ON THE SPECTRUM AND EIGENFUNCTIONS
OF THE OPERATOR \((Vf)(x) = \int_0^x f(t)dt\)

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1. Introduction. It is well known that the Volterra operator \(V : f \rightarrow \int_0^x f(t)dt\) defined on \(L^p(0,1) (C[0,1])\) is quasinilpotent, that is, \(\sigma(V) = \{0\}\). It was pointed out in [5]–[6] that the operator

\[(1) V_\phi : f \rightarrow \int_0^{\phi(x)} f(t)dt\]

which is a composition of integration and substitution with \(\phi \in C[0,1]\) is quasinilpotent on \(C[0,1]\) if \(\phi(x) \leq x\) for all \(x \in [0,1]\).

Let \(\phi : [0,1] \rightarrow [0,1]\) be a measurable function and let \(V_\phi : L^p(0,1) \rightarrow L^p(0,1) (1 \leq p < \infty)\) be defined by (1). It was proved in [12]–[13] that \(V_\phi\) is quasinilpotent on \(L^p(0,1)\) if and only if \(\phi(x) \leq x\) for almost all \(x \in [0,1]\). It was also noted in [13] and proved in [14] that the spectral radius of \(V_{x^\alpha}\) defined on \(L^p(0,1)\) or \(C[0,1]\) is \(1 - \alpha\) \((0 < \alpha < 1)\).

We note also paper [4], where the hypercyclicity of \(V_{x^\alpha}\) was proved on some Fréchet space.

In this note we find the spectrum of \(V_{x^\alpha}\) defined on \(L^2(0,1)\) and investigate some properties of its eigenfunctions.

Notations. Let \(X\) be a Banach space and let \(T\) be a bounded operator on \(X\). Then \(\ker T := \{x \in X : Tx = 0\}\) denotes a kernel of \(T\) and \(R(T) := \{Tx : x \in X\}\) denotes a range of \(T\). \(I\) denotes the identity operator on \(X\); \(\text{span} E\) denotes the closed linear span of the set \(E \subset X\); \(\mathbb{1}\) denotes the function \(f \equiv 1\) in \(L^2(0,1)\); \(\mathbb{Z}_+ := \{0,1,2,\ldots\}\). For simplicity we set \(\sum_{k=n}^m a_k := 0\) if \(n > m\).

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2. Auxiliary results. The following two Lemmas are well known. For the sake of completeness, proofs are given.

**Lemma 1.** The system $\{(\ln x)^n\}_{n=0}^{\infty}$ is complete in $L^2(0, 1)$.

**Proof.** Since the Laguerre functions $f_n(x) := e^{-x/2}\frac{1}{\pi}e^{x/2}x^n e^{-x} (n \in \mathbb{Z}_+)$ form [1] an orthonormal basis in $L^2(0, \infty)$, the system $\{x^n e^{-x/2}\}_{n=0}^{\infty}$ is complete in $L^2(0, \infty)$. Let the operator $T : L^2(0, \infty) \to L^2(0, 1)$ be defined by

$$(Tf)(x) := \frac{f(-\ln x)}{x^{1/2}}.$$ 

It is easily proved that $T$ is a surjective isometry. Thus the system $\{T(x^n e^{-x/2})\}_{n=0}^{\infty} = \{(\ln x)^n\}_{n=0}^{\infty}$ is complete in $L^2(0, 1)$.

**Remark 1.** Consider an operator $C : L^2(0, 1) \to L^2(0, 1)$ defined by $(Cf)(x) = f(x) - \int_x^1 \frac{f(t)}{t} \, dt$. It is well known [2] that $C$ is a simple unilateral shift. Since $\ker C^* = \{c \cdot 1 : c \in \mathbb{C}\}$, it follows [8] that the set $\{C^n 1\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(0, 1)$. It can easily be checked that $(C^n 1)(x) = P_n(ln x)$, where $P_n$ is a polynomial of degree $n$. Thus $L^2(0, 1) = \text{span}\{(\ln x)^n : n \geq 0\}$.

**Lemma 2.** Let $A$ be a compact operator defined on a Hilbert space $H$, $Af_n = \lambda_n f_n$ and $\text{span}\{f_n : n \geq 1\} = H$. Then

1) $\sigma_p(A) = \{\lambda_n\}_{n=1}^{\infty}$;

2) if $\lambda_i \neq \lambda_j$ for $i \neq j$ then for every eigenvalue of $A$ the algebraic multiplicity is equal to one.

**Proof.** 1) Let $\lambda \in \sigma_p(A)$ and $\lambda \neq \lambda_n$ for all $n = 1, 2, \ldots$. Then $\overline{\lambda} \in \sigma_p(A^*)$ and hence

$$H \neq (\ker(A^* - \overline{\lambda}I))^\perp = R(A - \lambda I) = \text{span}\{(A - \lambda I)f_n : n \geq 1\}$$

$$= \text{span}\{(\lambda_n - \lambda)f_n : n \geq 1\} = \text{span}\{f_n : n \geq 1\} = H.$$ 

This contradiction proves 1).

