfies* (2). *The following statements are equivalent*:

(i) $\limsup_{t\to\infty} \varphi(t)/t < 1$;

(ii) *given a metric space (X, d) and a φ-contractive map $T : X \to X$, T is a Rakotch contraction.*

Proof. (i)⇒(ii) follows from Theorem 1 ((vii)⇒(i)).

(ii)⇒(i). Define $X := \mathbb{R}_+$ and set for any $x, y \in X$,

$$d(x, y) := \max\{x, y\} \text{ if } x \ne y;\ d(x, x) := 0.$$

Then (X,d) is an ultrametric space. Further, set $T := \varphi$. By monotonicity of φ, we have for $x, y \in X$ with $x \neq y$,

$$d(Tx,Ty) \le \max\{\varphi(x),\varphi(y)\} = \varphi(\max\{x,y\}) = \varphi(d(x,y)),$$

so T is φ-contractive. By hypothesis, there is a decreasing function $\alpha : \mathbb{R}_+ \to [0,1]$ such that $\alpha(t) < 1$ for $t > 0$, and (1) holds. Setting in (1) $y := 0$, we get

$$\varphi(x) \le \alpha(x)x \text{ for } x \in \mathbb{R}_+$$

which implies $\limsup_{t\to\infty} \varphi(t)/t \le \lim_{t\to\infty} \alpha(t) < 1$ because of monotonicity of α. ∎

Example 1. Set $\varphi(t) := t^2/(t+1)$ for $t \in \mathbb{R}_+$. Then Proposition 1 yields the existence of a φ-contractive map in Browder's class which is not a Rakotch contraction.

With the help of the same trick as in the proof of Proposition 1 (a use of an ultrametric space), the following result was obtained in [Ja97P].

Proposition 2. *Assume that* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing and satisfies* (8). *The following statements are equivalent*:

(i) $\{t > 0 : \lim_{s\to t^+} \varphi(s) = t\} = \emptyset$;

(ii) *given a metric space* (X,d) *and a* φ*-contractive map* $T : X \to X$, T *is a Browder contraction.*

Hence and by Theorem 4 ((i)⇒(ix)), the class of Browder contractions is a proper subclass of Matkowski's class. Indeed, consider the following

Example 2. Define $\varphi(0) := 0$, $\varphi(t) := 1/(n+1)$ for $t \in (1/(n+1), 1/n]$ and $n \in \mathbb{N}$, $\varphi(t) := 1$ for $t > 1$. Then by Proposition 2, there is a φ-contractive map in Matkowski's class which is not a Browder contraction.

Finally, we would like to quote two interesting theorems of Dugundji and Granas [DG78] for Browder contractions. The first one is the following domain invariance theorem for weakly contractive fields. The letter I denotes the identity map.

Theorem 5 (Dugundji–Granas). *If* X *is a Banach space,* $U \subseteq X$ *is open and* $T : U \to X$ *is a Browder contraction, then the field* $F := I - T$ *is a homeomorphism from* U *onto* $F(U)$.

The second result concerns the problem of continuous dependence of fixed points on parameters.

Theorem 6 (Dugundji–Granas). *Let* (X,d) *be a complete metric space and* (Λ,ρ) *be a metric space. Let* $T : X \times \Lambda \to X$ *be a map such that* $\lambda \mapsto T(x,\lambda)$ *is continuous for all* $x \in X$, *and*

$$d(T(x,\lambda),T(y,\lambda)) \le d(x,y) - \Theta(x,y) \text{ for } x,y \in X \text{ and } \lambda \in \Lambda,$$

where Θ *(independent of* λ*) is compactly positive. Let* x_λ *be the unique fixed point of* T_λ, *where* $T_\lambda x := T(x,\lambda)$. *Then the map* $\lambda \mapsto x_\lambda$ *is continuous at each point where it is locally bounded.*

The last property means the restriction of the map to some neighbourhood of a point is bounded.

In Section 5 we give partial extensions of the above results for Dugundji contractions (see Theorems 10 and 11).

4. Boyd–Wong contractions. With the help of a result by Hegedüs and Szilágyi [HS80], we shall prove the following

LEMMA 4. *Let D and E_φ be as in Lemma* 1. *The following statements are equivalent*:

(i) *there is a right upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that $D \subseteq E_\varphi$*;

(ii) *given an $a > 0$, there are $b > a$ and $L \in [0, 1)$ such that $(t, u) \in D$ and $a \le t \le b$ imply $u \le Lt$*;

(iii) *given an $a > 0$, there are $b > a$ and $\delta > 0$ such that $(t, u) \in D$ and $a \le t \le b$ imply $u \le t - \delta$*;

(iv) *given an $a > 0$, there are $b > a$ and $\delta > 0$ such that $(t, u) \in D$ and $a \le t \le b$ imply $u \le a - \delta$*;

(v) *there is a right continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that $D \subseteq E_\varphi$*;

(vi) *there is a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that $\limsup_{s\to t^+} \varphi(s) < t$ for $t > 0$ and $D \subseteq E_\varphi$*;

(vii) *there is a function $\alpha : \mathbb{R}_+ \to [0, 1)$ such that for any sequence $(t_n)_{n\in\mathbb{N}}$*, (6) *holds, and $D \subseteq E_\varphi$, where $\varphi(t) := t\alpha(t)$ $(t \in \mathbb{R}_+)$*;

(viii) *there is a function $\alpha : \mathbb{R}_+ \to [0, 1)$ such that $\limsup_{s\to t^+} \alpha(s) < 1$ for $t > 0$, and $D \subseteq E_\varphi$, where $\varphi(t) := t\alpha(t)$ $(t \in \mathbb{R}_+)$.*

Proof. The equivalence of (i), (iv), (vi), (vii) and (viii) was shown in [HS80]. (v) implies (i) *a fortiori*. (i)⇒(v) follows from the separation theorem proved in [Ja97P] according to which if φ is as in (i), then there is a right continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(t) \le \psi(t) < t$ for $t > 0$. Thus it suffices to show (iv)⇒(ii)⇒(iii)⇒(iv).

