FIXED POINT THEORY AND ITS APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 77
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2007

# DEGREE OF $T$-EQUIVARIANT MAPS IN $\mathbb{R}^n$

JOANNA JANCZEWSKA
*Department of Technical Physics and Applied Mathematics*
*Gdańsk University of Technology*
*Narutowicza 11/12, 80-952 Gdańsk, Poland*
*E-mail: janczewska@mif.pg.gda.pl*

MARCIN STYBORSKI
*Department of Technical Physics and Applied Mathematics*
*Gdańsk University of Technology*
*Narutowicza 11/12, 80-952 Gdańsk, Poland*
*and Ph.D. student of the Institute of Mathematics, Polish Academy of Sciences*
*E-mail: marcins@impan.gov.pl*

**Abstract.** A special case of $G$-equivariant degree is defined, where $G = \mathbb{Z}_2$, and the action is determined by an involution $T : \mathbb{R}^p \oplus \mathbb{R}^q \to \mathbb{R}^p \oplus \mathbb{R}^q$ given by $T(u, v) = (u, -v)$. The presented construction is self-contained. It is also shown that two $T$-equivariant gradient maps $f, g : (\mathbb{R}^n, S^{n-1}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are $T$-homotopic iff they are gradient $T$-homotopic. This is an equivariant generalization of the result due to Parusiński.

**1. Introduction.** Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$. Consider a continuous map $f : \overline{\Omega} \to \mathbb{R}^n$ such that $f$ is not equal to 0 at any point on the boundary of $\Omega$. Then an integer $\deg(f, \Omega)$ called the Brouwer degree can be associated to $f$. The classical works on this subject are [3], [12], [13], and a modern one is [11]. It is well known that the Brouwer degree is an invariant of homotopy. This means that if $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ is a homotopy nowhere vanishing on $\partial\Omega \times [0, 1]$ then $\deg(h_t, \Omega) = \deg(h_0, \Omega)$ for all $t \in [0, 1]$, where $h_t(x) = h(x, t)$.

Let $G$ be a compact Lie group. Assume that $V$ is a real finite-dimensional representation of $G$ and $n = \dim V$. Take $\Omega \subset V$ and $f : \overline{\Omega} \to V$ as above. In addition, suppose that $\Omega$ is $G$-invariant ($gx \in \Omega$ for all $x \in \Omega$, $g \in G$) and $f$ is $G$-equivariant ($f(gx) = gf(x)$

2000 *Mathematics Subject Classification*: 47H11.

*Key words and phrases*: involution, $T$-equivariant degree, classification of $T$-equivariant maps.

Supported by Grant KBN no. 1 P03A 042 29.

The paper is in final form and no version of it will be published elsewhere.

for all $x \in \overline{\Omega}$, $g \in G$). In this case, the $G$-equivariant degree $\mathrm{Deg}_G(f, \Omega) \in B(G)$ is defined, where $B(G)$ stands for the Burnside ring of $G$. This degree was introduced by Ize, Massabó and Vignoli in [10]. Up till now, it was considered by many authors. See for instance [6], [11] and [15]. Of course, $G$-equivariant degree is an invariant of $G$-equivariant homotopy ($h(gx, t) = gh(x, t)$ for all $x \in \overline{\Omega}$, $t \in [0, 1]$, $g \in G$).

Let $G$ be equal to $\mathbb{Z}_2$. The action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ is determined by a decomposition of $\mathbb{R}^n$ onto the direct sum $\mathbb{R}^p \oplus \mathbb{R}^q$ and the involution $T(u, v) = (u, -v)$, where $n = p + q$, $p, q \in \mathbb{N} \cup \{0\}$ and $u \in \mathbb{R}^p$, $v \in \mathbb{R}^q$. In fact, to define the $\mathbb{Z}_2$-equivariant degree we do not need to use the representation theory. In this work we would like to describe a construction of this degree. We will call it $T$-equivariant degree.

Our approach is alternative to the one by Granas and Dugundji in [8]. There are two basic differences between our and their approach. Contrary to Granas and Dugundji, from the beginning we work with the family of $T$-equivariant maps (see Sec. 6, §20, Theorem 1.2, pp. 551–552 in [8], and Lemma 3.2, Conclusion 3.3 here). We also introduce notions of $T$-equivariant normal maps and homotopies, which are different from ones in [8]. Moreover, the proofs of all lemmas and propositions needed to define the degree are complete.

Our construction is divided into five main steps. Each step is a separate section.

In [14], Parusiński showed that if we have two gradient vector fields on the unit ball in $\mathbb{R}^n$ and nowhere vanishing on the sphere, then they are homotopic if and only if they are gradient homotopic. In the last section we will prove this theorem in $T$-equivariant case. Namely, consider two $T$-equivariant gradient vector fields $f$ and $g$ on the unit ball in $\mathbb{R}^n$ and nowhere vanishing on the sphere. It is shown that if there is a $T$-equivariant homotopy joining $f$ to $g$ then there is a $T$-equivariant gradient homotopy joining $f$ to $g$. Our result suggests that there is no interesting generalization of $T$-equivariant degree on gradient vector fields. The proof is based on the latest results by Ferrario (see [4]) and Dancer, Gęba and Rybicki (see [1]).

**2. $T$-equivariant maps and homotopy.** Let $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$, where $n \in \mathbb{N}$, $p, q \in \mathbb{N} \cup \{0\}$ and $n = p + q$. For every $x \in \mathbb{R}^n$ we write $x = (u, v)$, where $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$T(u, v) := (u, -v).$$

The map $T$ is a linear isomorphism and an involution, i.e. $T^2 = \mathrm{Id}_{\mathbb{R}^n}$.

DEFINITION 2.1. A set $X \subset \mathbb{R}^n$ is *$T$-invariant* if $T(X) \subset X$.

If a set $X$ is $T$-invariant then $T(X) = X$ and $T_{|X} : X \to X$ is an involution onto $X$.

DEFINITION 2.2. Let $X \subset \mathbb{R}^n$ be $T$-invariant.

1. A map $f : X \to \mathbb{R}^n$ is called *$T$-equivariant* if $f(Tx) = Tf(x)$ for all $x \in X$.
2. $h : X \times [0, 1] \to \mathbb{R}^n$ is called *a $T$-equivariant homotopy* if it is continuous and $h(Tx, t) = Th(x, t)$ for all $x \in X$ and $t \in [0, 1]$.
3. A function $\tau : X \to \mathbb{R}$ is called *$T$-equivariant* if $\tau(Tx) = \tau(x)$ for all $x \in X$.

Remark that if $f = (f_1, f_2)$, where $f_1 : X \to \mathbb{R}^p$ and $f_2 : X \to \mathbb{R}^q$, then $f$ is $T$-equivariant if and only if $f_1(u, -v) = f_1(u, v)$ and $f_2(u, -v) = -f_2(u, v)$ for all $(u, v) \in X$.

From now on, every open bounded $T$-invariant subset of $\mathbb{R}^n$ is said to be $T$-*admissible*.

