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TWO GENERIC RESULTS IN FIXED POINT THEORY

SIMEON REICH and ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion—Israel Institute of Technology
32000 Haifa, Israel
E-mail: sreich@tx.technion.ac.il, ajzasl@tx.technion.ac.il

Abstract. We give two examples of the generic approach to fixed point theory. The first example is concerned with the asymptotic behavior of infinite products of nonexpansive mappings in Banach spaces and the second with the existence and stability of fixed points of continuous mappings in finite-dimensional Euclidean spaces.

1. Introduction. The asymptotic behavior of infinite products of operators finds applications in many areas of Mathematics. See, for example, [1–5, 10–12, 14, 16–24] and the references mentioned therein. Given a bounded, closed and convex subset K of a Banach space and a sequence $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$ of self-mappings of K, we are interested in the convergence properties of the sequence of products $\{A_n \cdot \ldots \cdot A_1 x\}_{n=1}^{\infty}$, where $x \in K$. In the special case of a constant sequence $\mathbf{A}$, we are led to study the asymptotic behavior of a single operator and the possible convergence of its powers to a fixed point. In their seminal 1976 paper [7], De Blasi and Myjak show that the powers of a generic nonexpansive self-mapping of K do converge. Such an approach, when a certain property is investigated for a whole space of operators and not just for a single operator, has already been successfully applied in many areas of Analysis. For instance, in two recent papers [18, 22] we have extended the De Blasi–Myjak result in several directions to certain sequence spaces of nonexpansive mappings. One of these directions has involved weak ergodicity in the sense of population biology (see [6, 13, 15, 18, 25]). More precisely, we have shown that for most (in the sense of Baire category) sequences, the distances between the corresponding (random) infinite products with different initial points tend to zero, uniformly on K. The first main result of the present paper (Theorem 2.1 below) is a generic weak ergodic theorem for infinite products of nonself-mappings which takes into account computational errors. As a matter of fact, we use in this theorem the concept of

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porosity [8, 9, 24] which refines the notion of Baire's first category.

Our second and third main results (Theorems 3.1 and 3.2 below) establish generic existence and stability of fixed points for a class of nonself-mappings defined on certain closed (but not necessarily either convex or bounded) subsets of a finite-dimensional Euclidean space. In these theorems, we endow the relevant space of mappings with two topologies, one weaker than the other. In Theorem 3.1 we find an open (in the weak topology) and everywhere dense (in the strong topology) set such that each mapping in it possesses a fixed point. In Theorem 3.2 we construct a countable intersection of open (in the weak topology) and everywhere dense (in the strong topology) sets such that each mapping in this intersection has a stable fixed point.

2. Infinite products. Let $(X, \|\cdot\|)$ be a Banach space and let $K \subset X$ be a nonempty, bounded and closed subset of X.

Denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty}$ such that each $A_t : K \to X$, $t = 1, 2, \dots$, satisfies the following two conditions:

$$\|A_t x - A_t y\| \le \|x - y\| \text{ for each } x, y \in K, \quad t = 1, 2, \dots; \tag{2.1}$$

for any $\epsilon > 0$, there exists a sequence $\{x_t\}_{t=0}^{\infty} \subset K$ such that $\|x_{t+1} - A_{t+1}x_t\| \le \epsilon$, $t = 0, 1, 2, \dots$.

It is easy to see that if K is a compact set in the norm topology, then for each $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$, there is $\{x_t\}_{t=0}^{\infty} \subset K$ such that $x_{t+1} = A_{t+1}x_t$, $t = 0, 1, \dots$. Set

$$\operatorname{rad}(K) = \sup\{\|x\| : x \in K\}. \tag{2.2}$$

Proposition 2.1. *Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$. Then for each integer $t \ge 1$ and each $x \in K$,*

$$\|A_t x\| \le 3 \operatorname{rad}(K) + 1.$$

Proof. By definition, there exists a sequence $\{x_t\}_{t=0}^{\infty} \subset K$ such that

