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BOUNDARY VALUE PROBLEMS FOR NONLINEAR PERTURBATIONS OF SOME ϕ-LAPLACIANS

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Abstract. This paper surveys a number of recent results obtained by C. Bereanu and the author in existence results for second order differential equations of the form

$$(\phi(u'))' = f(t,u,u')$$

submitted to various boundary conditions. In the equation, $\phi : \mathbb{R} \to]-a,a[$ is a homeomorphism such that $\phi(0) = 0$. An important motivation is the case of the curvature operator, where $\phi(s) = s/\sqrt{1+s^2}$. The problems are reduced to fixed point problems in suitable function space, to which Leray–Schauder theory is applied.

1. Introduction. In recent years much work has been devoted to the study of various boundary value problems for differential equations of the form

$$(\phi(u'))' = f(t,u,u'), \tag{1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism such that $\phi(0) = 0$. We refer to the bibliographies of [8, 9] for references. The most studied example is $\phi(s) = |s|^{p-2}s$ if $s \neq 0$, $\phi(0) = 0$, for some $p > 1$, for which (1) is a perturbation of the p-Laplacian operator. A standard technique is the reduction of the problem to a fixed point problem in a suitable function space.

Much less attention has been paid to the case of homeomorphisms ϕ with bounded range or domain. The case of a bounded domain is not too different from the case of a homeomorphism of $\mathbb{R}$, and will not be considered here. The case of a bounded range, for example $\phi(s) = s/\sqrt{1+s^2}$, which occurs in some geometric and hydrodynamical problems, is more delicate, because of the occurence of ϕ^{-1} in the fixed point operators mentioned above. As a consequence, those operators cease to be defined everywhere, leading to some difficulties in the use of Leray–Schauder degree.

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After rapidly surveying, for the sake of introduction and comparison, the reduction to fixed point problems in the case of $\phi : \mathbb{R} \to \mathbb{R}$, we concentrate on the case where $\phi : \mathbb{R} \to \,]-a, a[$, describing some recent joint work with C. Bereanu [2, 3]. After finding necessary and sufficient conditions for the solvability of the forced problem

$$(\phi(u'))' = f(t)$$

under Dirichlet, Neumann or periodic boundary conditions, we consider similar problems for (1). In the case where $|f(t, u, v)|$ is bounded by a suitable constant depending upon a and T, we show that the associated fixed point operators are defined everywhere, and the classical Leray–Schauder continuation theorem (see e.g. [10]) can be applied. The situation is different when $f(t, u, v)$ is unbounded, and we overcome the difficulty, for f bounded from below or from above and Neumann or periodic boundary conditions, by using Leray–Schauder's degree homotopy invariance with parameter dependent domain. The requested a priori estimates are obtained by extending a technique of Ward [12] for periodic solutions of semilinear equations.

2. Perturbed ϕ-Laplacian with $\phi : \mathbb{R} \to \mathbb{R}$ and equivalent fixed point problems. For $h \in L^1(0, T)$, and $\phi : \mathbb{R} \to \mathbb{R}$, a homeomorphism such that $\phi(0) = 0$, let us consider the forced ϕ-Laplacian equation

$$(\phi(u'))' = h(t) \tag{2}$$

associated with the Dirichlet

$$u(0) = 0 = u(T), \tag{3}$$

the Neumann

$$u'(0) = 0 = u'(T) \tag{4}$$

or the periodic

$$u(0) - u(T) = 0 = u'(0) - u'(T) \tag{5}$$

boundary conditions. A solution of equation (2) is a function $u \in C^1([0, T])$ such that $(\phi(u'))$ is absolutely continuous, and which satisfies (2) almost everywhere on $[0, T]$. The solution of Dirichlet or periodic problem uses the following special case of a lemma proved in [8].

Lemma 1. *For each $h \in C[0, T]$, there exists a unique $\alpha := Q_\phi(h)$ such that*

$$\int_0^T \phi^{-1}(h(s) - \alpha)\, ds = 0.$$

Furthermore $Q_\phi : C[0, T] \to \mathbb{R}$ is completely continuous.