Assume that $\Omega \subset \mathbb{R}^n$ is $T$-admissible. It is obvious that $\overline{\Omega}$ is $T$-invariant. We say that $f : \overline{\Omega} \to \mathbb{R}^n$ is $T$-*admissible* if $f$ is continuous, $T$-equivariant and $f(x) \neq 0$ for all $x \in \partial\Omega$. We will denote by $\mathcal{A}_T(\Omega)$ the family of all $T$-admissible maps from $\overline{\Omega}$ into $\mathbb{R}^n$. In the same spirit we generalize the notion of homotopy. We say that a homotopy $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ is $T$-*admissible* if $h$ is $T$-equivariant and $h(x, t) \neq 0$ for all $x \in \partial\Omega$ and $t \in [0, 1]$. We will denote by $\mathcal{HA}_T(\Omega)$ the family of all $T$-admissible homotopies from $\overline{\Omega} \times [0, 1]$ into $\mathbb{R}^n$.

Definition 2.3. We say that $f$ is *homotopic to* $g$ *in* $\mathcal{A}_T(\Omega)$ and write $f \sim g$ in $\mathcal{A}_T(\Omega)$ if there exists $h \in \mathcal{HA}_T(\Omega)$ joining $f$ to $g$, i.e. $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$ for all $x \in \overline{\Omega}$.

It is easy to check that $\sim$ is an equivalence relation in $\mathcal{A}_T(\Omega)$. The homotopy class of $f \in \mathcal{A}_T(\Omega)$ under $\sim$ will be denoted by $[f]$. Finally, the set of all homotopy classes of the relation $\sim$ will be denoted by $\mathcal{A}_T[\Omega]$.

**3. $T$-equivariant generic maps.** Let $\Omega \subset \mathbb{R}^n$ be $T$-admissible. Here and subsequently,

$$\mathcal{A}_T^\infty(\Omega) := \{f \in \mathcal{A}_T(\Omega) : f_{|\Omega} \text{ is smooth}\},$$
$$\mathcal{HA}_T^\infty(\Omega) := \{h \in \mathcal{HA}_T(\Omega) : h_{t|\Omega} \text{ is smooth for } t \in [0, 1]\},$$

where $h_t : \overline{\Omega} \to \mathbb{R}^n$ is defined by $h_t(x) := h(x, t)$.

Definition 3.1. We say that $f$ is *homotopic to* $g$ *in* $\mathcal{A}_T^\infty(\Omega)$ and write $f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$ if there exists $h \in \mathcal{HA}_T^\infty(\Omega)$ joining $f$ to $g$.

The relation $\simeq$ is easily seen to be an equivalence relation in $\mathcal{A}_T^\infty(\Omega)$.

A map $f \in \mathcal{A}_T^\infty(\Omega)$ is said to be *generic* if $0 \in \mathbb{R}^n$ is a regular value of $f_{|\Omega}$, i.e. the derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism for all $x \in f^{-1}(\{0\})$.

In this section we show that under some restrictions on $\Omega$, every homotopy class in $\mathcal{A}_T[\Omega]$ possesses a $T$-equivariant generic map. For this purpose we prove now a few lemmas.

Let $K$ be a compact subset of $\mathbb{R}^n$. A map $f : K \to \mathbb{R}^n$ is called *smooth* if there exists an open set $X \subset \mathbb{R}^n$ such that $K \subset X$ and there exists a smooth map $\tilde{f} : X \to \mathbb{R}^n$ such that $\tilde{f}_{|K} = f$. Let $\{U_i\}_{i=1}^k$ be an open covering of $K$. Set $U = \bigcup_{i=1}^k U_i$. We call a family of smooth functions $\lambda_i : U \to [0, 1]$, where $i = 1, 2, \ldots, k$, *a smooth partition of unity* subordinate to the covering $\{U_i\}_{i=1}^k$, if this family satisfies the following conditions:

- $\operatorname{supp} \lambda_i = \overline{\{x \in \mathbb{R}^n : \lambda_i(x) \neq 0\}} \subset U_i$ for every $i = 1, 2, \ldots, k$,
- $\sum_{i=1}^k \lambda_i(y) = 1$ for every $y \in K$.

It is well known that such a partition exists (see [16]). Additionally, if $K$ and every $U_i$ are $T$-invariant sets and every $\lambda_i$ is a $T$-equivariant function then we say that $\{U_i\}_{i=1}^k$ is a $T$-invariant covering of $K$ and $\{\lambda_i\}_{i=1}^k$ is a $T$-equivariant partition of unity.

Lemma 3.1. *Assume that $K \subset \mathbb{R}^n$ is compact and $T$-invariant, and $\{U_i\}_{i=1}^k$ is an open $T$-equivariant covering of $K$. Then there exists a smooth $T$-equivariant partition of unity subordinate to the covering $\{U_i\}_{i=1}^k$.*

*Proof.* Let $\{\lambda_i\}_{i=1}^k$ be a smooth partition of unity subordinate to the covering $\{U_i\}_{i=1}^k$ of $K$. For every $i = 1, 2, \ldots, k$, let $\widehat{\lambda_i}$ be given by $\widehat{\lambda_i} = \frac{1}{2}(\lambda_i + \lambda_i T)$. It is obvious that each function $\widehat{\lambda_i}$ is smooth and $T$-equivariant. The family $\{\widehat{\lambda_i}\}_{i=1}^k$ is a desired one. ∎

Let $K \subset \mathbb{R}^n$ be compact and $T$-invariant. We say that $T$ *acts freely on* $K$ if $Tx \neq x$ for every $x \in K$, i.e. $K \cap \mathbb{R}^p = \emptyset$. Then

$$\mathrm{dist}(K, \mathbb{R}^p) := \inf\{|x - y| : x \in K,\ y \in \mathbb{R}^p\}$$

is a positive number.

Lemma 3.2. *Let $K \subset \mathbb{R}^n$ be a compact $T$-invariant set such that $T$ acts freely on $K$. If a map $f : K \to \mathbb{R}^n$ is continuous and $T$-equivariant then for every $\varepsilon > 0$ there is a smooth $T$-equivariant map $g : K \to \mathbb{R}^n$ such that*

$$\sup_{x \in K} |f(x) - g(x)| < \varepsilon.$$

From now on, $B(a, r)$ stands for an open ball of radius $r$, centered at a point $a \in \mathbb{R}^n$.

*Proof.* Fix $\varepsilon > 0$. Since $K$ is compact, $f$ is uniformly continuous. Hence, there is $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Set $\delta' = \min\{\delta, \mathrm{dist}(K, \mathbb{R}^p)\}$, and $U_x = B(x, \delta')$ for every $x \in K$. Then

- $TU_x = B(Tx, \delta')$,
- $U_x \cap TU_x = \emptyset$,
- $K \subset \bigcup_{x \in K}(U_x \cup TU_x)$.

