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FIXED POINT THEOREMS FOR SET-VALUED Y-CONTRACTIONS

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Abstract. The purpose of this paper is to present several fixed point theorems for the so-called set-valued Y-contractions. Set-valued Y-contractions in ordered metric spaces, set-valued graphic contractions, set-valued contractions outside a bounded set and set-valued operators on a metric space with cyclic representations are considered.

1. Introduction

1.1. *Basic notions and notation.* Throughout this paper, the standard notation and terminology in nonlinear analysis are used. For the convenience of the reader we recall some of them.

Let (X,d) be a metric space and $f : X \to X$ an operator. By $\operatorname{Fix} f := \{x \in X : x = f(x)\}$ we will denote the fixed point set of the operator f and by $\tilde{B}(x_0, r)$ the closed ball centered in $x_0 \in X$ with radius $r > 0$.

We will also use the following symbols:

$$P(X) := \{Y \subset X : Y \text{ is nonempty}\}, \qquad P_{\mathrm{cl}}(X) := \{Y \in P(X) : Y \text{ is closed}\},$$
$$P_{\mathrm{cp}}(X) := \{Y \in P(X) : Y \text{ is compact}\}, \quad P_{\mathrm{b}}(X) := \{Y \in P(X) : Y \text{ is bounded}\}.$$

In normed spaces, $P_{\mathrm{cv}}(X) := \{Y \in P(X) : Y \text{ is convex}\}$.

Let A and B be nonempty subsets of the metric space (X,d). The *gap* between these sets is

$$D(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}.$$

In particular, $D(x_0, B) = D(\{x_0\}, B)$ (where $x_0 \in X$) is called the *distance* from the point x_0 to the set B.

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The *Pompeiu–Hausdorff generalized distance* between the nonempty closed subsets A and B of the metric space (X,d) is defined by the formula

$$H(A,B) := \max\{\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\}.$$

If A and B are nonempty and bounded subsets of the metric space (X,d), then one defines

$$\delta(A,B) := \sup\{d(a,b) : a \in A,\ b \in B\}.$$

The symbol $T : X \multimap Y$ means $T : X \to P(Y)$, i.e., T is a set-valued operator from X to Y. We will denote by $\mathbb{M}^0(X,Y)$ the set of all set-valued operators $T : X \multimap Y$. If $X = Y$ then $\mathbb{M}^0(Y) := \mathbb{M}^0(Y,Y)$. For $K \in P(X)$, $T(K) := \bigcup_{x\in K} T(x)$ will denote the *image* of the set K, while $\mathrm{Graf}(T) := \{(x,y) \in X\times Y : y \in T(x)\}$ is the *graph* of T. Recall that the set-valued operator is called *closed* if $\mathrm{Graf}(T)$ is closed in $X \times Y$. If X and Y are Hausdorff topological spaces then $T^+(U) := \{x \in X : T(x) \subset U\}$, for $U \in P(Y)$. The set-valued operator $T : X \to P(Y)$ is said to be *upper semi-continuous* on X (briefly *u.s.c.*) if and only if $T^+(U)$ is open in X for each open subset U of Y.

For $T : X \to P(X)$ the set of all nonempty invariant subsets of X will be denoted by $I(T)$, i.e. $I(T) := \{Y \in P(X) : T(Y) \subset Y\}$. Also $\mathrm{Fix}(T) := \{x \in X : x \in T(x)\}$ denotes the *fixed point set* of the set-valued operator T, while $\mathrm{SFix}(T) := \{x \in X : \{x\} = T(x)\}$ is the *strict fixed point set* of T.

If (X,d) is a metric space, recall that $T : X \to P_{\mathrm{cl}}(X)$ is called a *set-valued a-contraction* if $a \in\ ]0,1[$ and $H(T(x_1),T(x_2)) \le a\cdot d(x_1,x_2)$, for each $x_1,x_2 \in X$. In the same setting, an operator $T : X \to P_{\mathrm{cl}}(X)$ is a *multivalued weakly Picard operator* (briefly *MWP* operator) (see [25]) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$;

(iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

A sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying the condition (ii) from the previous definition is called the *sequence of successive approximations of T starting from $x_0 \in X$*.