Notice that

$$Q_I(h) = \frac{1}{T}\int_0^T h := Q(h)$$

so that Q_ϕ can be seen as an extended mean value operator associated to ϕ.

If we define

$$H : C \to C, \quad h \mapsto \int_0^{\cdot} h(s)\,ds, \quad P : C \to C, \quad h \mapsto h(0)$$

$$u_L := \min_{[0,T]} u, \quad u_M := \max_{[0,T]} u, \quad \mathrm{Osc}_{[0,T]}\, u = u_M - u_L,$$

then the following results are easily proved, using Lemma 1.

PROPOSITION 1. *Problem* (2)(3) *has, for each* $h \in L^1(0,T)$, *a unique solution given by*

$$u = H \circ \phi^{-1} \circ (I - Q_\phi) \circ H(h).$$

Problem (2)(4) *is solvable if and only if* $Qh = 0$, *in which case the solutions are given by*

$$u = Pu + H \circ \phi^{-1} \circ H(h).$$

Problem (2)(5) *is solvable if and only if* $Qh = 0$, *in which case the solutions are given by*

$$u = Pu + H \circ \phi^{-1} \circ (I - Q_\phi) \circ H(h).$$

If $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous, let us consider now the nonlinearly perturbed ϕ-Laplacian

$$(\phi(u'))' = f(t,u,u'). \tag{6}$$

We associate to f its Nemytski operator

$$N_f : C^1[0,T] \to C[0,T], \quad N_f(u) := f(\cdot, u(\cdot), u'(\cdot))$$

and introduce the Banach spaces

$$\begin{aligned} C^1_0 &:= \{u \in C^1[0,T] : u(0) = 0 = u(T)\} \\ C^1_\# &:= \{u \in C^1[0,T] : u'(0) = 0 = u'(T)\} \\ C^1_{\mathrm{per}} &:= \{u \in C^1[0,T] : u(0) - u(T) = 0 = u'(0) - u'(T)\}. \end{aligned}$$

Using Proposition 1, one can obtain the following fixed point formulations of our boundary value problems for (6). The first operator was introduced in [5], the second one in [6, 7], and the third one in [8] (see e.g. [8, 9] for details).

PROPOSITION 2. *The solutions of* (6)(3) *are the functions* $u \in C^1_0$ *such that*

$$u = H \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ N_f(u) := \mathcal{M}_0(u).$$

The solutions of (6)(4) *are the functions* $u \in C^1_\#$ *such that*

$$u = Pu + Q \circ N_f(u) + H \circ \phi^{-1} \circ H \circ (I - Q) \circ N_f(u) := \mathcal{M}_\#(u).$$

The solutions of (6)(5) *are the functions* $u \in C^1_{\mathrm{per}}$ *such that*

$$u = Pu + Q \circ N_f(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ (I - Q) \circ N_f(u) := \mathcal{M}_{\mathrm{per}}(u).$$

3. Forced ϕ-Laplacian with $\phi : \mathbb{R} \to\,]-a,a[$. Let $\phi :]-a,a[\, \to \mathbb{R}$ be a homeomorphism such that $\phi(0) = 0$. An example is given by

$$\phi(s) = \frac{s}{\sqrt{1-s^2}}$$

occurring in special relativity. One can easily check that the construction of the mapping Q_ϕ can be done like in the case of $\phi : \mathbb{R} \to \mathbb{R}$. Furthermore, as ϕ^{-1} is defined on $\mathbb{R}$, the operators $\mathcal{M}_0$, $\mathcal{M}_\#$, $\mathcal{M}_{\rm per}$ are defined everywhere. Consequently, the treatment of this situation is quite similar to the case where $\phi : \mathbb{R} \to \mathbb{R}$, and will not be considered here.

The situation is different for a homeomorphism $\phi : \mathbb{R} \to\,]-a, a[$ such that $\phi(0) = 0$. An example is given by

$$\phi(s) = \frac{s}{\sqrt{1+s^2}}\,,$$

which is associated to the one-dimensional version of mean curvature and capillary problems. The following form of Lemma 1 is proved in [3].