The compactness of $K$ implies that there exist points $x_1, x_2, \ldots, x_k \in K$ such that $K \subset \bigcup_{i=1}^k (U_i \cup TU_i)$, where $U_i = U_{x_i}$. Consider a smooth $T$-equivariant partition of unity $\{\lambda_i\}_{i=1}^k$ subordinate to the covering $\{U_i \cup TU_i\}_{i=1}^k$ of $K$. Set $U = \bigcup_{i=1}^k (U_i \cup TU_i)$. For every $i \in \{1, 2, \ldots, k\}$, let $\pi_i : U \to \mathbb{R}^n$ be a map such that $\pi_i(U_i) = \{x_i\}$ and $\pi_i(TU_i) = \{Tx_i\}$. The function $g : U \to \mathbb{R}^n$ is defined by

$$g(x) = \sum_{i=1}^k \lambda_i(x) f(\pi_i(x)).$$

Take $x \in U$. If $x \in U_i \cup TU_i$ then in a sufficiently small neighbourhood of $x$ the map $f \circ \pi_i$ is constant. If $x \in \partial U_i \cup \partial TU_i$ then in a sufficiently small neighbourhood of $x$ the function $\lambda_i$ is equal to 0. Hence $g$ is smooth.

Take $x \in K$. If $x \in U_i$ then $\pi_i(x) = x_i$ and $|\pi_i(x) - x| < \delta$. If $x \in TU_i$ then $\pi_i(x) = Tx_i$ and $|\pi_i(x) - x| < \delta$. Finally, if $x \notin U_i \cup TU_i$ then $\lambda_i(x) = 0$. From this it follows that

$$|g(x) - f(x)| = \Big|\sum_{i=1}^k \lambda_i(x) f(\pi_i(x)) - \sum_{i=1}^k \lambda_i(x) f(x)\Big| \leq \sum_{i=1}^k \lambda_i(x) |f(\pi_i(x)) - f(x)| < \varepsilon.$$

Moreover,

$$g(Tx)=\sum_{i=1}^{k}\lambda_i(Tx)f(\pi_i(Tx))=\sum_{i=1}^{k}\lambda_i(x)f(T\pi_i(x))=\sum_{i=1}^{k}\lambda_i(x)Tf(\pi_i(x))=Tg(x),$$

which completes the proof. ∎

CONCLUSION 3.3. *Let $\Omega\subset\mathbb{R}^n$ be a $T$-admissible set such that $T$ acts freely on $\overline{\Omega}$. Then for every $f\in\mathcal{A}_T(\Omega)$ there exists $g\in\mathcal{A}_T^\infty(\Omega)$ such that $f\sim g$ in $\mathcal{A}_T(\Omega)$.*

*Proof.* By the assumption, $\overline{\Omega}$ is a $T$-invariant compact set and $T$ acts freely on $\overline{\Omega}$. Set $d=\inf\{|f(x)|:x\in\partial\Omega\}$. From Lemma 3.2 it follows that there exists $g\in\mathcal{A}_T^\infty(\Omega)$ such that

$$\sup_{x\in\overline{\Omega}}|f(x)-g(x)|<d.$$

Consider the linear homotopy $h:\overline{\Omega}\times[0,1]\to\mathbb{R}^n$ joining $f$ to $g$, i.e.

$$h(x,t)=tg(x)+(1-t)f(x).$$

It is trivial that $h$ is continuous and $h(Tx,t)=Th(x,t)$ for all $x\in\overline{\Omega}$ and $t\in[0,1]$. Take $x\in\partial\Omega$ and $t\in[0,1]$. Then

$$|h(x,t)|=|f(x)-t(f(x)-g(x))|\geq|f(x)|-t|f(x)-g(x)|\geq|f(x)|-|f(x)-g(x)|>0.$$

Hence $h\in\mathcal{HA}_T(\Omega)$. ∎

Let $U\subset\mathbb{R}^n$ be an open bounded set, and $K\subset U$ be compact. It is well known that there exists a smooth function $\eta:\mathbb{R}^n\to[0,1]$ such that

$$\eta(x)=\begin{cases}1 & \text{for } x\in K,\\ 0 & \text{for } x\in\mathbb{R}^n\setminus U.\end{cases}$$

In the mathematical literature, $\eta$ is called *the Urysohn function* (see [8]).

LEMMA 3.4. *Let $U$ and $U_0$ be $T$-admissible subsets of $\mathbb{R}^n$. Assume that $\overline{U}_0\subset U$. Then there exists a smooth $T$-equivariant function $\tilde{\eta}:\mathbb{R}^n\to[0,1]$ such that $\tilde{\eta}(x)=1$ for every $x\in\overline{U}_0$ and $\tilde{\eta}(x)=0$ for every $x\in\mathbb{R}^n\setminus U$.*

The proof is similar to that of Lemma 3.1. We leave it to the reader.

LEMMA 3.5. *Let $\Omega_0$ and $\Omega$ be $T$-admissible subsets of $\mathbb{R}^n$ such that $\Omega_0\subset\Omega$. Suppose that $f_0\simeq g_0$ in $\mathcal{A}_T^\infty(\Omega_0)$ and there is an $f\in\mathcal{A}_T^\infty(\Omega)$ such that $f_{|\Omega_0}=f_0$. Then there exist a map $g\in\mathcal{A}_T^\infty(\Omega)$ and a $T$-admissible set $U_0\subset\Omega_0$ satisfying the following conditions*:

1. *$f\simeq g$ in $\mathcal{A}_T^\infty(\Omega)$,*
2. *$g(x)=f(x)$ for every $x\in\overline{\Omega}\setminus\Omega_0$,*
3. *$g(x)=g_0(x)$ for every $x\in U_0$,*
4. *$g_0^{-1}(\{0\})\cap\Omega_0=g^{-1}(\{0\})\cap\Omega_0\subset U_0$.*

*Proof.* Let $\bar{h}\in\mathcal{HA}_T^\infty(\Omega_0)$ be a homotopy joining $f_0$ to $g_0$. Take an open $T$-invariant subset $U_0$ of $\mathbb{R}^n$ such that $\overline{U}_0\subset\Omega_0$ and $\bar{h}(x,t)\neq0$ for every $(x,t)\in(\overline{\Omega}_0\setminus U_0)\times[0,1]$. Consider an open $T$-invariant subset $U$ of $\mathbb{R}^n$ such that $\overline{U}_0\subset U\subset\overline{U}\subset\Omega_0$. Let $\eta:\mathbb{R}^n\to[0,1]$ be a smooth $T$-equivariant Urysohn function for the pair of sets $U_0$ and $U$, i.e. $\eta(x) = 1$ for every $x \in \overline{U}_0$ and $\eta(x) = 0$ for every $x \in \mathbb{R}^n \setminus U$. Let $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ be defined by

$$h(x,t) = \begin{cases} f(x) & \text{for } x \in \overline{\Omega} \setminus \overline{U}, \\ \bar{h}(x, t\eta(x)) & \text{for } x \in \Omega_0. \end{cases}$$

We check that $h \in \mathcal{HA}_T^\infty(\Omega)$ and $g(x) := h(x, 1)$, $x \in \overline{\Omega}$, satisfies the claim of our lemma.