(Here $\mathbb{N}$ denotes the set of all nonnegative integers. The set of all positive integers will be denoted by $\mathbb{N}^*$.)

Let X be a nonempty set. By definition (see [20]), the triple $(X,S(X),M^0)$ is a *fixed point structure* (briefly *f.p.s.*) if:

(i) $S(X) \subset P(X)$, $S(X) \neq \emptyset$;

(ii) $M^0 : P(X) \multimap \bigcup_{Y\in P(X)} \mathbb{M}^0(Y)$, with $Y \multimap M^0(Y) \subset \mathbb{M}^0(Y)$ is a mapping such that if $Z \subset Y$, $Z \neq \emptyset$ then $M^0(Z) \supset \{T_{|_Z} : T \in M^0(Y),\ Z \in I(T)\}$;

(iii) every $Y \in S(X)$ has the fixed point property with respect to $M^0(Y)$.

Let X be a nonempty set. Define $s(X) := \{(x_n)_{n\in\mathbb{N}} : x_n \in X,\ n \in \mathbb{N}\}$.

Let $c(X) \subset s(X)$ be a subset of $s(X)$ and $\mathrm{Lim} : c(X) \to X$ an operator. By definition the triple $(X,c(X),\mathrm{Lim})$ is called an *L-space* (Fréchet [4]) if the following conditions are satisfied:

(i) If $x_n = x$ for each $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $\mathrm{Lim}(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition an element of $c(X)$ is a convergent sequence and $x := \mathrm{Lim}(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we write $x_n \to x$ as $n \to \infty$.

In what follow we denote an L-space by $(X, \to)$. Hausdorff topological spaces, metric spaces, generalized metric spaces ($d(x,y) \in \mathbb{R}^m_+$, or $d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$, or $d(x,y) \in K$, K a cone in an ordered Banach space, or $d(x,y) \in E$, E an ordered linear space with a notion of linear convergence, etc.), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are examples of L-spaces.

1.2. *The aim of the paper.* The purpose of this work is to consider the following open question of I. A. Rus [21]:

I. *Let (X,d) be a metric space and $Y \subseteq X \times X$. The operator $T : X \to P_{\mathrm{cl}}(X)$ is called a* set-valued (Y,a)-contraction *if*

$$a \in \,]0,1[\ \textit{ and } H(T(x_1),T(x_2)) \le a \cdot d(x_1,x_2) \textit{ for each } (x_1,x_2) \in Y. \tag{1.1}$$

Construct a fixed point theory for set-valued (Y,a)-contractions.

Of course, in the previous definition, one can imagine generalized type conditions for T, such as

$$\text{there exists } \varphi : \mathbb{R}^5_+ \to \mathbb{R}_+ \text{ such that}$$
$$H(T(x_1),T(x_2)) \le \varphi\big(d(x_1,x_2), D(x_1,T(x_1)), D(x_2,T(x_2)), D(x_1,T(x_2)), D(x_2,T(x_1))\big)$$
$$\text{for each } (x_1,x_2) \in Y. \tag{1.2}$$

In connection with the above problem one can consider

II. *Let (X,d) be a metric space and $Y \subseteq X \times X$. The operator $T : X \to P_{\mathrm{b}}(X)$ is called a* set-valued (Y,a)-δ-contraction *if*

$$a \in \,]0,1[\ \textit{ and } \delta(T(x_1),T(x_2)) \le a \cdot d(x_1,x_2) \textit{ for each } (x_1,x_2) \in Y. \tag{1.3}$$

Construct a fixed point theory for set-valued (Y,a)-δ-contractions.