Lemma 2. *Let $B = \{h \in C[0,T] : \|h\|_\infty < a/2\}$. For each $h \in B$, there exists a unique $\alpha \in \mathbb{R}$ such that*

$$\int_0^T \phi^{-1}(h(s) - \alpha)\, ds = 0.$$

Moreover, if $\|h\|_\infty \le \varepsilon$, then $\alpha \in [-\varepsilon, \varepsilon]$. The function $Q_\phi : B \to \mathbb{R}$ defined by $Q_\phi(h) := \alpha$ is completely continuous.

As the mapping ϕ^{-1} is only defined on $]-a, a[$, the operators $\mathcal{M}_0$, $\mathcal{M}_\#$ and $\mathcal{M}_{\rm per}$ are not defined everywhere on their associated function space, which creates serious difficulties in the application of Leray–Schauder theory. To motivate the results for (6), we first analyze the simple case of the forced ϕ-Laplacian

$$(\phi(u'))' = f(t) \tag{7}$$

with $f \in L^1(0,T)$, submitted to Dirichlet, Neumann or periodic boundary conditions. For each $\tau \in [0,T]$, we define $F_\tau : [0,T] \to \mathbb{R}$ by

$$F_\tau(t) := \int_\tau^t f(s)\, ds, \tag{8}$$

so that

$$F_\tau(t) = F_0(t) - F_0(\tau).$$

Consider first the Neumann problem

$$(\phi(u'))' = f(t), \quad u'(0) = 0 = u'(T) \tag{9}$$

If (9) has a solution, then, integrating both members of (9) and using the boundary condition, we obtain

$$\int_0^T f(s)\, ds = 0. \tag{10}$$

Then (9) gives, for each $t \in [0,T]$,

$$\phi(u'(t)) = F_0(t), \tag{11}$$

which implies the second necessary condition for existence

$$\|F_0\|_\infty < a. \tag{12}$$

Now, if (10) and (12) hold, (11) is equivalent to

$$u'(t) = \phi^{-1}(F_0(t)),$$

and the functions u given by

$$u(t) = u(0) + \int_0^t \phi^{-1}(F_0(s))\, ds \tag{13}$$

are solutions of (9). Hence we have proved the following

PROPOSITION 3. *Problem* (9) *has a solution if and only if conditions* (10) *and* (12) *hold, in which case problem* (9) *has the family of solutions given by* (13).

When $f \in L^\infty(0,T)$, the sharp inequality

$$\|F_0\|_\infty \le \frac{T}{2}\, \|f\|_\infty$$

proved in [3] shows that condition (12) can be replaced by a condition upon f

$$\|f\|_\infty < \frac{2a}{T}\,. \tag{14}$$

EXAMPLE 1. The Neumann problem

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \alpha \cos t, \quad u'(0) = 0 = u'(\pi),$$

for which $F_0(t) = \alpha \sin t$, is solvable if and only if $|\alpha| < 1$. Condition (14) gives $|\alpha| < 2/\pi$.

EXAMPLE 2. The Neumann problem

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \alpha c(t), \quad u'(0) = 0 = u'(1)$$

with

$$c(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1/2 \\ -1 & \text{if } 1/2 < t \le 1, \end{cases}$$

for which

$$F_0(t) = \begin{cases} t & \text{if } 0 \le t \le 1/2 \\ 1/2 - t & \text{if } 1/2 < t \le 1. \end{cases}$$

is solvable if and only if $|\alpha| < 2$. Condition (14) also gives $|\alpha| < 2$.