(iv)⇒(ii). Let $a > 0$. There are $b > a$ and $\delta > 0$ as in (iv). Without loss of generality, we may assume $\delta \le a$. Set $L := (a - \delta)/a$. Clearly, $L \in [0, 1)$ and if $(t, u) \in D$ and $a \le t \le b$, then

$$u \le a - \delta = La \le Lt.$$

Thus (ii) holds.

(ii)⇒(iii). We use the same argument as in the proof of Lemma 3 ((ii)⇒(iii)).

(iii)⇒(iv). Let $a > 0$. There are $b > a$ and $\delta > 0$ as in (iii). Without loss of generality, we may assume $\delta \le 2(b - a)$. Set $\delta' := \delta/2$ and $b' := a + \delta'$. Clearly, $b' \le b$. Hence if $(t, u) \in D$ and $a \le t \le b'$, then $a \le t \le b$, so

$$u \le t - \delta \le b' - \delta = a - \delta/2 = a - \delta'.$$

Thus (iv) holds. ∎

THEOREM 7. *Let (X,d) be a metric space and T be a selfmap of X. The following statements are equivalent*:

(i) *T is a Boyd–Wong contraction*;

(ii) *given an $a>0$, there are $b>a$ and $L\in[0,1)$ such that for all $x,y\in X$,*
$$a\le d(x,y)\le b \text{ implies } d(Tx,Ty)\le Ld(x,y);$$

(iii) *given an $a>0$, there are $b>a$ and $\delta>0$ such that for all $x,y\in X$,*
$$a\le d(x,y)\le b \text{ implies } d(Tx,Ty)\le d(x,y)-\delta;$$

(iv) *given an $a>0$, there are $b>a$ and $\delta>0$ such that for all $x,y\in X$,*
$$a\le d(x,y)\le b \text{ implies } d(Tx,Ty)\le a-\delta;$$

(v) *T is φ-contractive, where φ is right continuous*;

(vi) *T is φ-contractive, where φ is such that* $\limsup_{s\to t^+}\varphi(s)<t$ *for* $t>0$;

(vii) *T is a Geraghty* (II) *contraction* (*cf.* (6));

(viii) *T satisfies* (1), *where $\alpha:\mathbb{R}_+\to[0,1)$ is such that* $\limsup_{s\to t^+}\alpha(s)<1$ *for* $t>0$.

Proof. Set
$$D:=\{(d(x,y),d(Tx,Ty)):x,y\in X\}$$
and apply Lemma 4. ■

REMARK 6. (iv) is a stronger version of the Meir–Keeler [MK69] condition in which the strict inequality $d(Tx,Ty)<a$ is substituted for $d(Tx,Ty)\le a-\delta$ in (iv). As shown in [MK69], these two conditions are not equivalent. (For a discussion of other variants of the Meir–Keeler condition, see [Ja95].) Nevertheless, it is possible to characterize the Meir–Keeler condition in terms of a function φ as recently done by Lim [Li01]. (v) was used in a fixed point theorem of Mukherjea [Mu77]; thus his result is equivalent to the one of Boyd and Wong [BW69].

REMARK 7. For a set-valued map $T:X\to \mathrm{CB}(X)$ (cf. Remark 3), (viii) appears in Reich's [Re74] problem: It is still unknown whether each such a multifunction has a fixed point. Applying Lemma 4 with D defined by
$$D:=\big\{\big(d(x,y),H(Tx,Ty)\big):x,y\in X\big\},$$
we get equivalent characterizations of set-valued maps considered by Reich.

The following result given in [Ja97P] explains relations between the classes of Browder and Boyd–Wong.

PROPOSITION 3. *Assume that $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is right upper semi-continuous and satisfies* (2). *The following statements are equivalent*:

(i) $\{t>0:\limsup_{s\to t^-}\varphi(s)=t\}=\emptyset$;

(ii) *given a metric space (X,d) and a φ-contractive map $T:X\to X$, T is a Browder contraction.*

Hence and by Theorem 4 ((i)⇒(vii)), the class of Browder contractions is a proper subclass of the Boyd–Wong class. In particular, consider the following

Example 3. Define $\varphi(t) := t^2$ for $t \in [0,1)$ and $\varphi(t) := t - 1/2$ for $t \geq 1$. Then Proposition 3 yields the existence of a φ-contractive map in the Boyd–Wong class which is not a Browder contraction.

On the other hand, the classes of Matkowski and Boyd–Wong are incomparable. To see that, consider first the following

Example 4. Let φ be as in Example 2, and define (X,d) and T as in the proof of Proposition 1. Then T is a Matkowski contraction. Suppose, on the contrary, that T is ψ-contractive, where ψ is right upper semi-continuous. In particular, $d(Tx, T0) \leq \psi(d(x,0))$ for $x \in X$, i.e., $\varphi(x) \leq \psi(x)$ for all $x \in \mathbb{R}_+$. Since $\lim_{x\to 1^+} \varphi(x) = 1$ and $\psi(x) < x$, we infer $\lim_{x\to 1^+} \psi(x) = 1$. On the other hand, by hypothesis,

$$\lim_{x\to 1^+} \psi(x) \leq \psi(1) < 1,$$

which yields a contradiction. Thus T is not a Boyd–Wong contraction.