III. *Let (X,d) be a metric space and $Y \subseteq X \times X$. The operator $T : X \to P_{\mathrm{cl}}(X)$ is called a* set-valued (Y,a)-Gap-contraction *if*

$$a \in \,]0,1[\ \textit{ and } D(T(x_1),T(x_2)) \le a \cdot d(x_1,x_2) \textit{ for each } (x_1,x_2) \in Y. \tag{1.4}$$

Construct a fixed point theory for set-valued (Y,a)-Gap-contractions.

Obviously, as in case I, generalized type conditions on T can be also considered.

The following examples are in connection with the above problems.

EXAMPLE 1.1. Let (X,d) be a metric space and $Y := X \times X$. Then a set-valued (Y,a)-contraction is a set-valued a-contraction ([1], [2], [3], [6], [8], [11], [16], [21], [26], etc.).

EXAMPLE 1.2. Let (X,d) be a metric space, $T : X \to P_{\mathrm{cl}}(X)$ and $Y := \mathrm{Graf}(T)$. Then a set-valued (Y,a)-contraction is a set-valued a-graphic contraction ([17], [18]).

Example 1.3. Let (X,d) be a metric space, $x_0 \in X$ and $r > 0$. Consider $Y := \tilde{B}(x_0,r) \times \tilde{B}(x_0,r)$. Then for $T : X \to P_{\mathrm{cl}}(X)$ the set-valued (Y,a)-contraction condition means that the following Frigon–Granas type condition (see [5], [12]) holds:

$$\text{there is } a \in \,]0,1[\; \text{ such that } H(T(x),T(y)) \le a \cdot d(x,y) \text{ for all } x,y \in \tilde{B}(x_0,r). \quad (1.5)$$

Example 1.4. Let (X,d) be a metric space, $Z \in P_{\mathrm{b}}(X)$ and $Y := (X \setminus Z) \times (X \setminus Z)$. Then a set-valued (Y,a)-contraction is a set-valued a-contraction outside a bounded subset, see [21], [19].

Example 1.5. Let $(X,d,\le)$ be an ordered metric space. Consider $Y := \{(x,y) \in X \times X : x \le y \text{ or } y \le x\}$. Then $T : X \to P_{\mathrm{cl}}(X)$ is a set-valued (Y,a)-contraction if (see also [15], [13]) if

$$a \in \,]0,1[\; \text{ and } H(T(x),T(y)) \le a \cdot d(x,y), \text{ for all } x,y \in X, x \le y. \quad (1.6)$$

Example 1.6. Let (X,d) be a metric space, $T : X \to P_{\mathrm{cl}}(X)$ and $A_i \in P(X)$, $i \in \{1,2,\dots,m\}$. Suppose that $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation with respect to the set-valued operator T, i.e. $T(A_i) \subset A_{i+1}$, $i \in \{1,2,\dots,m\}$ (where $A_{m+1} = A_1$).

Consider $Y := \bigcup_{i=1}^{m} (A_i \times A_{i+1})$. Then T is a set-valued (Y,a)-contraction ([9], [23]) if

$$a \in \,]0,1[\; \text{ and } H(T(x),T(y)) \le a \cdot d(x,y) \quad \text{for all } x \in A_i \text{ and all } y \in A_{i+1},\ i \in \{1,2,\dots,m\}. \quad (1.7)$$

2. Set-valued (Y,a)-contractions in ordered metric spaces. Let $(X,\le)$ be a partially ordered set. Let $X_{\le} := \{(x,y) \in X \times X : x \le y \text{ or } y \le x\}$. Also, if $x,y \in X$ with $x \le y$, then by $[x,y]_{\le}$ we will denote the ordered segment joining x and y, i.e. $[x,y]_{\le} := \{z \in X : x \le z \le y\}$.

Definition 2.1. Let X be a nonempty set. Then, by definition $(X,d,\le)$ is an *ordered metric space* if and only if:

(i) (X,d) is a metric space;

(ii) $(X,\le)$ is a partially ordered set;

(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \le y_n$ for each $n \in \mathbb{N} \Rightarrow x \le y$.