We now consider the Dirichlet problem

$$(\phi(u'))' = f(t), \quad u(0) = 0 = u(T). \tag{15}$$

If u is a solution of (15), the boundary condition implies the existence of $\tau \in [0,T]$ such that $u'(\tau) = 0$. Consequently, for each $t \in [0,T]$,

$$\phi(u'(t)) = F_\tau(t), \tag{16}$$

which implies the necessary condition

$$\|F_\tau\|_\infty < a. \tag{17}$$

If (17) holds, (16) is equivalent to

$$u'(t) = \phi^{-1}(F_\tau)(t),$$

and the boundary conditions give

$$\int_0^T \phi^{-1}(F_\tau(s))\,ds = 0. \tag{18}$$

Now, if there exists $\tau \in [0,T]$ such that (17) and (18) hold, it is easy to check that the function u given by

$$u(t) = \int_0^t \phi^{-1}(F_\tau(s))\,ds \tag{19}$$

is a solution of (15). Hence we have proved the following

PROPOSITION 4. *Problem* (15) *has a solution if and only if there exists* $\tau \in [0,T]$ *such that* (17) *and* (18) *hold, in which case the solution is given by* (19).

It follows from the inequalities

$$\frac{1}{2}\operatorname{Osc}_{[0,T]} F_0 \le \|F_\tau\|_\infty \le \operatorname{Osc}_{[0,T]} F_0 \le \|f\|_1 \tag{20}$$

that $\operatorname{Osc}_{[0,T]} F_0 < 2a$ is necessary and $\operatorname{Osc}_{[0,T]} F_0 < a$ or $\|f\|_1 < a$ are sufficient for the solvability of (15). When $f \in L^\infty(0,T)$ $\|f\|_\infty < a/T$ is also sufficient.

EXAMPLE 3. The Dirichlet problem

$$\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \alpha, \quad u(0) = 0 = u(1),$$

for which $F_\tau(t) = \alpha(t-\tau)$, hence $\displaystyle\int_0^1 \frac{\alpha(t-\tau)}{1-\alpha^2(t-\tau)^2}\,ds = 0$ if and only if $\tau = 1/2$, and $\|\alpha(\cdot - 1/2)\|_\infty = |\alpha|/2$, is solvable if and only if $|\alpha| < 2$, with the solution

$$u(t) = \frac{1}{\alpha}\left[\sqrt{1-\frac{\alpha^2}{4}} - \sqrt{1-\alpha^2\Big(t-\frac{1}{2}\Big)^2}\,\right].$$

We finally consider the periodic problem

$$(\phi(u'))' = f(t), \quad u(0) - u(T) = u'(0) - u'(T) = 0. \tag{21}$$

If u is a solution of (21), the second boundary condition implies that

$$\int_0^T f(s)\,ds = 0 \tag{22}$$

and the first boundary condition implies the existence of $\tau \in [0,T]$ such that $u'(\tau) = 0$. Hence, for each $t \in [0,T]$,

$$\phi(u'(t)) = F_\tau(t) \tag{23}$$

which implies the second necessary condition

$$\|F_\tau\|_\infty < a. \tag{24}$$

Then (23) is equivalent to

$$u'(t) = \phi^{-1}(F_\tau(t)),$$

and the first boundary condition gives the third necessary condition

$$\int_0^T \phi^{-1}(F_\tau(s))\,ds = 0. \tag{25}$$

Conversely, if (22) holds, as well as (24) and (25) for some $\tau \in [0,T]$, it is easily checked that

$$u(t) = u(0) + \int_0^t \phi^{-1}(F_\tau(s))\,ds. \tag{26}$$

is a solution of (21). Hence we have proved the following

PROPOSITION 5. *Problem* (21) *has a solution if and only if* (22) *holds, and if there exists* $\tau \in [0,T]$ *such that* (24) *and* (25) *hold, in which case the solutions are given by* (26).

Using (20), we can replace (24) by the more explicit conditions

$$\operatorname{Osc}_{[0,T]} F_0 < \frac{a}{2}$$

and, if $f \in L^\infty(0,T)$, by

$$\|f\|_\infty < \frac{a}{2T}\,.$$

REMARK 1. The existence of $\tau \in [0,T]$ such that

$$\int_0^T \phi^{-1}(F_\tau(s))\,ds = 0$$

i.e. such that

$$\int_0^T \phi^{-1}(F_0(s) - F_0(\tau))\,ds = 0$$

is equivalent to the existence of $c \in \operatorname{Range} F_0$ such that

$$\int_0^T \phi^{-1}(F_0(s) - c)\,ds = 0,$$

to which Lemma 2 can be applied.