Simultaneously, there exist Boyd–Wong contractions which do not satisfy Matkowski's condition. Instead of an example, we give here somewhat more general result which can easily be deduced from [Ja97P, Theorems 4 and 6].

Proposition 4. *Assume that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is right upper semi-continuous and satisfies* (2). *If*

$$\inf\{t > 0 : \limsup_{s\to t^-} \varphi(s) = t\} = 0,$$

then there is a φ-contractive map which is not a Matkowski contraction.

5. Dugundji contractions

Lemma 5. *Let D and E_φ be as in Lemma* 1. *The following statements are equivalent*:

(i) *given an $a > 0$, there is a $\delta > 0$ such that $(t,u) \in D$ and $t \geq a$ imply $u \leq t - \delta$, i.e.,*

$$\inf\big\{t - u : (t,u) \in D \text{ and } t \geq a\big\} > 0;$$

(ii) *there is a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that the function $t \mapsto t - \varphi(t)$ is increasing, and $D \subseteq E_\varphi$;*

(iii) *there is a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that for any sequence $(t_n)_{n\in\mathbb{N}}$, $t_n - \varphi(t_n) \to 0$ implies $t_n \to 0$, and $D \subseteq E_\varphi$;*

(iv) *there is a strictly increasing and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that $\lim_{t\to\infty}(t - \varphi(t)) > 0$, and $D \subseteq E_\varphi$;*

(v) *there is an increasing and right continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that $\lim_{t\to\infty}(t - \varphi(t)) > 0$, and $D \subseteq E_\varphi$;*

(vi) *there is an upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that $\liminf_{t\to\infty}(t - \varphi(t)) > 0$, and $D \subseteq E_\varphi$;*

(vii) *there is a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *such that $\limsup_{s\to t} \varphi(s) < t$ for $t > 0$, $\liminf_{t\to\infty}(t - \varphi(t)) > 0$, and $D \subseteq E_\varphi$.*

Proof. (i)⇒(ii). Set $\psi(0) := 0$ and

$$\psi(a) := \inf\{t - u : (t,u) \in D \text{ and } t \geq a\} \text{ for } a > 0.$$

By (i), we have $\psi(a) > 0$ for $a > 0$. Without loss of generality, we may assume D is unbounded; otherwise, we could replace D by the set $D \cup (\mathbb{R}_+ \times \{0\})$ which also satisfies (i). Then $\psi(a)$ is finite for all $a \in \mathbb{R}_+$. Clearly, ψ is increasing. Define

$$\varphi(t) := t - \psi(t) \text{ for } t \in \mathbb{R}_+.$$

Then φ satisfies (2) and $t \mapsto t - \varphi(t)$ $(= \psi(t))$ is increasing. Moreover, if $(t, u) \in D$ and $t > 0$, then $t - u \geq \psi(t)$, i.e., $u \leq \varphi(t)$. Hence and by (14), we get $D \subseteq E_\varphi$.

(ii)$\Rightarrow$(iii). Let φ be as in (ii), and let $t_n - \varphi(t_n) \to 0$. Suppose, on the contrary, $t_n \not\to 0$. Without loss of generality, we may assume $t_n \geq \delta$ for some $\delta > 0$. By hypothesis,

$$t_n - \varphi(t_n) \geq \delta - \varphi(\delta) > 0$$

which yields a contradiction.

(iii)$\Rightarrow$(iv). It is easily seen (iii) implies

$$\delta_n := \inf\{t - \varphi(t) : t \geq 1/n\} > 0 \text{ for } n \in \mathbb{N}.$$

Hence $\varphi(t) \leq t - \delta_n$ for $t \geq 1/n$. Define

$$\eta(0) := 0, \quad \eta(t) := t - \delta_{n+1} \text{ for } t \in [1/(n+1), 1/n) \text{ and } n \in \mathbb{N},$$
$$\text{and } \eta(t) := t - \delta_1 \text{ for } t \in [1, 2).$$

It is easily seen that $\limsup_{s\to t} \eta(s) < t$ for $t \in (0, 2)$, so by [Ja97P, Lemma 1], there is a strictly increasing and continuous $\gamma : [0, 2) \to \mathbb{R}_+$ such that $\eta(t) \leq \gamma(t) < t$ for $t \in (0, 2)$. Now we define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ setting

$$\psi(t) := \gamma(t) \text{ for } t \in [0, 1], \text{ and } \psi(t) := t - 1 + \gamma(1) \text{ for } t > 1.$$

Clearly, ψ is strictly increasing, continuous and satisfies (2). Since $\gamma(1) \geq \eta(1) = 1 - \delta_1$, we get $1 - \gamma(1) \leq \delta_1$ which yields

$$\psi(t) \geq t - \delta_1 \geq \varphi(t) \text{ for } t > 1.$$

For $t \in [0, 1]$, $\psi(t) = \gamma(t) \geq \eta(t) \geq \varphi(t)$. Thus $\varphi(t) \leq \psi(t)$ for all $t \in \mathbb{R}_+$ which gives $E_\varphi \subseteq E_\psi$. Since $D \subseteq E_\varphi$, we infer $D \subseteq E_\psi$.

Implications (iv)$\Rightarrow$(v)$\Rightarrow$(vi)$\Rightarrow$(vii) are obvious.