The main result of this section is

Theorem 2.2. *Let $(X,d,\le)$ be an ordered complete metric space and $T : X \to P_{\mathrm{cl}}(X)$ a set-valued operator. Suppose that the following assertions hold*:

(i) *there exists $x_0 \in X$ such that if $y \in T(x_0)$ then $(x_0,y) \in X_{\le}$;*

(ii) *for each $x,y \in X$, with $(x,y) \notin X_{\le}$, there exists $c(x,y) \in X$ such that $(x,c(x,y))$ and $(y,c(x,y))$ belong to $X_{\le}$;*

(iii) *$(x,y) \in X_{\le}$ implies ($u \in T(x)$ and $v \in T(y)$ then $(u,v) \in X_{\le}$);*

(iv) *T is a closed set-valued operator*;

(v) *T is a set-valued $(X_{\le},a)$-contraction.*

Then for each $x \in X$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations of T starting from x that converges to a fixed point of T.

Proof. Let $q > 1$ and $x_1 \in T(x_0)$ be arbitrary. Then there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) \le q \cdot H(T(x_0), T(x_1))$. From (i) we have $(x_0, x_1) \in X_{\le}$. Using (v) we obtain $H(T(x_0), T(x_1)) \le a \cdot d(x_0, x_1)$. As a consequence, $d(x_1, x_2) \le q \cdot a \cdot d(x_0, x_1)$. Taking (iii) into account we get $(x_1, x_2) \in X_{\le}$. Inductively we can construct a sequence $(x_n)_{n\in\mathbb{N}}$ having the following two properties:

(a) $x_{n+1} \in T(x_n)$ for $n \in \mathbb{N}$;

(b) $d(x_{n+1}, x_n) \le q \cdot a \cdot d(x_n, x_{n-1})$ for $n \in \mathbb{N}^*$.

If we choose $q \in \left]1, a^{-1}\right[$ then, by standard methods, we obtain the convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to some $x^* \in X$. From (iv) it follows that $x^* \in F_T$. Hence $\mathrm{Fix}(T) \neq \emptyset$.

We will prove now that for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(z_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from $(x, y) \in \mathrm{Graf}(T)$ such that (z_n) converges to a fixed point of T.

Case A. Let $z_0 \in X$ such that $(x_0, z_0) \in X_{\le}$. Then there exists $z_1 \in T(z_0)$ such that $d(x_1, z_1) \le q \cdot H(T(x_0), T(z_0)) \le q \cdot a \cdot d(x_0, z_0)$. Since $(x_1, z_1) \in X_{\le}$ we can continue this procedure and obtain a sequence $(z_n)_{n\in\mathbb{N}}$ such that $z_{n+1} \in T(z_n)$ for $n \in \mathbb{N}$ and $d(x_{n+1}, z_{n+1}) \le q \cdot a \cdot d(x_n, z_n)$. Choosing $q \in \left]1, a^{-1}\right[$ we deduce that $(z_n) \to x^* \in T(x^*)$.

Case B. Let $z_0 \in X$ such that $(x_0, z_0) \notin X_{\le}$. Then, from (ii) there exists $c(x_0, z_0) \in X$ such that

$$(x_0, c(x_0, z_0)) \in X_{\le} \tag{1}$$

and

$$(z_0, c(x_0, z_0)) \in X_{\le}. \tag{2}$$

From (1), using the method from Case A, we can construct a sequence of successive approximations for T starting from $c(x_0, z_0)$ which converges to x^*. From (2) and the above conclusion, we deduce in the same manner as in Case A that there exists a sequence (z_n) of successive approximations for T starting from z_0 such that $(z_n) \to x^*$. This finishes the proof. ■

REMARK 2.3. It is an open question whether T satisfying the hypotheses of Theorem 2.2 is a MWP operator.