Remark that $(\overline{\Omega} \setminus \overline{U}) \cap \Omega_0 = \Omega_0 \setminus \overline{U}$. If $x \in \Omega_0 \setminus \overline{U}$ then $\eta(x) = 0$, and hence $\bar{h}(x, t\eta(x)) = \bar{h}(x, 0) = f(x)$ for all $t \in [0, 1]$. In consequence, $h$ is smooth and $h(x, 0) = f(x)$ for every $x \in \overline{\Omega}$. If $x \in \partial\Omega$ then $h(x,t) = f(x) \neq 0$ for all $t \in [0, 1]$. Moreover, for $x \in \Omega_0$ and $t \in [0, 1]$ we have $h(Tx, t) = \bar{h}(Tx, t\eta(Tx)) = \bar{h}(Tx, t\eta(x)) = T\bar{h}(x, t\eta(x)) = Th(x, t)$. Thus $h$ is $T$-equivariant. Summarizing, $h \in \mathcal{HA}_T^\infty(\Omega)$ and it joins $f$ to $g$.

Take $x \in \overline{\Omega} \setminus \Omega_0$. Since $\overline{\Omega} \setminus \Omega_0 \subset \overline{\Omega} \setminus U$, we get $g(x) = h(x, 1) = f(x)$.

If $x \in U_0$ then $\eta(x) = 1$ and $g(x) = h(x, 1) = \bar{h}(x, 1) = g_0(x)$.

Finally, fix $x \in \Omega_0$. If $x \in \Omega_0 \setminus \overline{U}$ then $g(x) = \bar{h}(x, 0) = f_0(x)$. If $x \in \overline{U}$ then $g(x) = \bar{h}(x, \eta(x))$. Since $\{x \in \overline{\Omega}_0 : \bar{h}(x,t) = 0 \text{ for any } t \in [0, 1]\} \subset U_0$, we have $g_0^{-1}(\{0\}) \cap \Omega_0 = g^{-1}(\{0\}) \cap \Omega_0 \subset U_0$, which completes the proof. ∎

Let $K \subset \mathbb{R}^n$ be nonempty, compact and $T$-admissible. Set $k \in \mathbb{N}$. We call a family of open sets $\{U_i\}_{i=1}^k$ a $(T, k)$-*simple covering* of $K$ if it satisfies the following conditions:

1. $U_i \cap TU_i = \emptyset$ for every $i \in \{1, 2, \ldots, k\}$,
2. $K \subset \bigcup_{i=1}^k (U_i \cup TU_i)$.

We say that $K$ is *a* $(T, k)$-*simple set* if it possesses a $(T, k)$-simple covering. If $K = \emptyset$, it is said to be $(T, 0)$-*simple*.

Proposition 3.6. *Every nonempty compact $T$-invariant subset $K$ of $\mathbb{R}^n$ such that $T$ acts freely on $K$ is $(T, k)$-simple for a certain $k \in \mathbb{N}$.*

*Proof.* Since $T$ acts freely on $K$, $K \cap \mathbb{R}^p = \emptyset$. Set $l = \operatorname{dist}(K, \mathbb{R}^p)$. We have

$$K \subset \bigcup_{x \in K} B(x, l)$$

By compactness of $K$, there are $x_1, x_2, \ldots, x_k \in K$ such that

$$K \subset \bigcup_{i=1}^{k} B(x_i, l).$$

Let $U_i = B(x_i, l)$ for $i = 1, 2, \ldots, k$. It is evident that $U_i \cap TU_i = \emptyset$ and

$$K \subset \bigcup_{i=1}^{k} (U_i \cup TU_i).$$ ∎

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. For every $f : \overline{\Omega} \to \mathbb{R}^n$ such that $f_{|\Omega}$ is $C^r$-smooth, where $r \geq 1$, and $f(x) \neq 0$ for all $x \in \partial\Omega$, set

$$R(f) = \{x \in f^{-1}(\{0\}) : Df(x) \in \mathrm{GL}(\mathbb{R}^n)\}.$$

Lemma 3.7. *Assume that $\Omega \subset \mathbb{R}^n$ is $T$-admissible, $T$ acts freely on $\overline{\Omega}$, which is $(T,k)$-simple for a certain $k \in \mathbb{N}$. Let $f \in \mathcal{A}_T^\infty(\Omega)$. Then there exists $g \in \mathcal{A}_T^\infty(\Omega)$ such that*

(i) *$f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$,*

(ii) *$g^{-1}(\{0\}) \setminus R(g)$ is $(T,k-1)$-simple.*

*Proof.* Let $\{U_i\}_{i=1}^k$ be a $(T,k)$-simple covering of $\overline{\Omega}$. Set

$$K = \overline{\Omega} \setminus \bigcup_{i=2}^{k} (U_i \cup TU_i), \quad K_1 = K \cap U_1.$$

Let us remark that $K$ is $T$-invariant, $K \subset (U_1 \cup TU_1)$, $K$ and $K_1$ are compact. Thus $K$ is $(T,1)$-simple and $K = K_1 \cup TK_1$.

From the Sard theorem it follows that there exists a regular value $y_0$ of $f_{|\Omega\cap U_1}$ such that $|y_0| < \inf\{|f(x)| : x \in \partial\Omega\}$. Since $f \circ T = T \circ f$, we have $Df(Tx) = T \circ Df(x) \circ T$ for every $x \in \Omega$. Hence $Ty_0$ is also a regular value of $f_{|\Omega\cap TU_1}$. Moreover, $|Ty_0| = |y_0|$.

Let $\eta : \mathbb{R}^n \to [0,1]$ be a smooth function such that $\eta(x) = 1$ for all $x \in K_1$ and $\eta(x) = 0$ for all $x \in \mathbb{R}^n \setminus U_1$. Let $g : \overline{\Omega} \to \mathbb{R}^n$ be given by

$$g(x) = \begin{cases} f(x) - \eta(x)y_0 & \text{for } x \in U_1 \cap \overline{\Omega}, \\ f(x) - \eta(Tx)Ty_0 & \text{for } x \in TU_1 \cap \overline{\Omega}, \\ f(x) & \text{for } x \in \overline{\Omega} \setminus (U_1 \cup TU_1), \end{cases}$$

The map $g_{|\Omega}$ is easily seen to be smooth. Let

$$h(x,t) = f(x) + t(g(x) - f(x))$$

for all $(x,t) \in \overline{\Omega} \times [0,1]$. Take $x \in \overline{\Omega}$. If $x \in U_1 \cap \overline{\Omega}$ then $g(Tx) = f(Tx) - \eta(T^2x)Ty_0 = Tf(x) - \eta(x)Ty_0 = T(f(x) - \eta(x)y_0) = Tg(x)$. If $x \in TU_1 \cap \overline{\Omega}$ then $g(Tx) = f(Tx) - \eta(Tx)y_0 = Tf(x) - \eta(Tx)y_0 = Tf(x) - T^2\eta(Tx)y_0 = T(f(x) - \eta(Tx)Ty_0) = Tg(x)$. Finally, if $x \in \overline{\Omega} \setminus (U_1 \cup TU_1)$ then $g(Tx) = f(Tx) = Tf(x) = Tg(x)$. Consequently, $g$ is $T$-equivariant.