(vii)$\Rightarrow$(i). Suppose, on the contrary, $\inf\{t - u : (t, u) \in D,\ t \geq a\} = 0$. There are sequences $(t_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ and an $a > 0$ such that $(t_n, u_n) \in D$, $t_n \geq a$ and $t_n - u_n \to 0$. By (vii), $u_n \leq \varphi(t_n)$, so

$$0 < t_n - \varphi(t_n) \leq t_n - u_n$$

which yields $t_n - \varphi(t_n) \to 0$. Since by hypothesis, $\liminf_{t\to\infty}(t - \varphi(t)) > 0$, we infer $(t_n)_{n\in\mathbb{N}}$ is bounded. Thus, without loss of generality, we may assume $t_n \to t_0$ for some $t_0 \geq a$. Then $\varphi(t_n) \to t_0$ which yields $\limsup_{s\to t_0} \varphi(s) = t_0$, a contradiction. ∎

THEOREM 8. *Let (X, d) be a metric space and T be a selfmap of X. The following statements are equivalent*:

(i) *T is a Dugundji contraction*;

(ii) *given an $a > 0$, there is a $\delta > 0$ such that for any $x, y \in X$,*

$$d(x, y) \geq a \text{ implies } d(Tx, Ty) \leq d(x, y) - \delta;$$

(iii) *T is φ-contractive, where φ is such that $t \mapsto t - \varphi(t)$ ($t \in \mathbb{R}_+$) is increasing;*

(iv) *T is φ-contractive, where φ is such that for any sequence $(t_n)_{n\in\mathbb{N}}$, $t_n - \varphi(t_n) \to 0$ implies $t_n \to 0$;*

(v) *T is φ-contractive, where φ is strictly increasing, continuous and such that* $\lim_{t\to\infty}(t - \varphi(t)) > 0$;

(vi) *T is φ-contractive, where φ is increasing, right continuous and such that* $\lim_{t\to\infty}(t - \varphi(t)) > 0$;

(vii) *T is φ-contractive, where φ is upper semi-continuous and* $\liminf_{t\to\infty}(t-\varphi(t)) > 0$;

(viii) *T is φ-contractive, where φ is such that* $\limsup_{s\to t}\varphi(s) < t$ *for $t > 0$ and* $\liminf_{t\to\infty}(t - \varphi(t)) > 0$.

Proof. It is easily seen (i) and (ii) are equivalent. Next, define D as in the proof of Theorem 1 and apply Lemma 5. ■

REMARK 8. The limit condition $\liminf_{t\to\infty}(t-\varphi(t)) > 0$ was also used in some extension of the Banach theorem given in [JS99].

Now we give a counterpart of Proposition 1 to illuminate the relations between the classes of Dugundji and Browder.

PROPOSITION 5. *Assume that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, right continuous and satisfies* (2). *The following statements are equivalent*:

(i) $\liminf_{t\to\infty}(t - \varphi(t)) > 0$;

(ii) *given a metric space (X, d) and a φ-contractive map $T : X \to X$, T is a Dugundji contraction.*

Proof. (i)⇒(ii) follows from Theorem 8 ((vi)⇒(i)).

(ii)⇒(i). Define (X, d) and T as in the proof of Proposition 1. Then T is φ-contractive. By hypothesis and Theorem 8 ((i)⇒(v)), there is an increasing continuous function ψ such that $\lim_{t\to\infty}(t - \psi(t)) > 0$, and T is ψ-contractive. As in the proof of Proposition 1, we infer $\varphi(t) \le \psi(t)$ for $t \in \mathbb{R}_+$. Hence, $t - \varphi(t) \ge t - \psi(t)$ which implies (i). ■

EXAMPLE 5. Set $\varphi(t) := t/2$ for $t \in [0, 1]$ and $\varphi(t) := t - 1/(t + 1)$ for $t > 1$. Then Proposition 5 yields the existence of a φ-contractive map in Browder's class which is not a Dugundji contraction.

Our next purpose is to show that each Dugundji contraction has an AFP, or, equivalently, in the terminology of Reich and Zaslavski [RZ01], the fixed point problem for each Dugundji contraction is *well-posed*.

THEOREM 9. *Let (X, d) be a metric space and $T : X \to X$ be a Dugundji contraction. Then, given sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$,*

$$d(x_n, Tx_n) \to 0 \text{ and } d(y_n, Ty_n) \to 0 \text{ imply } d(x_n, y_n) \to 0.$$

In particular, if (X, d) is complete, then T has an AFP.

Proof. By Theorem 8 ((i)⇒(iv)), T is φ-contractive, where φ is such that for any $(t_n)_{n\in\mathbb{N}}$, $t_n - \varphi(t_n) \to 0$ implies $t_n \to 0$. Set $t_n := d(x_n, y_n)$. Then

$$t_n \le d(x_n, Tx_n) + d(Tx_n, Ty_n) + d(y_n, Ty_n) \le d(x_n, Tx_n) + \varphi(t_n) + d(y_n, Ty_n)$$

which implies $t_n - \varphi(t_n) \to 0$, so by the property of φ, $t_n \to 0$.

Now if (X, d) is complete, then by Dugundji's theorem, T has a CFP x_*. Applying the first part of Theorem 9 with $y_n := x_*$, we infer x_* is an AFP. ∎

Remark 9. Theorem 9 generalizes the result of Reich and Zaslavski [RZ01] who proved that each Rakotch contraction has an AFP. We do not know, however, if Theorem 9 could be extended by admitting maps from Browder's class.

Now we give a domain invariance theorem for Dugundji contractive fields which partially extends the result of Dugundji and Granas [DG78] in that a space need not be complete, and a domain need not be open. However, we slightly strengthen the assumption on a map substituting Dugundji's class for the class of Browder. (Compare Theorem 10 with Theorem 5.)