3. Set-valued graphic contractions. Let (X, d) be a complete metric space and $T : X \to P_b(X)$ be a set-valued operator. If there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$\delta(T(x), T(y)) \le a \cdot d(x, y) + b \cdot \delta(x, T(x)) + c \cdot \delta(y, T(y)) \text{ for each } x \in X \text{ and } y \in T(x),$$

then the problem is to study when $\mathrm{Fix}(T) = \mathrm{SFix}(T) \neq \emptyset$.

In connection with the above problem we have:

EXAMPLE 3.1. Let (X, d) be a metric space and $X = X_1 \cup \ldots \cup X_m$ be a partition of X. For each $i \in \{1, 2, \ldots, m\}$, let $x_i \in X_i$ be given elements. Consider $T : X \to P_b(X)$

defined by $T(x) := \{x_i\}$, if $x \in X_i$. Then T satisfies the above condition and $\mathrm{Fix}(T) = \mathrm{SFix}(T) = \{x_1, \ldots, x_m\}$.

The first result of this section is the following strict fixed point theorem for a set-valued operator satisfying to a certain contractive type condition on its graph.

THEOREM 3.2. *Let (X, d) be a complete metric space and $T : X \to P_b(X)$ be a closed set-valued operator. Suppose that*

there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that

$$\delta(T(x), T(y)) \le a \cdot d(x, y) + b \cdot \delta(x, T(x)) + c \cdot \delta(y, T(y)) \quad \text{for each } (x, y) \in \mathrm{Graf}(T). \tag{3.1}$$

Then $\mathrm{Fix}(T) = \mathrm{SFix}(T) \neq \emptyset$.

Proof. Let $q > 1$ and $x_0 \in X$ be arbitrary. Then there exists $x_1 \in T(x_0)$ such that $\delta(x_0, T(x_0)) \le q \cdot d(x_0, x_1)$. We have $\delta(x_1, T(x_1)) \le \delta(T(x_0), T(x_1)) \le a \cdot d(x_0, x_1) + b \cdot \delta(x_0, T(x_0)) + c \cdot \delta(x_1, T(x_1)) \le ad(x_0, x_1) + bqd(x_0, x_1) + c \cdot \delta(x_1, T(x_1))$. Hence $\delta(x_1, T(x_1)) \le \frac{a+bq}{1-c} \cdot d(x_0, x_1)$. By this procedure, we can obtain the sequence $(x_n)_{n\in\mathbb{N}}$ having the property $d(x_n, x_{n+1}) \le (\frac{a+bq}{1-c})^n \cdot d(x_0, x_1)$ for each $n \in \mathbb{N}$. If we choose $q > \frac{b}{1-a-c}$ then $\frac{a+bq}{1-c} < 1$. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete metric space (X, d). Denote by x^* the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. Since $\mathrm{Graf}(T)$ is a closed set in $X \times X$ we obtain the first conclusion $x^* \in T(x^*)$.

Let us establish now the relation $\mathrm{Fix}(T) = \mathrm{SFix}(T)$. It is enough to prove that $\mathrm{Fix}(T) \subset \mathrm{SFix}(T)$. For, let $x \in \mathrm{Fix}(T)$ be arbitrary. Then, using the hypothesis (with $y = x \in T(x)$) we get successively: $\delta(T(x)) \le (b+c) \cdot \delta(x, T(x)) \le (b+c) \cdot \delta(T(x))$. Suppose, by absurdum, that $\mathrm{card}\, T(x) > 1$. Then $\delta(T(x)) > 0$ and using the above relation we get $1 \le b + c$, a contradiction. Hence $\delta(T(x)) = 0$ and so $\{x\} = T(x)$. ∎

REMARK 3.3. Theorem 3.2 is an extension of some results given in S. Reich [16] and I. A. Rus [21].

Moreover, condition (3.1) can be replaced with a more general one, namely: there exist $a \in [0, 1[$ such that

$$\delta(T(x), T(y)) \le a \cdot \delta(x, T(x)) \text{ for each } (x, y) \in \mathrm{Graf}(T),$$

since $d(x, y) \le \delta(x, T(x))$ and $\delta(y, T(y)) \le \delta(T(x), T(y))$.