$$\|x_{t+1} - A_{t+1}x_t\| \le 1, \quad t = 0, 1, \dots. \tag{2.3}$$

By (2.1) and (2.3), for each integer $t \ge 0$ and each $x \in K$,

$$\begin{aligned}\|A_{t+1}x\| &\le \|A_{t+1}x - A_{t+1}x_t\| + \|A_{t+1}x_t - x_{t+1}\| + \|x_{t+1}\| \\ &\le \|x - x_t\| + 1 + \|x_{t+1}\| \le 3 \operatorname{rad}(K) + 1,\end{aligned}$$

as claimed. ∎

For each $\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty} \in \mathcal{M}$, set

$$\rho\big(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big) = \sup\big\{\|A_t x - B_t x\| : x \in K \text{ and } t = 1, 2, \dots\big\}. \tag{2.4}$$

By Proposition 2.1, $\rho\big(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big)$ is finite for each pair $\{A_t\}_{t=1}^{\infty}$, $\{B_t\}_{t=1}^{\infty} \in \mathcal{M}$. Clearly, ρ is a metric on $\mathcal{M}$.

Proposition 2.2. *The metric space $(\mathcal{M}, \rho)$ is complete.*

Proof. Let $\{A_t^{(n)}\}_{t=1}^{\infty}$, $n = 1, 2, \dots$, be a Cauchy sequence in $\mathcal{M}$. Clearly, for each $x \in K$ and each integer $t \ge 1$, the sequence $\{A_t^{(n)}x\}_{t=1}^{\infty}$ is a Cauchy sequence in $(X, \|\cdot\|)$ and therefore it converges to $A_t x \in X$ in the norm topology of X. Thus

$$A_t x = \lim_{n \to \infty} A_t^{(n)} x \text{ for each point } x \in K \text{ and each integer } t \ge 1. \tag{2.5}$$

It is not difficult to see that for each integer $t \geq 1$,

$$\|A_t x - A_t y\| \leq \|x - y\| \text{ for all } x, y \in K. \tag{2.6}$$

Let $\epsilon > 0$. Since $\{A_t^{(n)}\}_{t=1}^{\infty}$, $n = 1, 2, \ldots$, is a Cauchy sequence, there exists a natural number q such that for each pair of integers $m, n \geq q$,

$$\|A_t^{(m)} x - A_t^{(n)} x\| \leq \epsilon/8 \tag{2.7}$$

for all integers $t \geq 1$ and all points $x \in K$. By (2.7) and (2.5),

$$\|A_t^{(m)} x - A_t x\| \leq \epsilon/8 \tag{2.8}$$

for each integer $m \geq q$, each integer $t \geq 1$, and each $x \in K$. Since $\{A_t^{(q)}\}_{t=1}^{\infty} \in \mathcal{M}$, there is a sequence $\{x_t\}_{t=0}^{\infty} \subset K$ such that

$$\|x_{t+1} - A_{t+1}^{(q)} x_t\| \leq \epsilon/4, \quad t = 0, 1, 2, \ldots. \tag{2.9}$$

In view of (2.9) and (2.8), for each integer $t \geq 0$,

$$\|x_{t+1} - A_{t+1} x_t\| \leq \|x_{t+1} - A_{t+1}^{(q)} x_t\| + \|A_{t+1}^{(q)} x_t - A_{t+1} x_t\| \leq \epsilon/4 + \epsilon/8 < \epsilon/2.$$

Since ϵ is an arbitrary positive number, we conclude that

$$\{A_t\}_{t=1}^{\infty} \in \mathcal{M}.$$

In view of (2.8),

$$\rho\big(\{A_t\}_{t=1}^{\infty}, \{A_t^{(m)}\}_{t=1}^{\infty}\big) \leq \epsilon/8 \text{ for each integer } m \geq q.$$