Since $|g(x) - f(x)| < |y_0|$ for all $x \in \overline{\Omega}$, we conclude that $h$ is a homotopy joining $f$ to $g$ in $\mathcal{A}_T^\infty(\Omega)$.

Remark that $g^{-1}(\{0\}) \setminus R(g)$ is a compact set and $g^{-1}(\{0\}) \subset \bigcup_{i=2}^k (U_i \cup TU_i) \cup K$. Take $x \in K$. If $x \in K_1$ then $g(x) = f(x) - y_0$. If $x \in TK_1$ then $g(x) = f(x) - Ty_0$. From this $K \cap g^{-1}(\{0\}) \subset R(g)$, and so $g^{-1}(\{0\}) \setminus R(g) \subset \bigcup_{i=2}^k (U_i \cup TU_i)$ is $(T,k-1)$-simple. ■

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^n$ be a $T$-admissible set such that $T$ acts freely on $\overline{\Omega}$. Assume that $f \in \mathcal{A}_T^\infty(\Omega)$ and $f^{-1}(\{0\}) \setminus R(f)$ is $(T,k)$-simple for a certain $k \in \mathbb{N}$. Then there exists a map $g \in \mathcal{A}_T^\infty(\Omega)$ such that*

(i) *$f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$,*

(ii) *$g^{-1}(\{0\}) \setminus R(g)$ is $(T,k-1)$-simple.*

*Proof.* Since $f^{-1}(\{0\}) \setminus R(f)$ is $(T,k)$-simple, there is an open and $T$-invariant subset $\Omega_0$ of $\Omega$ such that

(a) $f^{-1}(\{0\}) \setminus R(f) \subset \Omega_0$,

(b) $R(f) \subset \Omega \setminus \overline{\Omega}_0$,

(c) $\overline{\Omega}_0$ is $(T,k)$-simple.

Set $f_0 = f_{|\Omega_0}$. Combining $(a)$ with $(b)$, we see that $f_0 \in \mathcal{A}_T^\infty(\Omega_0)$. By Lemma 3.7 it follows that there is $g_0 \in \mathcal{A}_T^\infty(\Omega_0)$ such that $f_0 \simeq g_0$ in $\mathcal{A}_T^\infty(\Omega_0)$ and $g_0^{-1}(\{0\}) \setminus R(g_0)$ is $(T,k-1)$-simple. From Lemma 3.5 we have that there is $g \in \mathcal{A}_T^\infty(\Omega)$ such that $f \simeq g$ in $\mathcal{A}_T^\infty(\Omega)$ and $g^{-1}(\{0\})\setminus R(g) = g_0^{-1}(\{0\})\setminus R(g_0)$. Thus $g^{-1}(\{0\})\setminus R(g)$ is $(T,k-1)$-simple. ∎

Applying the mathematical induction, Lemma 3.8 and Conclusion 3.3, one can immediately prove the next theorem.

THEOREM 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a $T$-admissible set such that $T$ acts freely on $\overline{\Omega}$. If $f \in \mathcal{A}_T(\Omega)$ then there exists a generic map $g \in \mathcal{A}_T^\infty(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$.*

CONCLUSION 3.10. *Let $\Omega \subset \mathbb{R}^n$ be a $T$-admissible set such that $T$ acts freely on $\overline{\Omega}$. If $f \in \mathcal{A}_T(\Omega)$ then* $\deg(f,\Omega) \in 2\mathbb{Z}$.

Here and subsequently, $\deg(f,\Omega)$ stands for the Brouwer degree of $f$ on $\Omega$.

*Proof.* Fix $f \in \mathcal{A}_T(\Omega)$. By Theorem 3.9 there is a generic map $g \in \mathcal{A}_T^\infty(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$. Hence $\deg(f,\Omega) = \deg(g,\Omega)$. Since $T\circ g = g\circ T$, we have $Dg(x) = T\circ Dg(Tx)\circ T$ for every $x \in \Omega$. From this

$$g^{-1}(\{0\}) \cap \Omega = \{x_1, x_2, \ldots, x_m\} \cup \{Tx_1, Tx_2, \ldots, Tx_m\}$$

and $\operatorname{sign}\det Dg(x_i) = \operatorname{sign}\det Dg(Tx_i)$ for $i = 1,2,\ldots,m$. In consequence,

$$\deg(g,\Omega) = \sum_{i=1}^{m} \operatorname{sign}\det Dg(x_i) + \sum_{i=1}^{m} \operatorname{sign}\det Dg(Tx_i) = 2\sum_{i=1}^{m} \operatorname{sign}\det Dg(x_i),$$

which completes the proof. ∎

**4. $T$-equivariant normal maps.** Let $\Omega \subset \mathbb{R}^n$ be $T$-admissible and let $\varepsilon > 0$. Define

$$\Omega(\varepsilon) = \{(u,v) \in \Omega : |v| < \varepsilon\}.$$

DEFINITION 4.1. Let $f = (f_1, f_2) \in \mathcal{A}_T(\Omega)$, where $f_1 : \overline{\Omega} \to \mathbb{R}^p$, and $f_2 : \overline{\Omega} \to \mathbb{R}^q$.

1. A map $f$ is said to be *$\varepsilon$-normal* if there exists $\varepsilon > 0$ such that

$$f(u,v) = (f_1(u,0), v)$$

for all $(u,v) \in \Omega(\varepsilon)$.

2. A map $f$ is called *normal* if there exists $\varepsilon > 0$ such that $f$ is $\varepsilon$-normal.

We will denote by $\mathcal{NA}_T(\Omega)$ the family of all normal maps from $\overline{\Omega}$ into $\mathbb{R}^n$.

DEFINITION 4.2. Let $\Omega \subset \mathbb{R}^n$ be a $T$-admissible set.

1. A homotopy $h \in \mathcal{HA}_T(\Omega)$ is called *normal* if there exists $\varepsilon > 0$ such that $h_t : \overline{\Omega} \to \mathbb{R}^n$ is $\varepsilon$-normal for every $t \in [0,1]$.

2. We say that $f$ is *homotopic to $g$ in $\mathcal{NA}_T(\Omega)$* and write $f \approx g$ in $\mathcal{NA}_T(\Omega)$ if there exists a normal homotopy joining $f$ to $g$.

We will denote by $\mathcal{HNA}_T(\Omega)$ the family of all normal homotopies from $\overline{\Omega} \times [0,1]$ into $\mathbb{R}^n$. The homotopy class of $f \in \mathcal{NA}_T(\Omega)$ under $\approx$ will be denoted by $[[f]]$. Finally, the set of all homotopy classes of the relation $\approx$ will be denoted by $\mathcal{NA}_T[\Omega]$.

The construction of the degree for maps in $\mathcal{A}_T(\Omega)$, which will be described in the next section, is based on the following theorem.