4. Bounded perturbations. Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and such that

$$|f(t,u,v)| \le c \tag{27}$$

for some $c \ge 0$ and all $(t,u,v) \in [0,T] \times \mathbb{R}^2$.

Let us first consider the Dirichlet problem

$$(\phi(u'))' = f(t,u,u'), \quad u(0) = 0 = u(T). \tag{28}$$

The following result is proved in [3].

THEOREM 1. *Problem* (28) *has a solution if* (27) *holds with*

$$c < \frac{a}{2T}\,. \tag{29}$$

Sketch of the proof. To use Leray–Schauder degree we introduce the homotopy

$$(\phi(u'))' = \lambda f(t,u,u'), \quad u(0) = 0 = u(T), \quad \lambda \in [0,1]. \tag{30}$$

One can verify that each solution $u \in C^1_0$ of the equation

$$u = H \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ (\lambda N_f)(u) := \mathcal{M}_1(\lambda, u), \quad \lambda \in [0,1], \tag{31}$$

is a solution of (28). Because of condition (29), $\mathcal{M}_1$ is defined and completely continuous on $[0,1] \times C^1_0$. It is not difficult to check that the set of possible solutions of (30) verifies the a priori estimate

$$\|u\| < (T+1)M$$

where

$$\|u\| = \|u\|_\infty + \|u'\|_\infty,$$

and

$$M = \max\big\{|\phi^{-1}(-2cT)|, |\phi^{-1}(2cT)|\big\}.$$

As $\mathcal{M}_1(0,\cdot) = I$, the conclusion follows from Leray–Schauder's theory (see e.g. [10]). ■

Remark 2. Returning to Example 3, we see that if $f(t) = \alpha$ and $T = 1$, Theorem 1 gives the sufficient condition $|\alpha| < 1/2$, instead of the necessary and sufficient condition $|\alpha| < 2$.

In the case of Neumann or periodic boundary conditions, the boundedness condition upon f must be supplemented by a sign condition which corresponds to the necessary condition (10) when $f = f(t)$. Let us consider the problems

$$(\phi(u'))' = f(t,u,u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T), \tag{32}$$

and

$$(\phi(u'))' = f(t,u,u'), \quad u'(0) = 0 = u'(T). \tag{33}$$

The following result of [3] improves, in the Neumann case, a result of [2].

Theorem 2. *Assume that the following conditions are satisfied.*

(B) (27) *holds with*

$$c < a/T \quad (\textit{resp. } c < 2a/T).$$

(S) *There exist $R > 0$ and $\epsilon \in \{-1,1\}$ such that*

$$\epsilon \int_0^T f(t,u(t),u'(t))\,dt > 0 \quad \textit{if } u_L \geq R,\ |u'|_\infty \leq M,$$
$$\epsilon \int_0^T f(t,u(t),u'(t))\,dt < 0 \quad \textit{if } u_M \leq -R,\ |u'|_\infty \leq M,$$
$$\textit{with } M = \max\{|\phi^{-1}(-cT)|, |\phi^{-1}(cT)|\}$$
$$(\textit{resp. } M = \max\{|\phi^{-1}(-2cT)|, |\phi^{-1}(2cT)|\}).$$

Then the problem (32) (*resp.* (33)) *has at least one solution.*

Sketch of the proof. Let us consider the periodic case, the Neumann one being similar and slightly simpler. To use Leray–Schauder degree we introduce the homotopy

$$\begin{aligned}&(\phi(u'))' = \lambda N_f(u) + (1-\lambda) Q N_f(u),\\ &u(0) - u(T) = 0 = u'(0) - u'(T), \quad \lambda \in [0,1].\end{aligned} \tag{34}$$