This completes the proof of Proposition 2.2. ∎

Denote by $\mathcal{E}$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ for which there exists a sequence $\{x_t\}_{t=1}^{\infty} \subset K$ such that

$$A_{t+1} x_t = x_{t+1} \text{ for all integers } t \geq 0.$$

PROPOSITION 2.3. *The set $\mathcal{E}$ is an everywhere dense subset of $\mathcal{M}$.*

Proof. Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and $\epsilon > 0$. By definition, there exists a sequence $\{x_t\}_{t=0}^{\infty} \subset K$ such that

$$\|A_{t+1} x_t - x_{t+1}\| \leq \epsilon/4, \quad t = 0, 1, 2 \ldots.$$

For each $t = 1, 2, \ldots$, define

$$B_t x = A_t x - A_t x_{t-1} + x_t, \quad x \in K.$$

It is not difficult to see that

$$B_t x_{t-1} = x_t, \quad t = 1, 2, \ldots, \quad \{B_t\}_{t=1}^{\infty} \in \mathcal{E}$$

and

$$\rho(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \leq \epsilon.$$

Proposition 2.3 is proved. ∎

Before stating our first main result we recall the notion of porosity [8, 9, 24].

Let (Y, d) be a complete metric space. We denote by $B(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called *porous* in (Y, d) if there exist $\alpha \in (0, 1)$

and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y, r) \setminus E.$$

A subset of the space Y is called *σ-porous* in (Y, d) if it is a countable union of porous subsets in (Y, d).

Since porous sets are nowhere dense, all σ-porous sets are of the first Baire category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure zero. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure zero and are of the first category.

THEOREM 2.1. *There exists a set $\mathcal{F} \subset \mathcal{M}$ such that $\mathcal{M} \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}, \rho)$ and such that for each $\{A_t\}_{t=1}^{\infty} \in \mathcal{F}$, the following property holds*:

(P1) *For each $\epsilon > 0$, there exist $\delta > 0$ and a natural number T_0 such that if the integers $m_1, m_2 > T_0$ and if $\{x_t\}_{t=0}^{m_1}$, $\{y_t\}_{t=0}^{m_2} \subset K$ satisfy*

$$\begin{aligned} \|x_{t+1} - A_{t+1}x_t\| &\le \delta, \quad t = 0, \ldots, m_1 - 1, \\ \|y_{t+1} - A_{t+1}y_t\| &\le \delta, \quad t = 0, \ldots, m_2 - 1, \end{aligned} \tag{2.10}$$

then

$$\|x_t - y_t\| \le \epsilon, \quad t = T_0 + 1, \ldots, \min\{m_1, m_2\}. \tag{2.11}$$

Proof. For each integer $n \ge 1$, denote by $\mathcal{F}_n$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ for which the following property holds:

(P2) there exist $\delta > 0$ and a natural number T_0 such that if the integers $m_1, m_2 > T_0$ and $\{x_t\}_{t=0}^{m_1}$, $\{y_t\}_{t=0}^{m_2} \subset K$ satisfy (2.10), then

$$\|x_t - y_t\| \le 1/n, \quad t = T_0 + 1, \ldots, \min\{m_1, m_2\}.$$

Set

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n.$$

It is not difficult to see that for each $\{A_t\}_{t=1}^{\infty} \in \mathcal{F}$, property (P1) holds.

In order to complete the proof of the theorem, it is sufficient to show that for each integer $n \ge 1$, $\mathcal{M} \setminus \mathcal{F}_n$ is a porous subset of $(\mathcal{M}, \rho)$.