Theorem 10. *Let X be a normed linear space, and A be an arbitrary non-empty subset of X. Let $T : A \to X$ be a Dugundji contraction, and F be the field associated with T, i.e., $F = I - T$. Then $F : A \to F(A)$ is a homeomorphism; moreover, F and F^{-1} are uniformly continuous. If X is a Banach space, then we have:*

(a) *if A is closed, then $F(A)$ is closed in X;*

(b) *if A is open, then $F(A)$ is open in X.*

Proof. By Theorem 8 ((i)⇒(ii)), T is φ-contractive, where φ is such that $t \mapsto t - \varphi(t)$ is increasing. First we show $F : A \to F(A)$ is a bijection. Let $x, y \in A$ and $Fx = Fy$. Then $x - y = Tx - Ty$. Suppose, on the contrary, $x \neq y$. Then by (2), we have

$$\|x - y\| = \|Tx - Ty\| \leq \varphi(\|x - y\|) < \|x - y\|$$

which yields a contradiction.

Since T is uniformly continuous, so is F. Now we show $F^{-1} : F(A) \to A$ is uniformly continuous. Clearly, this is equivalent to the condition: Given an $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in A$, $\|Fx - Fy\| < \delta$ implies $\|x - y\| < \varepsilon$ which, in turn, is equivalent to the following one:

$$\inf\big\{\|Fx - Fy\| : x, y \in A,\ \|x - y\| \geq \varepsilon\big\} > 0. \tag{16}$$

Fix an $\varepsilon > 0$. Let $x, y \in A$ and $\|x - y\| \geq \varepsilon$. Then by hypothesis and the property of φ, we have

$$\|Fx - Fy\| = \|(x - y) - (Tx - Ty)\| \geq \|x - y\| - \|Tx - Ty\| \geq \\ \|x - y\| - \varphi(\|x - y\|) \geq \varepsilon - \varphi(\varepsilon)$$

which yields (16) since φ satisfies (2). Thus $F : A \to F(A)$ is a homeomorphism.

To prove (a), assume $(y_n)_{n\in\mathbb{N}}$ is such that $y_n \in F(A)$ and $y_n \to y$ for some $y \in X$. Then there is an $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in A$ and $y_n = Fx_n$. Since F^{-1} is uniformly continuous and $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so is $(x_n)_{n\in\mathbb{N}}$. By completeness of X and closeness of A, we get $x_n \to x$ for some $x \in A$. Since F is continuous, we have $Fx_n \to Fx$, i.e., $y_n \to Fx$, and hence $y = Fx$, so $y \in F(A)$.

Part (b) follows from the Dugundji–Granas [DG78] result since, in particular, T is a Browder contraction. ∎

A novelty here is the fact that with the help of Theorem 10, we may prove the following result by a simple topological argument instead of using a fixed point theorem as done in [GD03, p. 12, Corollary (2.2)] and [DG78].

COROLLARY 1. *Let X be a Banach space and $T : X \to X$ be a Dugundji contraction. Then the field F associated with T is a homeomorphism.*

Proof. By Theorem 10, $F : X \to F(X)$ is a homeomorphism, and $F(X)$ is both closed and open in X. Since X is connected, we infer $F(X) = X$. ■

The last result of this section is a theorem on continuous dependence of fixed points on parameters for a family of Dugundji contractions.

THEOREM 11. *Let (X, d) be a complete metric space, and (Λ, ρ) be a metric space. Let $T : X \times \Lambda \to X$ be a map, and set*

$$T_\lambda x := T(x, \lambda) \text{ for } x \in X \text{ and } \lambda \in \Lambda.$$

Assume that $\{T_\lambda : \lambda \in \Lambda\}$ is a family of Dugundji contractions which satisfy (9) *uniformly with respect to λ, i.e., given an $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x, y \in X$ and all $\lambda \in \Lambda$,*

$$d(x, y) - d(T_\lambda x, T_\lambda y) < \delta \text{ implies } d(x, y) < \varepsilon. \tag{17}$$

For $\lambda \in \Lambda$, let x_λ be the unique fixed point of T_λ. If each map $\lambda \mapsto T(x, \lambda)$ $(x \in X)$ is continuous, then $\lambda \mapsto x_\lambda$ is continuous on Λ.

Proof. Let $\lambda_0, \lambda \in \Lambda$. Then we have

$$d(x_\lambda, x_{\lambda_0}) = d(T_\lambda x_\lambda, T_{\lambda_0} x_{\lambda_0}) \le d(T_\lambda x_\lambda, T_\lambda x_{\lambda_0}) + d(T_\lambda x_{\lambda_0}, T_{\lambda_0} x_{\lambda_0}),$$

and hence,

$$d(x_\lambda, x_{\lambda_0}) - d(T_\lambda x_\lambda, T_\lambda x_{\lambda_0}) \le d(T(x_{\lambda_0}, \lambda), T(x_{\lambda_0}, \lambda_0)). \tag{18}$$

Let $\varepsilon > 0$. By hypothesis, there is a $\delta > 0$ such that (17) holds. By the continuity of $\lambda \mapsto T(x_{\lambda_0}, \lambda)$ at λ_0, we infer there is a neighbourhood U of λ_0 such that for all $\lambda \in U$,

$$d\big(T(x_{\lambda_0}, \lambda), T(x_{\lambda_0}, \lambda_0)\big) < \delta.$$

Now (18) and (17) imply $d(x_\lambda, x_{\lambda_0}) < \varepsilon$ for all $\lambda \in U$ which completes the proof. ■

REMARK 10. Theorem 11 partially extends Theorem 6 of Dugundji and Granas: We slightly strengthened the contractive condition, but thanks to that we could drop the assumption on local boundedness of the map $\lambda \mapsto x_\lambda$ which, in practice, is rather hardly verifiable. We do not know whether this condition is essential in Theorem 6.