One can verify that each solution $u \in C^1_{\text{per}}$ of the equation

$$u = Pu + QN_f(u) + H \circ \phi^{-1} \circ (I - Q_\phi) \circ H \circ (I - Q) \circ (\lambda N_f)(u)$$
$$:= \mathcal{M}_2(\lambda, u), \quad \lambda \in [0,1] \tag{35}$$

is a solution of (32). Because of Assumption (B) and the inequality (see [2])

$$\|H(I-Q)v\|_\infty \le \frac{T}{2}\,\|v\|_\infty \quad \text{for all } v \in C[0,T],$$

we have

$$\lambda \|H(I-Q)N_f(u)\|_\infty \le \frac{cT}{2} < \frac{a}{2}\,,$$

so that, using Lemma 2, we see that $\mathcal{M}_2$ is defined and completely continuous on $[0,1] \times C^1_{\text{per}}$. If u is a possible solution of (35), we obtain, from Assumption (B),

$$\|\phi(u')\|_\infty = \lambda \|H(I-Q)N_f(u)\|_\infty \le \frac{cT}{2} < \frac{a}{2}\,,$$

which gives

$$\|u'\|_\infty < M := \max\bigl\{|\phi^{-1}(-cT/2)|, |\phi^{-1}(cT/2)|\bigr\}, \tag{36}$$

and hence

$$u_M - u_L \le TM. \tag{37}$$

Now, we also have

$$\int_0^T f(t, u(t), u'(t))\,dt = 0,$$

and Assumption (S) implies that

$$u_M > -R \quad \text{and} \quad u_L < R. \tag{38}$$

It then follows from (36), (37) and (38) that

$$\|u\| < R + (T+1)M := \rho.$$

Consequently, denoting by d_{LS} the Leray–Schauder degree and by d_{B} the Brouwer degree, we have [10]

$$\begin{aligned}
d_{\text{LS}}[I - \mathcal{M}_2(1,\cdot), B(\rho), 0] &= d_{\text{LS}}[I - \mathcal{M}_2(0,\cdot), B(\rho), 0] \\
d_{\text{LS}}[I - \mathcal{M}_2(0,\cdot), B(\rho), 0] &= d_{\text{LS}}[I - (P + QN_f), B_\rho, 0] \\
d_{\text{LS}}[I - (P + QN_f), B_\rho, 0] &= d_{\text{B}}[-QN_f, (-\rho, \rho), 0] \\
&= \frac{\operatorname{sign}(-QN_f(\rho)) - \operatorname{sign}(QN_f(-\rho))}{2}\,.
\end{aligned} \tag{39}$$

Now Assumption (S) implies that $QN_f(-\rho) \cdot QN_f(+\rho) < 0$, so that

$$\bigl|d_{\text{LS}}[I - \mathcal{M}_2(1,\cdot), B(\rho), 0]\bigr| = 1, \tag{40}$$

and the existence of a solution follows from Leray–Schauder's continuation theorem. ■

Remark 3. By using a perturbation argument, Assumption (S) can be weakened into

(S*) There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt \geq 0 \quad \text{if } u_L \geq R,\ |u'|_\infty \leq M,$$

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt \leq 0 \quad \text{if } u_M \leq -R,\ |u'|_\infty \leq M,$$

with $M = \max\{|\phi^{-1}(-cT)|, |\phi^{-1}(cT)|\}$
(resp. $M = \max\{|\phi^{-1}(-2cT)|, |\phi^{-1}(2cT)|\}$).

Notice that, for $f = f(t)$, (S*) reduces to (10).

5. Perturbations bounded from below or above. Let us first consider the Neumann problem

$$(\phi(u'))' = f(t, u, u'), \quad u'(0) = 0 = u'(T), \tag{41}$$

with $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ continuous and possibly unbounded. We use the homotopy

$$(\phi(u'))' = \lambda N_f(u) + (1 - \lambda) Q N_f(u), \quad \lambda \in [0, 1]. \tag{42}$$

One can verify that each solution of the fixed point problem in $C^1_\#$

$$u = Pu + QN_f(u) + H \circ \phi^{-1} \circ [\lambda H(I - Q) N_f](u) := \mathcal{M}_3(\lambda, u) \tag{43}$$

is a solution of (42). The operator $\mathcal{M}_3$ is well defined on the nonempty open subset of $[0, 1] \times C^1_\#$

$$\Omega := \{(\lambda, u) \in [0, 1] \times C^1_\# : \|\lambda H(I - Q) N_f(u)\|_\infty < a\}.$$

The following lemma provides a priori estimates for the possible solutions of (43) by extending a technique of Ward [12] for semilinear periodic problems. See [3] for details. For $v \in C[0, T]$, let $v^+ = \max(v, 0)$, $v^- = \max(-v, 0)$.