Indeed, let $n \ge 1$ be an integer. Choose a positive number

$$\alpha < (2^{10}n)^{-1}(\mathrm{rad}(K) + 1)^{-1}. \tag{2.12}$$

Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and $r \in (0, 1]$. By Proposition 2.3, there exists $\{B_t\}_{t=1}^{\infty} \in \mathcal{E}$ such that

$$\rho\big(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big) \le r/16. \tag{2.13}$$

By the definition of $\mathcal{E}$, there exists $\{x_t\}_{t=0}^{\infty} \subset K$ such that

$$B_{t+1}x_t = x_{t+1}, \quad t = 0, 1, 2, \ldots. \tag{2.14}$$

Set

$$\gamma = 32n\alpha r,$$

and let $t \ge 1$ be an integer. For each $x \in K$, set

$$C_t x = \gamma x_t + (1 - \gamma)B_t x. \tag{2.15}$$

It is easy to see that $\{C_t\}_{t=1}^{\infty} \in \mathcal{M}$ and that

$$C_t x_{t-1} = x_t, \quad t = 1, 2 \dots . \tag{2.16}$$

By (2.15), (2.2) and Proposition 2.1,

$$\begin{aligned}\rho(\{C_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) &\le \sup\{\|C_t z - B_t z\| : z \in K \text{ and } t = 1, 2, \dots\} \\ &= \sup\{\gamma \|x_t - B_t z\| : t = 1, 2, \dots \text{ and } z \in K\} \le \gamma(4\,\mathrm{rad}(K) + 1).\end{aligned} \tag{2.17}$$

Choose a natural number T_0 such that

$$T_0 \alpha r > \mathrm{rad}(K) + 1 \tag{2.18}$$

and a positive number

$$\delta < \min\{\alpha r, (8n)^{-1} T_0^{-1}\}/2. \tag{2.19}$$

Assume that

$$\{D_t\}_{t=1}^{\infty} \in \mathcal{M} \quad \text{and} \quad \rho(\{D_t\}_{t=1}^{\infty}, \{C_t\}_{t=1}^{\infty}) \le \alpha r. \tag{2.20}$$

By (2.20), (2.17), (2.13) and (2.15),

$$\begin{aligned}&\rho\big(\{D_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}\big) \\ &\quad \le \rho\big(\{D_t\}_{t=1}^{\infty}, \{C_t\}_{t=1}^{\infty}\big) + \rho\big(\{C_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}\big) + \rho\big(\{B_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}\big) \\ &\quad \le \alpha r + 4\gamma(\mathrm{rad}(K) + 1) + r/16 \le \alpha r + 32 n \alpha r \cdot (4\,\mathrm{rad}(K) + 1) + r/16 \\ &\qquad\qquad \le r\big[8 \cdot 32 n \alpha (\mathrm{rad}(K) + 1) + 1/16\big] \le r/2.\end{aligned}$$

Thus

$$\rho\big(\{D_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}\big) \le r/2. \tag{2.21}$$

Assume that the integers $m_1, m_2 > T_0$ and that $\{x_t\}_{t=0}^{m_1}$, $\{y_t\}_{t=0}^{m_2} \subset K$ satisfy

$$\begin{aligned}\|x_{t+1} - D_{t+1} x_t\| &\le \delta, \quad t = 0, \dots, m_1 - 1, \\ \|y_{t+1} - D_{t+1} y_t\| &\le \delta, \quad t = 0, \dots, m_2 - 1.\end{aligned} \tag{2.22}$$

We now show that

$$\|x_t - y_t\| \le 1/n, \quad t = T_0 + 1, \dots, \min\{m_1, m_2\}. \tag{2.23}$$

Indeed, let an integer T satisfy

$$T_0 + 1 \le T \le \min\{m_1, m_2\}. \tag{2.24}$$

We claim that there is an integer $j \in \{T - T_0, \dots, T - 1\}$ such that

$$\|x_j - y_j\| \le (4n)^{-1}. \tag{2.25}$$

Let us assume the converse. Then for each $j = T - T_0, \dots, T - 1$,

$$\|x_j - y_j\| > (4n)^{-1}. \tag{2.26}$$

It follows from (2.22), (2.20), (2.25) and (2.1) that for $j = T - T_0, \dots, T-1$,