THEOREM 4.1. *The map* $\tau : \mathcal{NA}_T[\Omega] \to \mathcal{A}_T[\Omega]$, $[[f]] \mapsto [f]$ *is a bijection.*

*Proof.*

*Step 1.* We show that $\tau$ is a surjection.

Fix $f = (f_1, f_2) \in \mathcal{A}_T(\Omega)$. We show that there is $g \in \mathcal{NA}_T(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$. Set $d = \inf\{|f(x)| : x \in \partial\Omega\}$. Since $f$ is $T$-equivariant, $f_2(u, 0) = 0$ for every $(u, 0) \in \overline{\Omega}$. By the continuity of $f$, there is $0 < \varepsilon \le d/12$ such that if $x, y \in \overline{\Omega}$ and $|x - y| < 2\varepsilon$ then $|f_i(x) - f_i(y)| < d/6$ for $i = 1, 2$.

Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\eta(t) = 1$ for every $|t| \le \varepsilon$ and $\eta(t) = 0$ for every $|t| \ge 2\varepsilon$. Let $h : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ be defined by

$$h(u, v, t) = (1 - t\eta(|v|))f(u, v) + t\eta(|v|)(f_1(u, 0), v).$$

Then

$$\begin{aligned} h(T(u,v),t) = h(u,-v,t) &= (1 - t\eta(|v|))f(u,-v) + t\eta(|v|)(f_1(u,0),-v) \\ &= (1 - t\eta(|v|))Tf(u,v) + t\eta(|v|)T(f_1(u,0),v) = Th(u,v,t), \end{aligned}$$

for every $(u, v) \in \overline{\Omega}$ and $t \in [0, 1]$.

Take $(u, v) \in \partial\Omega$ and $t \in [0, 1]$. If $|v| \ge 2\varepsilon$ then $h(u, v, t) = f(u, v) \ne 0$. If $|v| < 2\varepsilon$ then

$$\begin{aligned} |h(u,v,t)| &= |f(u,v) - t\eta(|v|)\big(f_1(u,v) - f_1(u,0), f_2(u,v) - v\big)| \\ &\ge |f(u,v)| - t\eta(|v|)|f_1(u,v) - f_1(u,0)| - t\eta(|v|)|f_2(u,v) - v| \\ &\ge |f(u,v)| - |f_1(u,v) - f_1(u,0)| - |f_2(u,v) - v| \\ &\ge |f(u,v)| - \big(|f_1(u,v) - f_1(u,0)| + |f_2(u,v)| + |v|\big) \\ &> d - 3\frac{d}{6} = \frac{d}{2} > 0. \end{aligned}$$

In consequence, $h \in \mathcal{HA}_T(\Omega)$. Set $g := h_1$. If $|v| \le \varepsilon$ then $g(u, v) = h(u, v, 1) = (f_1(u, 0), v)$. Thus $g$ is normal.

*Step 2.* We show that $\tau$ is an injection.

Take $f = (f_1, f_2) \in \mathcal{NA}_T(\Omega)$ and $g = (g_1, g_2) \in \mathcal{NA}_T(\Omega)$ such that $f \sim g$ in $\mathcal{A}_T(\Omega)$. We prove that $f \approx g$ in $\mathcal{NA}_T(\Omega)$. Let $h = (h_I, h_{II}) \in \mathcal{A}_T(\Omega)$ be a homotopy joining $f$ to $g$ in $\mathcal{A}_T(\Omega)$. Set $d = \inf\{|h(x, t)| : x \in \partial\Omega \wedge t \in [0, 1]\}$. Since $h$ is $T$-equivariant, we get $h_{II}(u, 0, t) = 0$ for every $(u, 0) \in \overline{\Omega}$ and $t \in [0, 1]$. Take $0 < \varepsilon \le d/12$ such that $f$, $g$ are $2\varepsilon$-normal, and if $x, y \in \overline{\Omega}$ and $|x-y| < 2\varepsilon$ then $|h(x, t)-h(y, t)| < d/6$. Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth function such that $\eta(t) = 1$ for every $|t| \le \varepsilon$ and $\eta(t) = 0$ for every $|t| \ge 2\varepsilon$. Let $\hat{h} : \overline{\Omega} \times [0, 1] \to \mathbb{R}^n$ be given by

$$\hat{h}(u, v, t) = (1 - \eta(|v|))h(u, v, t) + \eta(|v|)(h_I(u, 0, t), v).$$

We check at once that $\hat{h}$ is a normal homotopy joining $f$ to $g$. ■

**5. $T$-equivariant degree.** In this section we introduce the degree of $T$-equivariant maps in $\mathbb{R}^n$, called the $T$-equivariant degree. First we define this degree for $T$-equivariant normal maps, and next for all $T$-admissible ones.

Let $\Omega \subset \mathbb{R}^n$ be a $T$-admissible set and let $f = (f_1, f_2) \in \mathcal{NA}_T(\Omega)$, where $f_1 : \overline{\Omega} \to \mathbb{R}^p$, $f_2 : \overline{\Omega} \to \mathbb{R}^q$. Set $\Omega_0 = \Omega \cap \mathbb{R}^p$. Assume that $\Omega_0 \neq \emptyset$. The map $g_0 : \overline{\Omega}_0 \to \mathbb{R}^p$ is given by $g_0(u) = f_1(u, 0)$. Since $f(u, v) \neq 0$ for all $(u, v) \in \partial\Omega$ and $f_2(u, 0) = 0$ for all $(u, 0) \in \overline{\Omega}$, we conclude that $g_0(u, v) \neq 0$ for all $(u, v) \in \partial\Omega_0$. Define

$$d_0 = \begin{cases} \deg(g_0, \Omega_0) & \text{if } \Omega_0 \neq \emptyset, \\ 0 & \text{if } \Omega_0 = \emptyset. \end{cases}$$

Since $f$ is normal, there is $\varepsilon > 0$ such that $f(x) \neq 0$ for all $x \in \partial\Omega(\varepsilon)$. Set $\Omega_1 = \Omega \setminus \overline{\Omega(\varepsilon)}$. Let us remark that $T$ acts freely on $\overline{\Omega}_1$. Define

$$g_1(x) = f(x),$$

where $x \in \overline{\Omega}_1$. It is evident that $g_1 \in \mathcal{A}_T(\Omega_1)$. By Conclusion 3.10 there exists an integer $d_1$ such that $\deg(g_1, \Omega_1) = 2d_1$. The $T$-equivariant degree of $f$ on $\Omega$ is given as follows:

$$\deg_T(f, \Omega) = (d_0, d_1) \in \mathbb{Z} \oplus \mathbb{Z}.$$

Let us denote by $\mathcal{N}$ the set of all pairs $(f, \Omega)$ such that $f \in \mathcal{NA}_T(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is $T$-admissible.

THEOREM 5.1. *The map* $\deg_T : \mathcal{N} \to \mathbb{Z} \oplus \mathbb{Z}$, $(f, \Omega) \mapsto \deg_T(f, \Omega)$, *possesses the following properties*:

1. *Homotopy invariance*:
*If* $h \in \mathcal{HNA}_T(\Omega)$ *then* $\deg_T(h_t, \Omega) = \deg_T(h_0, \Omega)$ *for every* $t \in (0, 1]$.