Lemma 3. *Assume that the following conditions hold.*

(LB) *There exists $c \in C[0, T]$ such that*

$$\|c^-\|_1 < \frac{a}{2}$$

and

$$f(t, u, v) \geq c(t)$$

for all $(t, u, v) \in [0, T] \times \mathbb{R}^2$.

(S) *There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt > 0 \quad \text{if } u_L \geq R,\ |u'|_\infty \leq M,$$

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt < 0 \quad \text{if } u_M \leq -R,\ |u'|_\infty \leq M,$$

with $M = \max\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\}$).

If $(\lambda, u) \in \Omega$ is such that $u = \mathcal{M}_3(\lambda, u)$, then

$$\|H(I-Q)N_f(u)\|_\infty \leq 2\|c^-\|_1, \quad \|u\| < R + M(T+1).$$

Sketch of the proof. If $u = \mathcal{M}_3(\lambda, u)$, then $QN_f(u) = 0$. Furthermore, Assumption (LB) implies that

$$|f(t,u,v)| \leq f(t,u,v) + 2c^-(t).$$

Consequently,

$$\begin{aligned}\|\phi(u')\|_\infty = \|\lambda H(I-Q)N_f(u)\|_\infty \leq \|N_f(u)\|_1 \\ \leq \int_0^T N_f(u)(s)ds + 2\|c^-\|_1 = 2\|c^-\|_1 < a,\end{aligned}$$

and hence

$$\|u'\|_\infty \leq M, \quad u_M - u_L \leq TM.$$

Now Assumption (S) implies that $u_M > -R$ and $u_L < R$, which gives $\|u\|_\infty < R + TM$. ■

Let now K, ρ be such that

$$2\|c^-\|_1 < K < \frac{a}{2}, \quad \rho > R + M(T+1),$$

and define

$$V := \big\{(\lambda, u) \in [0,1] \times C^1_\# : \|\lambda H(I-Q)N_f(u)\|_\infty < K, \ \|u\| < \rho\big\}.$$

$V \neq \emptyset$ is open, and $\overline{V} \subset \Omega$. Classical arguments show that $\mathcal{M}_3$ is compact on $\overline{V}$. Lemma 3 implies that $u \neq \mathcal{M}_3(\lambda, u)$ for all $(\lambda, u) \in \partial V$. Hence, we can use the generalized homotopy invariance of Leray–Schauder degree (with varying domain) (see e.g. [11]), to obtain the following existence result (see [3] for details).

Theorem 3. *Assume that the conditions* (LB) *and* (S) *of Lemma* 3 *hold. Then* (41) *has at least one solution.*

Sketch of the proof. We find, like in (39) and (40), with $V_1 = \{u \in C^1_\# : (1,u) \in V\}$,

$$|d_{\mathrm{LS}}[I - \mathcal{M}_3(1,\cdot), V_1, 0]| = \left|\frac{\operatorname{sign}(-QN_f(\rho)) - \operatorname{sign}(QN_f(-\rho))}{2}\right| = 1,$$

Consequently, $V_1 \neq \emptyset$ and there exists $u \in V_1$ such that $u = \mathcal{M}(1,u)$. ■

Remark 4. Assumption (LB) can be replaced by

(UB) There exists $c \in C[0,T]$ such that

$$\|c^+\|_1 < \frac{a}{2}$$

and

$$f(t,u,v) \leq c(t)$$

for all $(t,u,v) \in [0,T] \times \mathbb{R}^2$.