REMARK 11. Applying Lemma 5 with D defined as

$$D := \big\{d(x, y), d\big(T(x, \lambda), T(y, \lambda)\big) : x, y \in X,\ \lambda \in \Lambda\big\},$$

we may get other equivalent characterizations of the family $\{T_\lambda : \lambda \in \Lambda\}$ from Theorem 11, e.g., the following one: All T_λ are φ-contractive (with φ independent of λ), where φ is upper semi-continuous and such that $\liminf_{t\to\infty}(t - \varphi(t)) > 0$. Then it is clear Theorem 11 yields a well-known parametrized version of the Banach theorem (see, e.g., [GD03, (A.4), p. 18]).

6. Matkowski–Walter contractions. As far as we know, condition (12) first appeared in the paper of Matkowski [Ma77] who proved that if a map T has a *contractive iterate at a point*, i.e., given an $x \in X$, there is an $n(x) \in \mathbb{N}$ such that

$$d(T^{n(x)}x, T^{n(x)}y) \leq \varphi(d(x,y)) \text{ for all } y \in X,$$

where φ is increasing and satisfies (8) and (12), then T has a CFP. Subsequently, Benedykt and Matkowski [BM81] showed (12) was essential in the above result. Independently, Walter [Wa81] studied the class of maps satisfying the following condition

$$d(Tx, Ty) \leq \varphi(M(x,y)) \text{ for } x, y \in X,$$

where M is as in Remark 3, and φ is increasing, continuous and satisfies (2) and (12). Then, by Walter's theorem [Wa81, Theorem 4], T has a CFP. Moreover, (12) is also essential here as shown in [Ja95, Example 2].

LEMMA 6. *Let D and E_φ be as in Lemma* 1. *The following statements are equivalent*:

(i) *there is an upper semi-continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *and* (12) *such that $D \subseteq E_\varphi$*;

(ii) *there is an increasing and right continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *and* (12) *such that $D \subseteq E_\varphi$*;

(iii) *there is a strictly increasing and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying* (2) *and* (12) *such that $D \subseteq E_\varphi$.*

Proof. (i)⇒(iii) follows from a separation theorem [JJ04, Theorem A.1 and Corollary A.1]. (iii)⇒(ii)⇒(i) are obvious. ∎

As an immediate consequence, we obtain the following

THEOREM 12. *Let (X,d) be a metric space and T be a selfmap of X. The following statements are equivalent*:

(i) *T is a Matkowski–Walter contraction*;

(ii) *T is φ-contractive, where φ is strictly increasing, continuous and satisfies* (12);

(iii) *T is φ-contractive, where φ is increasing, right continuous and satisfies* (12);

(iv) *T is φ-contractive, where φ is upper semi-continuous and satisfies* (12).

The following two results illuminate relations between the classes of Matkowski–Walter, Dugundji and Rakotch. We omit their proofs since they use similar arguments as the proofs of Propositions 1, 2 and 5.

PROPOSITION 6. *Assume that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous and satisfies* (2) *and* (12). *The following statements are equivalent*:

(i) $\limsup_{t\to\infty} \varphi(t)/t < 1$;

(ii) *given a metric space (X,d) and a φ-contractive map $T : X \to X$, T is a Rakotch contraction.*

EXAMPLE 6. Set $\varphi(t) := t/2$ for $t \in [0,1]$, and $\varphi(t) := t - \sqrt{t} + 1/2$ for $t > 1$. Then Proposition 6 yields the existence of a φ-contractive map in the Matkowski–Walter class which is not a Rakotch contraction.

PROPOSITION 7. *Assume that* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing and right continuous, satisfies* (2) *and* $\liminf_{t\to\infty}(t - \varphi(t)) > 0$. *The following statements are equivalent*:

(i) $\lim_{t\to\infty}(t - \varphi(t)) = \infty$;

(ii) *given a metric space* (X, d) *and a* φ*-contractive map* $T : X \to X$, T *is a Matkowski–Walter contraction.*

EXAMPLE 7. Set $\varphi(t) := t/2$ for $t \in [0, 1]$ and $\varphi(t) := t - 1/2$ for $t > 1$. By Proposition 7, there is a φ-contractive map in Dugundji's class which is not a Matkowski–Walter contraction.

For Matkowski–Walter contractions, the following result concerning invariant sets of infinite iterated function systems was obtained in [GJ05].

THEOREM 13. *Let* (X, d) *be a complete metric space, and* $F_n : X \to X$ $(n \in \mathbb{N})$ *be Matkowski–Walter contractions with a function* φ *independent of* $n \in \mathbb{N}$. *The following statements are equivalent*:

(i) *there exists an* $x_0 \in X$ *such that* $(F_n x_0)_{n\in\mathbb{N}}$ *is bounded*;

(ii) $(x_n)_{n\in\mathbb{N}}$ *is bounded, where* x_n *is a unique fixed point of* F_n;

(iii) *there is a non-empty bounded and separable set* $K \subseteq X$ *which is invariant with respect to the family* $\{F_n : n \in \mathbb{N}\}$, *i.e.*,

$$K = \bigcup_{n\in\mathbb{N}} F_n(K).$$

Moreover, if (i) *holds, then for any sequence* $(\sigma_n)_{n\in\mathbb{N}}$ *of positive integers and* $x \in X$, *the limit*

$$\Gamma(\sigma) := \lim_{n\to\infty} F_{\sigma_1} \circ \ldots \circ F_{\sigma_n}(x)$$

exists and does not depend on x. *Furthermore, the set* $\Gamma(\mathbb{N}^{\mathbb{N}})$ *is the greatest bounded set which is invariant with respect to* $\{F_n : n \in \mathbb{N}\}$.