2. *Excision*:
*Assume that* $\Omega_0 \subset \Omega$ *is* $T$*-invariant and* $f^{-1}(\{0\}) \cap \Omega \subset \Omega_0$. *Then*

$$\deg_T(f, \Omega) = \deg_T(f_{|\Omega_0}, \Omega_0).$$

3. *Additivity*:
*Assume that* $\Omega_1$, $\Omega_2$ *are disjoint open* $T$*-invariant subsets of* $\Omega$ *such that* $f^{-1}(\{0\}) \cap \Omega \subset \Omega_1 \cup \Omega_2$. *Then*

$$\deg_T(f, \Omega) = \deg_T(f_{|\Omega_1}, \Omega_1) + \deg_T(f_{|\Omega_2}, \Omega_2).$$

4. *Existence*:
*If* $\deg_T(f, \Omega) \neq 0$ *then there exists a point* $x \in \Omega$ *such that* $f(x) = 0$.

We call $\deg_T : \mathcal{N} \to \mathbb{Z} \oplus \mathbb{Z}$ *the* $T$*-equivariant degree* of normal maps. Its properties follow directly from the definition. It is worth pointing out that if $f \in \mathcal{NA}_T(\Omega)$ then there is the following dependence between $\deg(f, \Omega)$ and $\deg_T(f, \Omega)$:

$$\deg(f, \Omega) = d_0 + 2d_1, \qquad d_0 = \deg(f, \Omega(\varepsilon)).$$

Let $\mathcal{E}$ denote the family of all pairs $(f, \Omega)$ such that $f \in \mathcal{A}_T(\Omega)$ and $\Omega$ is $T$-admissible. Applying Theorem 4.1 one can extend the $T$-equivariant degree over $\mathcal{E}$. Consider $f \in \mathcal{A}_T(\Omega)$. There exists $g \in \mathcal{NA}_T(\Omega)$ such that $[g] = [f]$. Set

$$\mathrm{Deg}_T(f, \Omega) = \deg_T(g, \Omega).$$

From Theorem 4.1 it follows that the above formula does not depend on the choice of $g$.

DEFINITION 5.1. The map $\mathrm{Deg}_T : \mathcal{E} \to \mathbb{Z} \oplus \mathbb{Z}$, $(f, \Omega) \mapsto \mathrm{Deg}_T(f, \Omega)$, is called *the $T$-equivariant degree.*

The next theorem is a natural consequence of Definition 5.1 and Theorem 5.1.

THEOREM 5.2. *The $T$-equivariant degree possesses the following properties*:

1. *If* $h \in \mathcal{HA}_T(\Omega)$ *then* $\mathrm{Deg}_T(h_t, \Omega) = \mathrm{Deg}_T(h_0, \Omega)$ *for every* $t \in (0, 1]$.
2. *Assume that* $\Omega_0 \subset \Omega$ *is $T$-invariant and* $f^{-1}(\{0\}) \cap \Omega \subset \Omega_0$. *Then*
$$\mathrm{Deg}_T(f, \Omega) = \mathrm{Deg}_T(f_{|\Omega_0}, \Omega_0).$$
3. *Assume that* $\Omega_1$ *and* $\Omega_2$ *are disjoint open $T$-invariant subsets of* $\Omega$ *such that* $f^{-1}(\{0\}) \cap \Omega \subset \Omega_1 \cup \Omega_2$. *Then*
$$\mathrm{Deg}_T(f, \Omega) = \mathrm{Deg}_T(f_{|\Omega_1}, \Omega_1) + \mathrm{Deg}_T(f_{|\Omega_2}, \Omega_2).$$
4. *If* $\mathrm{Deg}_T(f, \Omega) \neq 0$ *then there exists a point* $x \in \Omega$ *such that* $f(x) = 0$.

**6. $T$-homotopies versus gradient $T$-homotopies.** In this section we prove the Parusiński theorem in $T$-invariant case.

From now on, we assume that $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$ and $p \geq 2$. Let $B^n$ denote the open unit ball, $S^{n-1}$ the unit sphere, and $D^n$ the unit disc in $\mathbb{R}^n$ centered at 0. We have $D^n = B^n \cup S^{n-1}$. It is trivial that these sets are $T$-invariant. Set $D^p = D^n \cap \mathbb{R}^p$, $B^p = B^n \cap \mathbb{R}^p$ and $S^{p-1} = S^{n-1} \cap \mathbb{R}^p$.

Among many generalizations of the Brouwer degree there is the stable equivariant degree. It was considered by several authors (see [2, 4, 17] and the references given there). The stable equivariant degree of the $T$-equivariant continuous map $f : S^{n-1} \to S^{n-1}$ is the element $d_T(f) \in \mathbb{Z} \oplus \mathbb{Z}$ given by
$$d_T(f) = (\deg(f, S^{p-1}), \deg(f, S^{n-1})).$$
Let $[S^{n-1}, S^{n-1}]_T$ denote the set of all $T$-equivariant homotopy classes of $T$-equivariant continuous self-maps of $S^{n-1}$. Let $[f]_T$ stands for the $T$-equivariant homotopy class of $f : S^{n-1} \to S^{n-1}$. D. Ferrario proved that the stable equivariant degree $d_T$ classifies $T$-equivariant continuous self-maps of $S^{n-1}$ (see Theorem 7.1 in [4]). This means that the map
$$[S^{n-1}, S^{n-1}]_T \ni [f]_T \longmapsto d_T(f) \in \mathbb{Z} \oplus \mathbb{Z}$$
is an injection.

PROPOSITION 6.1. *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a $T$-equivariant continuous map such that* $f(S^{n-1}) \subset \mathbb{R}^n \setminus \{0\}$. *Then there exist a $T$-equivariant continuous map* $\hat{f} : \mathbb{R}^n \to \mathbb{R}^n$ *and a $T$-equivariant homotopy* $h : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ *such that*

- $h_0 = f$, $h_1 = \hat{f}$,
- $\hat{f}(S^{n-1}) \subset S^{n-1}$,
- $\hat{f}(D^n) \subset D^n$,
- $h(S^{n-1} \times [0, 1]) \subset \mathbb{R}^n \setminus \{0\}$.