REMARK 5. Assumption (S) can be weakened into

(S*) There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt \geq 0 \quad \text{if } \min u \geq R,\ |u'|_\infty \leq M,$$

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt \leq 0 \quad \text{if } \max u \leq -R,\ |u'|_\infty \leq M,$$

$$\text{with } M = \max\bigl\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\bigr\}).$$

EXAMPLE 4. The Neumann problems, with $\alpha \in C[0, T]$ positive,

$$\left(\frac{u'}{\sqrt{1 + u'^2}}\right)' - \alpha(t) \exp u + h(t) = 0, \qquad u'(0) = 0 = u'(T),$$

$$\left(\frac{u'}{\sqrt{1 + u'^2}}\right)' + \alpha(t) \exp u - h(t) = 0, \qquad u'(0) = 0 = u'(T),$$

have at least one solution if

$$\|h^-\|_1 < \|h^+\|_1 < \frac{1}{2}\,.$$

The condition $\|h^-\|_1 < \|h^+\|_1$ is necessary.

Corresponding results can be obtained in a similar way for periodic boundary conditions (see [2] for details).

THEOREM 4. *Assume that the following conditions hold.*

(ULB) *There exists $c \in C[0, T]$ such that*

$$\|c^-\|_1 < a/4 \quad (\textit{resp. } \|c^+\|_1 < a/4) \tag{44}$$

and

$$f(t, u, v) \geq c(t) \quad (\textit{resp. } f(t, u, v) \leq c(t))$$

for all $(t, u, v) \in [0, T] \times \mathbb{R}^2$.

(S*) *There exist $R > 0$ and $\epsilon \in \{-1, 1\}$ such that*

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt \geq 0 \quad \textit{if } u_L \geq R,\ |u'|_\infty \leq M,$$

$$\epsilon \int_0^T f(t, u(t), u'(t))\, dt \leq 0 \quad \textit{if } u_M \leq -R,\ |u'|_\infty \leq M,$$

$$\textit{with } M = \max\bigl\{|\phi^{-1}(2\|c^-\|_1)|, |\phi^{-1}(-2\|c^-\|_1)|\bigr\}$$
$$(\textit{resp. } M = \max\bigl\{|\phi^{-1}(2\|c^+\|_1)|, |\phi^{-1}(-2\|c^+\|_1)|\bigr\}).$$

Then the problem

$$(\phi(u'))' = f(t, u, u'), \quad u(0) - u(T) = 0 = u'(0) - u'(T) \tag{45}$$

has at least one solution.

The proof is similar to the Neumann case and details can be found in [2].

6. Final remarks and open problems. The case of periodic solutions of *systems* of the form

$$(\phi(u'))' = f(t, u, u')$$

with $\phi : \mathbb{R}^N \to B(0, 1)$ a homeomorphism has been recently and independently considered by Benevieri, do Ó and de Medeiros [1], using a similar approach. They pay a special attention to the corresponding extension of Lemma 1 and obtain the corresponding generalization of the continuation theorem in [8].

When $\phi : \mathbb{R} \to]-a, a[$ is a *diffeomorphism* such that $\phi(0) = 0$, then u is a solution of (1) if and only if u is of class C^2 and is a solution of

$$u'' = [\phi'(u')]^{-1} f(t, u, u'). \tag{46}$$

Then the fixed point approach used here can be replaced by the use of classical continuation theorem for semilinear equations. This simplifies the treatment and also allows to improve estimate (29) of Theorem 1 into

$$c < \frac{a}{T}$$

and estimate (44) of Theorem 4 into

$$\|c^-\|_1 < a/2 \quad (\text{resp. } \|c^+\|_1 < a/2).$$

For details see [4].

Many problems remain open in this area, for example

1. Improve estimates on f when they are not sharp.
2. Study the Dirichlet problem with nonlinearities only bounded from below or from above.
3. Use a variational approach when $f = f(t, u)$.
4. Study the corresponding problems for the partial differential case

$$\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}} \right) = f(x, u, \nabla u)$$

with Dirichlet, Neumann or periodic boundary conditions.

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