Finally, let us note that the class of increasing and continuous functions satisfying (2) and (12) also appeared in a fixed point theorem for asymptotic contractions which was given in [JJ04].

7. Retrospect. Here we select characterizations of the classes of contractions given in the previous sections by choosing those formulated in a style of Krasnosel'skiĭ. By using this style, the class of Banach contractions may be described in the following way.

T is a Banach contraction ($T \in \mathtt{Ba}$) *iff*

$$\exists L \in [0, 1)\ \forall a > 0\ \forall x, y \in X \quad \big(a \le d(x, y) \Rightarrow d(Tx, Ty) \le Ld(x, y)\big).$$

Below we give appropriate characterizations of other classes of maps.

T is a Rakotch contraction ($T \in \mathtt{Ra}$) *iff*

$$\forall a > 0\ \exists L \in [0, 1)\ \forall x, y \in X \quad \big(a \le d(x, y) \Rightarrow d(Tx, Ty) \le Ld(x, y)\big).$$

T is a Browder contraction ($T \in \mathtt{Br}$) *iff*

$$\forall a > 0\ \forall b > a\ \exists L \in [0, 1)\ \forall x, y \in X \quad \big(a \le d(x, y) \le b \Rightarrow d(Tx, Ty) \le Ld(x, y)\big),$$

or, equivalently,

$$\forall a > 0\ \forall b > a\ \exists \delta > 0\ \forall x, y \in X \quad \big(a \le d(x,y) \le b \Rightarrow d(Tx,Ty) \le d(x,y) - \delta\big).$$

T is a Boyd–Wong contraction ($T \in$ `BW`) iff

$$\forall a > 0\ \exists b > a\ \exists L \in [0,1)\ \forall x, y \in X \quad \big(a \le d(x,y) \le b \Rightarrow d(Tx,Ty) \le Ld(x,y)\big),$$

or, equivalently,

$$\forall a > 0\ \exists b > a\ \exists \delta > 0\ \forall x, y \in X \quad \big(a \le d(x,y) \le b \Rightarrow d(Tx,Ty) \le d(x,y) - \delta\big).$$

T is a Dugundji contraction ($T \in$ `Du`) iff

$$\forall a > 0\ \exists \delta > 0\ \forall x, y \in X \quad \big(a \le d(x,y) \Rightarrow d(Tx,Ty) \le d(x,y) - \delta\big).$$

Let `Bu`, `DG`, `Ge I`, `Ge II`, `Ge III`, `Kr`, `Ma`, `MW`, `MK`, `Mu`, `Ta` denote, respectively, the classes of Burton, Dugundji–Granas, Geraghty (I), (II) and (III), Krasnosel'skiĭ, Matkowski, Matkowski–Walter, Meir–Keeler (cf. Remark 6), Mukherjea (cf. Remark 6) and Tasković (cf. Remark 5). In view of the results of previous sections the following relations hold.

$$\begin{array}{ccccccccccc}
 & & & & \mathtt{Ba} & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & \mathtt{Bu} & = & \mathtt{Ra} & = & \mathtt{Ge\,I} & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & \mathtt{MW} & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & \mathtt{Du} & & & & & & \\
 & & & & \downarrow & & & & & & \\
\mathtt{Ta} & = & \mathtt{DG} & = & \mathtt{Br} & = & \mathtt{Kr} & = & \mathtt{Ge\,III} & & \\
 & & & \swarrow & & \searrow & & & & & \\
 & & \mathtt{Ma} & & & & \mathtt{BW} & = & \mathtt{Ge\,II} & = & \mathtt{Mu} \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \mathtt{Le} & & \longleftarrow & & \mathtt{MK} & & & &
\end{array}$$

In the above diagram `Le` is the class of *Leader contractions* (cf. [Le83]): $T \in \mathtt{Le}$ if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \exists r \in \mathbb{N}\ \forall x, y \in X \quad \big(d(x,y) < \varepsilon + \delta\ \Rightarrow\ d(T^r x, T^r y) < \varepsilon\big).$$

As shown in [Le83], the last condition is close to a necessary and sufficient condition for the existence of a CFP.

Acknowledgements. We are grateful to Professor Simeon Reich for sending us some reprints of his papers. Also, our sincere thanks are due to the referee for many useful suggestions.

References

[BM81] Z. Benedykt, J. Matkowski, *Remarks on some fixed point theorems*, Demonstratio Math. 14 (1981), 227–232.

[BW69] D. W. Boyd, J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. 20 (1969), 458–464.

[Br68] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Nederl. Akad. Wetensch. Proc. Ser. A 71=Indag. Math. 30 (1968), 27–35.

[Bu96] T. A. Burton, *Integral equations, implicit functions, and fixed points*, Proc. Amer. Math. Soc. 124 (1996), 2383–2390.

[BRZ06] D. Butnariu, S. Reich, A. J. Zaslavski, *Asymptotic behavior of inexact orbits for a class of operators in complete metric spaces*, J. Appl. Anal., accepted for publication.

[Ch00] Y.-Z. Chen, *Inhomogeneous iterates of contraction mappings and nonlinear ergodic theorems*, Nonlinear Anal. 39 (2000), Ser. A: Theory Methods, 1–10.