Next, we present a strict fixed point theorem.

THEOREM 3.4. *Let (X, d) be a complete metric space, and $T : X \to P_{\mathrm{b}}(X)$ be a set-valued operator. Suppose that*

there exist $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that for each $x \in X$

$$\text{there exists } y \in T(x) \text{ with } \delta(y, T(y)) \le a \cdot d(x, y) + b \cdot \delta(x, T(x)). \tag{3.2}$$

If the map $f : X \to \mathbb{R}_+$, defined by $f(x) := \delta(x, T(x))$, is lower semicontinuous, then $\mathrm{SFix}(T) \neq \emptyset$.

Proof. From (3.2), for each $x \in X$ there is a $y \in T(x)$ such that $\delta(y, T(y)) \le (a+b) \times \delta(x, T(x))$. Then, for each $x_0 \in X$ we can construct inductively a sequence $(x_n)_{n\in\mathbb{N}}$ of successive approximations for T starting from x_0, having the property $\delta(x_n, T(x_n)) \le (a+b)^n \cdot \delta(x_0, T(x_0))$. Hence, we will obtain $d(x_n, x_{n+1}) \le \delta(x_n, T(x_n)) \to 0$ as $n \to +\infty$. As a consequence, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Denote by $x^* \in X$ the limit of this sequence.

If we denote $f(x_n) := \delta(x_n, T(x_n))$, then using the lower semicontinuity of f we can write:

$$0 \le f(x^*) \le \liminf_{n\to+\infty} f(x_n) = 0.$$

So, $f(x^*) = 0$ and the conclusion $\{x^*\} = T(x^*)$ follows. ∎

Remark 3.5. If, instead of the lower semicontinuity of f, we suppose that $\mathrm{Graf}(T)$ is closed, then, since $(x_n)_{n\in\mathbb{N}}$ is a sequence of successive approximations for T, we immediately get that $x^* \in T(x^*)$. So, the conclusion of the above result is $\mathrm{Fix}(T) \neq \emptyset$. It is an open question if the above fixed point is a strict fixed point for T.

Remark 3.6. In Theorem 3.4 condition (3.2) can be replaced with a more general one: there exists $a \in [0,1[$ such that for each $x \in X$ there exists $y \in T(x)$ with $\delta(y, T(y)) \le a \cdot \delta(x, T(x))$, since again $d(x,y) \le \delta(x, T(x))$.

Remark 3.7. An open problem is to construct a fixed point theory for set-valued Y-contractions in K-metric spaces (see [27]).

4. Set-valued contractions outside a bounded set. Let X be a Banach space and $T : X \to P_b(X)$. By definition (see [10], [7]), the operator T is called *quasi-bounded* if there exist $m, M > 0$ such that

$$\|y\| \le m \cdot \|x\| + M \text{ for each } (x,y) \in \mathrm{Graf}(T). \tag{4.1}$$

The number

$$|T| := \inf\{m > 0 : \text{there exists } M > 0 \text{ such that relation (4.1) holds}\},$$

is called the *quasi-norm* of T.

Let us denote by α_K the Kuratowski measure of noncompactness on X and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a comparison function (i.e. φ is non-decreasing and $(\varphi^n(t))_{n\in\mathbb{N}}$ goes to 0, for each $t \ge 0$ as $n \to +\infty$). An operator $T : X \to P_b(X)$ is said to be an (α_K, φ)-*contraction* if T is a bounded operator, φ is a comparison function and $\alpha_K(T(A)) \le \varphi(\alpha_K(A))$ for each $A \in P_b(X) \cap I(T)$.