$$\begin{aligned}\|x_{j+1} - y_{j+1}\| &\le \|x_{j+1} - D_{j+1}x_j\| + \|D_{j+1}x_j - D_{j+1}y_j\| + \|D_{j+1}y_j - y_{j+1}\| \\ &\le \delta + \|D_{j+1}x_j - D_{j+1}y_j\| + \delta \\ &\le 2\delta + 2\rho(\{D_t\}_{t=1}^\infty, \{C_t\}_{t=1}^\infty) + \|C_{j+1}x_j - C_{j+1}y_j\| \\ &\le 2\delta + 2\alpha r + (1-\gamma)\|B_{j+1}x_j - B_{j+1}y_j\| \\ &\le 2\delta + 2\alpha r + (1-\gamma)\|x_j - y_j\|.\end{aligned}$$

By this inequality, (2.26), (2.19) and (2.15),

$$\|x_j - y_j\| - \|x_{j+1} - y_{j+1}\| \ge \gamma\|x_j - y_j\| - 2\delta - 2\alpha r \ge \gamma(4n)^{-1} - 4\alpha r \ge 4\alpha r.$$

Together with (2.2) and (2.18), this implies that

$$2\,\mathrm{rad}(K) \ge \|x_{T-T_0} - y_{T-T_0}\| - \|x_T - y_T\| \ge T_0 4\alpha r > 4\,\mathrm{rad}(K) + 4.$$

The contradiction we have reached demonstrates that indeed there is an integer $j \in \{T - T_0, \dots, T-1\}$ such that (2.25) holds.

By (2.1) and (2.22), for each integer i satisfying $j \le i \le T-1$,

$$\begin{aligned}\|x_{i+1} - y_{i+1}\| \le \|x_{i+1} - D_{i+1}x_i\| + \|D_{i+1}x_i - D_{i+1}y_i\| + \|D_{i+1}y_i - y_{i+1}\| \\ \le \delta + \|x_i - y_i\| + \delta.\end{aligned}$$

When combined with (2.25) and (2.19), this relation implies that

$$\|x_T - y_T\| \le \|x_j - y_j\| + 2\delta(T-j) \le \|x_j - y_j\| + 2\delta T_0 \le (4n)^{-1} + (4n)^{-1}.$$

Therefore

$$\|x_T - y_T\| \le (2n)^{-1}$$

for all integers T satisfying (2.24). Thus we have shown that each sequence $\{D_t\}_{t=1}^\infty \in \mathcal{M}$ satisfying (2.20) possesses property (P2) and consequently belongs to $\mathcal{F}_n$. Therefore $\mathcal{M} \setminus \mathcal{F}_n$ is indeed a porous subset of $(\mathcal{M}, \rho)$. This completes the proof of Theorem 2.1. ■

3. Existence and stability. Let $K \subset R^n$ be a nonempty, closed subset of the n-dimensional Euclidean space $(R^n, \|\cdot\|)$. We suppose that K is the closure of its nonempty interior $\mathrm{int}(K)$.

For each $x \in R^n$ and each $r > 0$, set $B(x,r) = \{y \in R^n : \|x - y\| \le r\}$ and fix $\theta \in K$.

Denote by $\mathcal{M}$ the set of all continuous mappings $A : K \to R^n$. We equip the space $\mathcal{M}$ with the uniformity determined by the base

$$\mathcal{E}_w(N,\epsilon) = \big\{(A,B) \in \mathcal{M} \times \mathcal{M} : \|Ax - Bx\| \le \epsilon \text{ for all } x \in B(\theta, N) \cap K\big\}, \tag{3.1}$$

where $N, \epsilon > 0$.

Clearly, the space $\mathcal{M}$ with this uniformity is metrizable and complete. We equip the space $\mathcal{M}$ with the topology induced by this uniformity. This topology will be called the *weak topology*.