[DM76] F. S. De Blasi, J. Myjak, *Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach*, C. R. Acad. Sci. Paris Sér. A–B 283 (1976), 185–187.

[Du76] J. Dugundji, *Positive definite functions and coincidences*, Fund. Math. 90 (1976), 131–142.

[DG78] J. Dugundji, A. Granas, *Weakly contractive maps and elementary domain invariance theorem*, Bull. Greek Math. Soc. 19 (1978), 141–151.

[Ge73] M. A. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc. 40 (1973), 604–608.

[Ge74] M. A. Geraghty, *An improved criterion for fixed points of contraction mappings*, J. Math. Anal. Appl. 48 (1974), 811–817.

[GD03] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer Monogr. Math., Springer, New York 2003.

[GJ05] G. Gwóźdź-Łukawska, J. Jachymski, *The Hutchinson–Barnsley theory for infinite iterated function systems*, Bull. Austral. Math. Soc. 72 (2005), 441–454.

[HS80] M. Hegedüs, T. Szilágyi, *Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings*, Math. Japon. 25 (1980), 147–157.

[HP57] E. Hille, R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. 31, Amer. Math. Soc., Providence 1957.

[Ja95] J. Jachymski, *Equivalent conditions and the Meir–Keeler type theorems*, J. Math. Anal. Appl. 194 (1995), 293–303.

[Ja97A] J. Jachymski, *An extension of A. Ostrowski's theorem on the round-off stability of iterations*, Aequationes Math. 53 (1997), 242–253.

[Ja97P] J. Jachymski, *Equivalence of some contractivity properties over metrical structures*, Proc. Amer. Math. Soc. 125 (1997), 2327–2335.

[Ja99] J. Jachymski, *On iterative equivalence of some classes of mappings*, Ann. Math. Sil. 13 (1999), 149–165.

[JJ04] J. Jachymski, I. Jóźwik, *On Kirk's asymptotic contractions*, J. Math. Anal. Appl. 300 (2004), 147–159.

[JS99] J. Jachymski, J. D. Stein, Jr., *A minimum condition and some related fixed-point theorems*, J. Austral. Math. Soc. Ser. A 66 (1999), 224–243.

[KS01] W. A. Kirk, B. Sims (eds.), *Handbook of Metric Fixed Point Theory*, Kluwer Acad. Publ., Dordrecht 2001.

[KV72] M. A. Krasnosel'skiĭ, G. M. Vaĭnikko, P. P. Zabreĭko, Ya. B. Rutitskiĭ, V. Ya. Stetsenko, *Approximate Solution of Operator Equations*, Wolters–Noordhoff Publ., Groningen 1972.

[Le83] S. Leader, *Equivalent Cauchy sequences and contractive fixed points in metric spaces*, Studia Math. 76 (1983), 63–67.

[Li01] T. C. Lim, *On characterizations of Meir–Keeler contractive maps*, Nonlinear Anal. 46 (2001), Ser. A: Theory Methods, 113–120.

[Ma75] J. Matkowski, *Integrable Solutions of Functional Equations*, Dissertationes Math. (Rozprawy Mat.) 127 (1975).

[Ma77] J. Matkowski, *Fixed point theorems for mappings with a contractive iterate at a point*, Proc. Amer. Math. Soc. 62 (1977), 344–348.

[Ma93] J. Matkowski, *Nonlinear contractions in metrically convex space*, Publ. Math. Debrecen 45 (1994), 103–114.

[MR03] E. Matoušková, S. Reich, *Reflexivity and approximate fixed points*, Studia Math. 159 (2003), 403–415.

[MK69] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. 28 (1969), 326–329.

[Mu77] A. Mukherjea, *Contractions and completely continuous mappings*, Nonlinear Anal. 1 (1977), 235–247.

[Os67] A. M. Ostrowski, *The round-off stability of iterations*, Z. Angew. Math. Mech. 47 (1967), 77–81.

[Ra62] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. 13 (1962), 459–465.

[Re72] S. Reich, *Fixed points of contractive functions*, Boll. Un. Mat. Ital. (4) 5 (1972), 26–42.

[Re74] S. Reich, *Some fixed point problems*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 57 (1974), 194–198.

[RZ00] S. Reich, A. J. Zaslavski, *Almost all nonexpansive mappings are contractive*, C. R. Math. Acad. Sci. Soc. R. Can. 22 (2000), 118–124.

[RZ01] S. Reich, A. J. Zaslavski, *Well-posedness of fixed point problems*, Far East J. Math. Sci. 2001, Special Volume, Part III, 393–401.

[RZ01C] S. Reich, A. J. Zaslavski, *The set of noncontractive mappings is σ-porous in the space of all nonexpansive mappings*, C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), 539–544.

[RZ02] S. Reich, A. J. Zaslavski, *Convergence of iterates of nonexpansive set-valued mappings*, in: Set Valued Mappings with Applications in Nonlinear Analysis, Ser. Math. Anal. Appl. 4, Taylor & Francis, London 2002, 411–420.

[RPP02] I. A. Rus, A. Petruşel, G. Petruşel, *Fixed Point Theory 1950–2000: Romanian Contributions*, House of the Book of Science, Cluj–Napoca 2002.

[Ta78] M. R. Tasković, *A generalization of Banach's contraction principle*, Publ. Inst. Math. (Beograd) (N.S.) 23(37) (1978), 179–191.

[Wa81] W. Walter, *Remarks on a paper by F. Browder about contraction*, Nonlinear Anal. 5 (1981), 21–25.