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OPEN SET OF EIGENVALUES AND SVEP

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The single-valued extension property (SVEP) plays an important role in the local spectral theory. A bounded linear operator T acting on a complex Banach space X is said to have SVEP if there is no analytic function $f : U \to X$ defined on a nonempty open subset U of the complex plane such that $f \not\equiv 0$ and $(T - z)f(z) \equiv 0$ $(z \in U)$.

By definition, every operator without SVEP has a nonempty open set consisting of eigenvalues. So if the point spectrum of an operator has empty interior, then the operator has automatically SVEP.

On the other hand, it is easy to construct an operator T with SVEP such that the interior of the point spectrum $\sigma_p(T)$ is nonempty. As a simple example, see [LN], p. 15, let $\mathbb{D} := \{z \in \mathbb{C};\ |z| < 1\}$ be the open unit disc, let X be the Banach space of all bounded complex-valued functions on $\mathbb{D}$ with the supremum norm and let $T \in B(X)$ be the operator of multiplication by the independent variable z. It is easy to see that T has SVEP and $\sigma_p(T) = \mathbb{D}$. Moreover, T is decomposable.

Note that the Banach space in the last example is non-separable. At first glance it seems that a separable Banach space is too "small" for the existence of an operator T with SVEP and with nonempty interior of the point spectrum. This motivates the following question which was raised at the Workshop on Operator Theory in Warsaw, 2004.

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Problem. Does there exist a separable Banach space and a bounded linear operator with SVEP on it such that the interior of the point spectrum of the operator is not empty?

The aim of this note is to give a positive answer to this question.

Theorem. *There exists a separable Banach space X and a bounded linear operator $T \in B(X)$ with SVEP such that the interior of the point spectrum of T is not empty.*

Proof. Let m be the restriction of the planar Lebesgue measure to the disc $\mathbb{D}$, let $L^1(m)$ be the Banach space of all complex valued functions on $\mathbb{D}$ that are absolutely integrable with respect to m, and let $A^1(m)$ be the Bergman space of all functions in $L^1(m)$ that are analytic on $\mathbb{D}$. It is well known that $A^1(m)$ is the closure of all polynomials [HKZ], p. 4.

Define a linear operator M on $L^1(m)$ by $(Mf)(z) := zf(z) \quad (f \in L^1(m))$. It is easy to see that M is a bounded linear operator on $L^1(m)$ with $\|M\| \le 1$. Let $X = L^1(m)/A^1(m)$ and let T be the quotient operator induced by M on X. We claim that T is an operator with SVEP and that $\operatorname{Int}\sigma_p(T) \neq \emptyset$.

Let us check the second assertion first. Choose and fix $\lambda \in \mathbb{D}$. Since the constant function 1 is not in $(z-\lambda)A^1(m)$ the function $f_\lambda(z) := (z-\lambda)^{-1}$ cannot be in $A^1(m)$. On the other hand, it is straightforward that $f_\lambda \in L^1(m)$. Since

$$(T-\lambda)(f_\lambda + A^1(m)) = 1 + A^1(m) = 0$$

in X, we conclude that $f_\lambda + A^1(m) \in X$ is a nonzero eigenvector of T at λ. This proves the inclusion $\mathbb{D} \subseteq \operatorname{Int}\sigma_p(T)$. Since $\|T\| \le 1$, we have $\sigma(T) = \overline{\mathbb{D}}$.

Suppose that T does not have SVEP. Then there exist $\lambda \in \mathbb{D}$, an open disc $U \subseteq \mathbb{D}$ centered at λ, and a nonzero analytic function $F:\ U \to X$ such that $(T-z)F(z) = 0$ for all $z \in U$. Let $F(z) = \sum_{n=0}^{\infty} x_n(z-\lambda)^n$ be the Taylor expansion of F in U. Then the following relations hold between the coefficients

$$(T-\lambda)x_0 = 0 \qquad \text{and} \qquad (T-\lambda)x_n = x_{n-1} \quad (n \ge 1).$$

Since F is nontrivial, some of the coefficients are nonzero; without loss of generality we may assume that $x_0 \neq 0$. Thus,

$$(T-\lambda)^2 x_1 = (T-\lambda)x_0 = 0 \qquad \text{and} \qquad (T-\lambda)x_1 = x_0 \neq 0.$$

Let $f \in L^1(m)$ be such that $x_1 = f + A^1(m)$. Then there are $\alpha \in \mathbb{C}$ and $h \in A^1(m)$ that satisfy the equality

$$(z-\lambda)^2 f(z) = \alpha + (z-\lambda)h(z)$$

for almost all $z \in \mathbb{D}$. It follows that

$$f(z) = \alpha(z-\lambda)^{-2} + (z-\lambda)^{-1}h(z),$$

which is possible only if $\alpha = 0$ since $f(z)$ and $(z-\lambda)^{-1}h(z)$ are in $L^1(m)$ and $(z-\lambda)^{-2}$ is not in $L^1(m)$. We get

$$(T-\lambda)x_1 = (T-\lambda)(f + A^1(m)) = h + A^1(m) = 0$$

in X, a contradiction. Therefore T has SVEP. ∎

Note that the operator M from the above proof is decomposable. Therefore T has the decomposition property (δ). It is unknown whether or not T is decomposable. However, it is true that T is not spectral because there do not exist spectral operators on separable Banach spaces whose point spectrum has nonempty interior. Indeed, assume that S is a spectral operator on a separable Banach space X with spectral measure E which is bounded by a constant $c > 0$. See chapter XV, section 2 of [DS] for details about spectral measures and spectral operators. If $\lambda \neq \mu$ are eigenvalues of S and x and y are the corresponding eigenvectors, then $E(\{\lambda\})(x-y) = x$ and therefore $\|x\| \leq c\|x-y\|$. Let $\{x_n\}_{n=1}^{\infty} \subset X$ be a dense subset that exists by separability. It follows that in any ball centered at x_n with radius $1/2c$, there exists at most one eigenvector with norm 1, i.e., there are at most countably many eigenvalues of S.

References

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