We also equip the space $\mathcal{M}$ with the uniformity determined by the base

$$\mathcal{E}_s(\epsilon) = \big\{(A,B) \in \mathcal{M} \times \mathcal{M} : \|Ax - Bx\| \le \epsilon \text{ for all } x \in K\big\}, \tag{3.2}$$

where $\epsilon > 0$. Clearly, the space $\mathcal{M}$ with this uniformity is also metrizable and complete. The topology induced by this uniformity on $\mathcal{M}$ will be called the *strong topology.*

Denote by $\mathcal{M}_f$ the set of all $A \in \mathcal{M}$ which have approximate fixed points. In other words, the set $\mathcal{M}_f$ consists of all $A \in \mathcal{M}$ such that

$$\inf\{\|x - Ax\| : x \in K\} = 0. \tag{3.3}$$

It is clear that $\mathcal{M}_f$ is a closed subset of $\mathcal{M}$ with the strong topology.

Note that if the set K is bounded, then $\mathcal{M}_f$ consists of all those elements of $\mathcal{M}$ which have fixed points. Every self-mapping of K which is a strict contraction, that is, has a Lipschitz constant strictly less than one, clearly belongs to $\mathcal{M}_f$.

If K is bounded and convex, and a continuous mapping $A : K \to R^n$ satisfies the Leray–Schauder condition with respect to $w \in \mathrm{int}(K)$, that is, $Ay - w \neq m(y - w)$ for all y on the boundary of K and $m > 1$, then it also belongs to $\mathcal{M}_f$. If such an A is a strict contraction, then this continues to be true even if K is neither bounded nor convex.

We endow the topological subspace $\mathcal{M}_f \subset M$ with both the relative weak and strong topologies.

Theorem 3.1. *Let $\gamma \in (0,1)$. There exists an open (in the weak topology), everywhere dense (in the strong topology) set $\mathcal{F}_\gamma \subset \mathcal{M}_f$ such that for each $A \in \mathcal{F}_\gamma$, there are $x_A \in \mathrm{int}(K)$, $r_A \in (0,1)$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that*

$$B(x_A, r_A) \subset K \quad \textit{and} \quad Ax_A = x_A,$$

and for each $C \in \mathcal{U}$, there is $x_C \in K$ such that $Cx_C = x_C$ and $\|x_C - x_A\| \le \gamma r_A$.

Theorem 3.2. *There exists a set $\mathcal{F} \subset \mathcal{M}_f$ which is a countable intersection of open (in the weak topology), everywhere dense (in the strong topology) subsets of $\mathcal{M}_f$ such that for each $A \in \mathcal{F}$ and each $\gamma \in (0,1)$, there exist $x_A \in \mathrm{int}(K)$, $r_A \in (0,1)$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that*

$$B(x_A, r_A) \subset K \quad \textit{and} \quad Ax_A = x_A,$$

and for each $C \in \mathcal{U}$, there is $x_C \in K$ such that $Cx_C = x_C$ and $\|x_C - x_A\| \le \gamma r_A$.

Example. Let $n = 1$, $K = \bigcup_{j=0}^{\infty}[2j, 2j+1]$, and define, for each integer $j \ge 1$ and each $x \in [2j, 2j+1]$, $Ax = x + 2^{-j}$. Clearly, $\inf\{|x - Ax| : x \in K\} = 0$, but A is fixed point free.

4. Auxiliary results. Denote by $\mathcal{E}$ the set of all $A \in \mathcal{M}_f$ for which there exist

$$x_A \in \mathrm{int}(K) \quad \text{and} \quad r_A \in (0,1) \tag{4.1}$$

such that

$$B(x_A, r_A) \subset K \quad \text{and} \quad Ay = x_A \text{ for all } y \in B(x_A, r_A/4). \tag{4.2}$$

Lemma 4.1. *The set $\mathcal{E}$ is an everywhere dense subset of $\mathcal{M}_f$ with the strong topology.*

Proof. Let $A \in \mathcal{M}_f$ and $\epsilon > 0$. By the definition of $\mathcal{M}_f$ (see (3.3)), there exists $x_0 \in K$ such that

$$\|Ax_0 - x_0\| < \epsilon/16. \tag{4.3}$$

Since K is the closure of $\mathrm{int}(K)$ and A is continuous, there is $x_1 \in \mathrm{int}(K)$ such that

$$\|x_1 - x_0\| < \epsilon/16 \quad \text{and} \quad \|Ax_1 - Ax_0\| < \epsilon/16. \tag{4.4}$$