The first main result of this section is

Theorem 4.1. *Let X be a Banach space, $Z \in P_b(X)$ and $T : X \to P_{\mathrm{cp,cv}}(X)$. Suppose that the following assertions hold*:

(i) *T is u.s.c. and compact* (*i.e. T sends bounded sets into relatively compact sets*);

(ii) *there exists $a \in]0,1[$ such that*

$$H(T(x_1), T(x_2)) \le a \cdot \|x_1 - x_2\| \textit{ for each } (x_1, x_2) \in (X \setminus Z) \times (X \setminus Z).$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

Proof. From (ii) the operator T is quasi-bounded with the quasi-norm $|T| = a < 1$. For, consider first $x \in Z$. Then $\|T(x)\| := \sup_{y \in T(x)} \|y\| \leq \|T(Z)\| < +\infty$. If $x \in X \setminus Z$ then consider an arbitrary but fixed $x_0 \in X \setminus Z$. We have $\|T(x)\| = H(T(x), \{0\}) \leq H(T(x), T(x_0)) + H(T(x_0), \{0\}) \leq a \cdot \|x - x_0\| + \|T(x_0)\| \leq a \cdot \|x\| + a \cdot \|x_0\| + \|T(x_0)\|$. Hence, for all $x \in X$ we get $\|T(x)\| \leq a \cdot \|x\| + \max(a \cdot \|x_0\| + \|T(x_0)\|, \|T(Z)\|)$. From Lemma 2.1 in [10] (p. 59) there exists $R > 0$ such that $T(\widetilde{B}(0, R)) \subset \widetilde{B}(0, R))$. If we take into account that $T(\widetilde{B}(0, R))$ is an invariant subset for T, the proof follows now from the Bohnenblust–Karlin fixed point theorem (see [20] and the references therein). ∎

A more general result of this type is

THEOREM 4.2. *Let X be a Banach space, $Z \in P_b(X)$ and $T : X \to P_{\mathrm{cp,cv}}(X)$. Suppose that the following assertions hold*:

(i) *T is an u.s.c. (α_K, φ)-contraction*;

(ii) *there exists $a \in]0, 1[$ such that*

$$H(T(x_1), T(x_2)) \leq a \cdot \|x_1 - x_2\| \text{ for each } (x_1, x_2) \in (X \setminus Z) \times (X \setminus Z).$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

Proof. The proof runs in a similar way to the above one, with the only difference that, instead of the Bohnenblust–Karlin fixed point theorem, we use Theorem 4.7 in [20]. ∎

In terms of the fixed point structures we can prove the following abstract result.

THEOREM 4.3. *Let $(X, S(X), M^0)$ be a f.p.s. on a Banach space X, $T : X \multimap X$ and $Z \in P_b(X)$. We suppose that*

(i) $\widetilde{B}(0, R)) \in S(X)$ *for each* $R > 0$;

(ii) $T \in M^0(Y)$ *for each* $Y \in S(X)$;

(iii) *there exists $a \in]0, 1[$ such that*

$$H(T(x_1), T(x_2)) \leq a \cdot \|x_1 - x_2\| \text{ for each } (x_1, x_2) \in (X \setminus Z) \times (X \setminus Z).$$

Then $\mathrm{Fix}(T) \neq \emptyset$.

The above considerations give rise to the following open question:

PROBLEM 4.4. *Which generalized set-valued contractions $T : X \to P_{\mathrm{b,cl,cv}}(X)$ are quasi-bounded with the quasi-norm $|T| < 1$?*

For the single-valued case see [24] (p. 20) and the references therein (M. C. Anisiu (1983), F. Aldea (2002)).

5. Cyclic representation of an invariant subset with respect to a set-valued operator. Let us observe first that the results in Kirk, Srinivasan, Veeramani [9] and I. A. Rus [23] give rise to the following concept:

DEFINITION 5.1. Let X be a nonempty set and $T : X \to P(X)$ a set-valued operator. By definition, $X = \bigcup_{i=1}^{m} X_i$ (where $X_i \subset X$ for each $i \in \{1, 2, \ldots, m\}$) is a *cyclic representation* of X with respect to T if $T(X_1) \subset X_2, \ldots, T(X_{m-1}) \subset X_m, T(X_m) \subset X_1$.