*Proof.* We check at once that $\hat{f}:\mathbb{R}^n\to\mathbb{R}^n$ given by

$$\hat{f}(x)=\begin{cases}|x|\left|f\left(\frac{x}{|x|}\right)\right|^{-1}f\left(\frac{x}{|x|}\right) & \text{for } x\neq 0,\\ 0 & \text{for } x=0\end{cases}$$

and $h(x,t)=t\hat{f}(x)+(1-t)f(x)$ satisfy all the claims. ∎

Fix $f\in\mathcal{A}_T(B^n)$. Clearly, $f$ can be extended to a $T$-equivariant continuous map over $\mathbb{R}^n$. Moreover, two different extensions of $f$ are linear homotopic and the linear homotopy joining these extensions has no zeroes on $S^{n-1}\times[0,1]$. Therefore we identify $f$ with its extension. Let $\hat{f}$ be a map as in Proposition 6.1. Then $\mathrm{Deg}_T(f,B^n)=\mathrm{Deg}_T(\hat{f},B^n)$. Remark that there is one-to-one correspondence between $\mathrm{Deg}_T(f,B^n)=(d_0,d_1)$ and $d_T(\hat{f})=(\deg(\hat{f},S^{p-1}),\deg(\hat{f},S^{n-1}))$. Namely,

$$d_0=\deg(\hat{f},S^{p-1}),\qquad d_1=\frac{1}{2}\big(\deg(\hat{f},S^{n-1})-\deg(\hat{f},S^{p-1})\big).$$

Therefore the map

$$\mathcal{A}_T[B^n]\ni[f]\longmapsto d_T(\hat{f})\in\mathbb{Z}\oplus\mathbb{Z}$$

is an injection.

Conclusion 6.2. *The $T$-equivariant degree* $\mathrm{Deg}_T(f,B^n)$ *classifies $T$-admissible maps from $D^n$ into $\mathbb{R}^n$.*

Another generalization of the Brouwer degree is the $T$-equivariant degree for gradient $T$-equivariant maps from $\mathbb{R}^n$ into $\mathbb{R}^n$. This degree was considered in [7, 5, 1].

Assume that $f:\mathbb{R}^n\to\mathbb{R}^n$ is a $T$-equivariant continuous map. We say that $f$ is $\nabla_T$-admissible if $f(S^{n-1})\subset\mathbb{R}^n\setminus\{0\}$ and there exists a $T$-equivariant $C^1$ function $\varphi:\mathbb{R}^n\to\mathbb{R}$ such that $f=\nabla\varphi$. We will denote by $\nabla\mathcal{A}_T(B^n)$ the set of all $\nabla_T$-admissible maps. In the same spirit we introduce the notion of $\nabla_T$-admissible homotopy. Let $h:\mathbb{R}^n\times[0,1]\to\mathbb{R}^n$ be a $T$-equivariant homotopy. We say that $h$ is $\nabla_T$-admissible if $h(S^{n-1}\times[0,1])\subset\mathbb{R}^n\setminus\{0\}$ and there exists a $T$-equivariant $C^1$ function $\chi:\mathbb{R}^n\times[0,1]\to\mathbb{R}$ such that $h(x,t)=\nabla_x\chi(x,t)$ for all $x\in\mathbb{R}^n$ and $t\in[0,1]$.

$f$ is homotopic to $g$ in $\nabla\mathcal{A}_T(B^n)$, if there is a $\nabla_T$-admissible homotopy $h$ joining $f$ to $g$. The $\nabla_T$-admissible homotopy class of $f\in\nabla\mathcal{A}_T(B^n)$ will be denoted by $[f]_\nabla$. The set of all $\nabla_T$-admissible homotopy classes in $\nabla\mathcal{A}_T(B^n)$ will be denoted by $\nabla\mathcal{A}_T[B^n]$.

The $T$-equivariant degree of $f\in\nabla\mathcal{A}_T(B^n)$ is the element $\nabla_T\deg(f,B^n)\in\mathbb{Z}\oplus\mathbb{Z}$. From the construction made by Gęba in [5] (see formula 3.5, Theorems 3.2 and 3.3), it follows that

$$\nabla_T\deg(f,B^n)=\mathrm{Deg}_T(f,B^n).$$

Dancer, Gęba and Rybicki proved that this degree classifies $\nabla_T$-admissible maps. More precisely, the map

$$\nabla\mathcal{A}_T[B^n]\ni[f]_\nabla\longmapsto\nabla_T\deg(f,B^n)\in\mathbb{Z}\oplus\mathbb{Z}$$

is a bijection (see Corollary 4.1 and Remark 4.1 in [1]).

Conclusion 6.3. *The $T$-equivariant degree* $\mathrm{Deg}_T(f,B^n)$ *classifies $\nabla_T$-admissible maps.*

From Conclusions 6.2 and 6.3 we get a nice theorem.

THEOREM 6.4. *Assume that* $f, g \in \nabla\mathcal{A}_T(B^n)$. *If* $f$ *is homotopic to* $g$ *in* $\mathcal{A}_T(B^n)$ *then* $f$ *is homotopic to* $g$ *in* $\nabla\mathcal{A}_T(B^n)$.

*Acknowledgments.* We wish to express our sincere thanks to K. Gęba and M. Izydorek for their useful hints and comments.

## References

[1] E. N. Dancer, K. Gęba, S. M. Rybicki, *Classification of homotopy classes of equivariant gradient maps*, Fund. Math. 185 (2005), 1–18.

[2] T. tom Dieck, *Transformation Groups*, Walter de Gruyter, Berlin 1987.

[3] J. Dugundji, A. Granas, *Fixed Point Theory*, Vol. I, Monografie Matematyczne 61, PWN, Warszawa 1982.

[4] D. L. Ferrario, *On the equivariant Hopf theorem*, Topology 42 (2003), 447–465.

[5] K. Gęba, *Degree for gradient equivariant maps and equivariant Conley index*, in: Topological Nonlinear Analysis II (Frascati 1995), Progr. Nonlinear Differential Equations Appl. 27, Birkhäuser, Boston 1997, 247–272.

[6] K. Gęba, W. Krawcewicz, J. Wu, *An equivariant degree with applications to symmetric bifurcation problems.* I. *Construction of the degree*, Proc. London Math. Soc. (3) 69 (1994), 377–398.

[7] K. Gęba, I. Massabó, A. Vignoli, *On the Euler characteristic of equivariant gradient vector fields*, Boll. Un. Mat. Ital. A (7) 4 (1990), 243–251.

[8] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer Monogr. Math., Springer, New York 2003.

[9] J. Ize, *Equivariant degree*, in: Handbook of Topological Fixed Point Theory, Springer, Dordrecht 2005, 301–337.

[10] J. Ize, I. Massabó, A. Vignoli, *Degree theory for equivariant maps* I, Trans. Amer. Math. Soc. 315 (1989), 443–510.

[11] W. Krawcewicz, J. Wu, *Theory of Degrees with Applications to Bifurcation and Differential Equations*, Wiley, New York 1997.

[12] J. W. Milnor, *Topology from the Differentiable Viewpoint*, The Univ. Press of Virginia, Charlottesville 1965.

[13] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Lecture Notes, 1973–1974, Courant Institute of Mathematical Sciences, New York University, New York 1974.

[14] A. Parusiński, *Gradient homotopies of gradient vector fields*, Studia Math. 96 (1990), 73–80.

[15] G. Peschke, *Degree of certain equivariant maps into a representation sphere*, Topology Appl. 59 (1994), 137–156.

[16] M. Spivak, *Calculus on Manifolds*, PWN, Warsaw 1977 (in Polish).

[17] H. Ulrich, *Fixed Point Theory of Parametrized Equivariant Maps*, Lecture Notes in Math. 1343, Springer, Berlin 1988.