Set

$$A_1 y = Ay - Ax_1 + x_1, \quad y \in K. \tag{4.5}$$

Clearly, $A_1 \in \mathcal{M}$. In view of (4.5),

$$A_1 x_1 = x_1. \tag{4.6}$$

By (4.5), (4.4) and (4.3), for each $y \in K$,

$$\|Ay - A_1 y\| = \|Ax_1 - x_1\| \le \|Ax_1 - Ax_0\| + \|Ax_0 - x_0\| + \|x_0 - x_1\| < 3\epsilon/16. \tag{4.7}$$

Since A_1 has a fixed point (see (4.6)), it is clear that $A_1 \in \mathcal{M}_f$. Since A_1 is continuous and $x_1 \in \mathrm{int}(K)$, there exists $r_1 \in (0,1)$ such that

$$B(x_1, r_1) \subset K \quad \text{and} \quad \|A_1 x - A_1 x_1\| \le \epsilon/16 \text{ for all } x \in B(x_1, r_1). \tag{4.8}$$

Define

$$\psi(t) = \begin{cases} 1, & t \in [0, r_1/2], \\ 2(r_1 - t) r_1^{-1}, & t \in (r_1/2, r_1), \\ 0, & t \in [r_1, \infty), \end{cases} \tag{4.9}$$

and

$$By = \psi(\|y - x_1\|) x_1 + \big(1 - \psi(\|y - x_1\|)\big) A_1 y, \quad y \in K. \tag{4.10}$$

Clearly, $B \in \mathcal{M}$. It follows from (4.10) and (4.9) that for each $y \in B(x_1, r_1/2)$,

$$By = x_1. \tag{4.11}$$

Therefore $B \in \mathcal{E}$. We will now show that

$$\|By - Ay\| \le \epsilon \text{ for all } x \in K.$$

Indeed, let $y \in K$. There are two cases to be considered:

$$\|x_1 - y\| \le r_1; \tag{4.12}$$

$$\|x_1 - y\| > r_1. \tag{4.13}$$

If (4.13) holds, then (4.13), (4.10), (4.9) and (4.7) imply that

$$By = A_1 y \quad \text{and} \quad \|By - Ay\| = \|A_1 y - Ay\| < \epsilon/4. \tag{4.14}$$

Let (4.12) hold. Then by (4.12), (4.10), (4.9), (4.6) and (4.8),

$$\|By - A_1 y\| = \big\|\psi(\|y - x_1\|)(x_1 - A_1 y)\big\| \le \|x_1 - A_1 y\| = \|A_1 x_1 - A_1 y\| < \epsilon/16.$$

When combined with (4.7), this inequality implies that

$$\|By - Ay\| \le \|By - A_1 y\| + \|A_1 y - Ay\| \le \epsilon/16 + 3\epsilon/16 = \epsilon/4.$$

Thus

$$\|By - Ay\| \le \epsilon/4 \text{ for all } y \in K.$$

This completes the proof of Lemma 4.1. ∎

LEMMA 4.2. *Let $A \in \mathcal{E}$, $x_A \in \mathrm{int}(K)$, $r_A \in (0,1)$ satisfy (4.2), and let $\gamma \in (0,1)$. Then there exists a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that for each $B \in \mathcal{U}$, there is $x_B \in K$ such that $\|x_B - x_A\| \le \gamma r_A/4$ and $Bx_B = x_B$.*

Proof. Set

$$\Delta = \gamma r_A/4 \tag{4.15}$$

and put

$$\mathcal{U} = \big\{B \in \mathcal{M}_f : \|Bz - Az\| \le \Delta \text{ for each } z \in B(x_A, r_A)\big\}. \tag{4.16}$$

Clearly, $\mathcal{U}$ is a neighborhood of A in $\mathcal{M}_f$ with the weak topology.