EXAMPLE 5.2. Let X be a nonempty set and $T : X \to P(X)$ a set-valued operator. We suppose that there exist $A \subset X$ and $n_0 \in \mathbb{N}^*$ such that $T^{n_0}(A) \subset A$. Then $B := \bigcup_{i=1}^{n_0-1} T^i(A)$ is a cyclic representation of B with respect to T. Moreover $B \in I(T)$.

REMARK 5.3. If $X := \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to $T : X \to P(X)$, then X_i $(i \in \{1, 2, \dots, m\})$ are invariant subsets for T^m.

The following abstract result is important in our considerations.

THEOREM 5.4. *Let $(X, \to)$ be an L-space, $T : X \to P(X)$ a set-valued operator and $X := \bigcup_{i=1}^{m} X_i$ be a cyclic representation of X with respect to T. Suppose that*

(i) *$X_i \in P_{\mathrm{cl}}(X)$ for each $i \in \{1, 2, \dots, m\}$;*

(ii) *there exists a convergent sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in X$, $x_{n+1} \in T(x_n)$ for each $n \in \mathbb{N}$.*

Then $\bigcap_{i=1}^{m} X_i \neq \emptyset$.

Proof. Denote by x^* the limit of the sequence $(x_n)_{n\in\mathbb{N}}$. Because $x_n \in \bigcup_{i=1}^{m} X_i$ for each $n \in \mathbb{N}$, it follows that infinitely many elements of this sequence lie in each X_i, $i \in \{1, 2, \dots, m\}$. From (i) one sees that $x^* \in X_i$, $i \in \{1, 2, \dots, m\}$. Hence $\bigcap_{i=1}^{m} X_i \neq \emptyset$. ■

As consequence of the above result we have

THEOREM 5.5. *Let $(X, S(X), M^0)$ be a fixed point structure, where $(X, \to)$ is an L-space. Let $A_i \in P_{\mathrm{cl}}(X)$ for each $i \in \{1, 2, \dots, m\}$. Define $Y := \bigcup_{i=1}^{m} A_i$ and consider $T : Y \to P(Y)$. Suppose that*

(i) *$Y := \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;*

(ii) *there exists a convergent sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n \in X$, $x_{n+1} \in T(x_n)$ for each $n \in \mathbb{N}$;*

(iii) *if $A := \bigcap_{i=1}^{m} A_i \neq \emptyset$ then $A \in S(X)$ and $T_{|A} \in M(A)$.*

Then $\mathrm{Fix}(T) \neq \emptyset$.

Proof. From (i), (ii) and Theorem 5.4 it follows that $A \neq \emptyset$. From the definition of the fixed point structure and using the assertion (iii) we have $\mathrm{Fix}(T) \neq \emptyset$. ■

THEOREM 5.6. *Let $(X, S(X), M^0)$ be a fixed point structure, where X is a nonempty set. Let $A_i \in P(X)$ for each $i \in \{1, 2, \dots, m\}$. Define $Y := \bigcup_{i=1}^{m} A_i$ and consider $T : Y \to P(Y)$. Suppose that*

(i) *$(A_i)_{i\in\{1,2,\dots,m\}}$ is a cyclic representation of Y with respect to T;*

(ii) *$A_i \in S(X)$ for some $i \in \{1, 2, \dots, m\}$;*

(iii) *$G_1, G_2 \in M^0$ implies $G_1 \circ G_2 \in M^0$.*

Then $\mathrm{Fix}(T^m) \neq \emptyset$.

Proof. Let, for example, $A_i \in S(X)$. From Definition 5.1 and (iii) it follows that $T^m \in M^0(A_i)$. The proof follows now from Remark 5.3. ∎

Some applications of the above results can be obtained by taking in Theorems 5.5 and 5.6 some particular fixed point structures (see I. A. Rus [20]).

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