Let $B \in \mathcal{U}$. It follows from (4.16), (4.2) and (4.15) that for each $z \in B(x_A, \gamma r_A/4)$,

$$\|Bz - x_A\| \le \|Bz - Az\| + \|Az - x_A\| \le \Delta + \|Az - x_A\| = \Delta = \gamma r_A/4.$$

Thus

$$B\big(B(x_A, \gamma r_A/4)\big) \subset B(x_A, \gamma r_A/4).$$

Since the mapping B is continuous, there is $x_B \in B(x_A, \gamma r_A/4)$ such that

$$Bx_B = x_B.$$

Lemma 4.2 is proved. ∎

5. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1. Let $A \in \mathcal{E}$. There exist $x_A \in \mathrm{int}(K)$ and $r_A \in (0,1)$ such that (4.2) holds. By Lemma 4.2, there exists an open neighborhood $\mathcal{U}(A)$ of A in $\mathcal{M}_f$ with the weak topology such that the following property holds:

(P3) For each $B \in \mathcal{U}(A)$, there is $x_B \in K$ such that

$$Bx_B = x_B \quad \text{and} \quad \|x_B - x_A\| \le \gamma r_A/8. \tag{5.1}$$

Set

$$\mathcal{F}_\gamma = \bigcup\{\mathcal{U}(A) : A \in \mathcal{E}\}. \tag{5.2}$$

By Lemma 4.1, $\mathcal{F}_\gamma$ is an open (in the weak topology), everywhere dense (in the strong topology) subset of $\mathcal{M}_f$.

Let $B \in \mathcal{F}_\gamma$. By (5.2), there is $A \in \mathcal{E}$ such that

$$B \in \mathcal{U}(A). \tag{5.3}$$

By property (P3), for each $C \in \mathcal{U}(A)$, there is $x_C \in K$ such that

$$Cx_C = x_C \quad \text{and} \quad \|x_C - x_A\| \le \gamma r_A/8. \tag{5.4}$$

Clearly,

$$\|x_B - x_A\| \le \gamma r_A/8. \tag{5.5}$$

It follows from (5.5) and (4.2) that

$$B(x_B, r_A/2) \subset B(x_A, r_A) \subset K. \tag{5.6}$$

By (5.4) and (5.5), for each $C \in \mathcal{U}(A)$,

$$\|x_C - x_B\| \le \|x_C - x_A\| + \|x_A - x_B\| \le \gamma r_A/8 + \gamma r_A/8 = \gamma r_A/4.$$

This completes the proof of Theorem 3.1. ∎

Proof of Theorem 3.2. For each integer $n \geq 1$, let $\mathcal{F}_n$ be as guaranteed in Theorem 3.1 with $\gamma = (2n)^{-1}$. Set

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \mathcal{F}_n. \tag{5.7}$$

Clearly, $\mathcal{F}$ is a countable intersection of open (in the weak topology), everywhere dense (in the strong topology) subsets of $\mathcal{M}_f$.

Let $A \in \mathcal{F}$ and $\gamma \in (0, 1)$. Choose a natural number n such that

$$n^{-1} < \gamma/8. \tag{5.8}$$

Since $A \in \mathcal{F}_n$ and the assertion of Theorem 3.1 holds with $\gamma = (2n)^{-1}$ and $\mathcal{F}_\gamma = \mathcal{F}_n$, there are $x_A \in \mathrm{int}(K)$, $r_A \in (0, 1)$, and a neighborhood $\mathcal{U}$ of A in $\mathcal{M}_f$ with the weak topology such that $B(x_A, r_A) \subset K$, $Ax_A = x_A$, and for each $C \in \mathcal{U}$ there is $x_C \in K$ such that $Cx_C = x_C$ and

$$\|x_C - x_A\| \leq r_A(2n)^{-1} < r_A\gamma.$$

Thus Theorem 3.2 is also established. ∎

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