2) Let $\lambda_k \in \sigma_p(A)$. Since $A$ is a compact operator and $\text{span}\{f_n : n \geq 1\} = H$, we obtain

$$\text{dim ker}(A - \lambda_k I)^m = \dim R(A - \lambda_k I)^m = \dim (\text{span}\{(\lambda_n - \lambda_k)^m f_n : n \geq 0\})^\perp = \dim (\text{span}\{f_n : n \geq 0, n \neq k\})^\perp = 1, \quad m = 1, 2, \ldots.$$ 

Hence the algebraic multiplicity of $\lambda_k$ is equal to one.

The following Lemma is a rephrasing of Problems I.50, V.161, V.162 from [9].

**Lemma 3.** Let $|q| < 1$ then

1) $F_q(z) := \prod_{k=1}^{\infty} (1 - q^k z) = 1 + \sum_{k=1}^{\infty} \frac{q^k (k+1)/2}{(q - 1) \cdots (q^k - 1)} z^k$ is an entire function.

2) The polynomials $P_n(z) := 1 + \sum_{k=1}^{n} \frac{n!}{(n-k)!} \frac{q^k (k+1)/2}{(q - 1) \cdots (q^k - 1)} z^k$ have only real positive zeroes.
3. Main results

**Proposition 1.** Let $0 < \alpha < 1$ and $V_\alpha := V_{x^\alpha}$ be defined on $L^2(0, 1)$. Then

1) $\sigma_p(V_\alpha) = \{(1 - \alpha)^n\}_{n=1}^\infty$;

2) the algebraic multiplicity of every eigenvalue of $V_\alpha$ is equal to one;

3) $f_{n+1}(x) = x^{\frac{\alpha}{1-\alpha}} \left( \ln^n x + \sum_{k=1}^{n} \frac{n! \alpha^{(k-1)/2} (1 - \alpha)^k}{(n-k)! (1-\alpha) \cdots (1-\alpha^k)} \ln^{n-k} x \right)$, \quad $n \in \mathbb{Z}_+$

is an eigenfunction for the operator $V_\alpha$ with eigenvalue $\lambda_{n+1} := (1 - \alpha)\alpha^n$;

4) $g_{n+1}(x) = 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\alpha^{(k-1)(k-2)/2} (1 - \alpha)^{k-1}}{(1 - \alpha) \cdots (1 - \alpha^{k-1})} \frac{1}{x^{(1 - \alpha)^{k-1}}}$, \quad $n \in \mathbb{Z}_+$

is an eigenfunction for the operator $V_\alpha^*$ with eigenvalue $\lambda_{n+1} := (1 - \alpha)\alpha^n$.

5) the system $\{f_n\}_{n=1}^\infty$ is complete in $L^2(0, 1)$;

6) the system $\{g_n\}_{n=1}^\infty$ is not complete in $L^2(0, 1)$.

7) the operator $V_\alpha$ does not admit a spectral synthesis, i.e. there exists an invariant subspace $E$ such that $V_\alpha|E$ is quasinilpotent.

**Proof.** 3) Since $x^\varepsilon \ln^m x \in C[0, 1]$ for all $\varepsilon > 0$ and $m \in \mathbb{Z}_+$, we have that $f_{n+1} \in L^2(0, 1)$.

Let us check that $f_{n+1}(x)$ is an eigenfunction of $V_\alpha$ corresponding to the eigenvalue $\lambda_{n+1} := (1 - \alpha)\alpha^n$. By definition, put

$$C_{n-k}(\alpha) := \frac{n! \alpha^{(k-1)/2} (1 - \alpha)^k}{(n-k)! (1-\alpha) \cdots (1-\alpha^k)}, \quad k = 1 \ldots n.$$

Then

$$\frac{\alpha}{1 - \alpha} C_{n-k}(\alpha) + (n-k+1) C_{n-k+1}(\alpha) = \frac{n! \alpha^{(k-1)(k-2)/2} (1 - \alpha)^{k-1}}{(n-k)! (1 - \alpha) \cdots (1 - \alpha^{k-1})} \left( \frac{\alpha^k}{1 - \alpha^k} + 1 \right)$$

$$= \frac{n! \alpha^{(k-1)(k-2)/2} (1 - \alpha)^{k-1}}{(n-k)! (1 - \alpha) \cdots (1 - \alpha^{k-1})}, \quad k = 1 \ldots n.$$

Further,

$$f_{n+1}(x) = \alpha x^{\alpha-1} \ln^n x \sum_{k=1}^{n} C_{n-k}(\alpha) \ln^{n-k} x$$

$$= \alpha x^{\alpha-1} \ln^n x \sum_{k=1}^{n} \frac{n! \alpha^{(k-1)/2} (1 - \alpha)^k}{(n-k)! (1-\alpha) \cdots (1-\alpha^k)} \ln^{n-k} x$$

$$= (1 - \alpha)\alpha^n x^{\frac{2\alpha-1}{1-\alpha}} \left( \alpha \ln^n x + \sum_{k=1}^{n} \frac{n! \alpha^{(k-1)(k-2)/2} (1 - \alpha)^{k-1}}{(n-k)! (1 - \alpha) \cdots (1 - \alpha^{k-1})} \ln^{n-k} x \right), \quad n \in \mathbb{Z}_+,$$

and

$$f'_{n+1}(x) = \frac{\alpha}{1 - \alpha} x^{\frac{\alpha}{1-\alpha}-1} \left( \ln^n x + \sum_{k=1}^{n} C_{n-k}(\alpha) \ln^{n-k} x \right)$$
\[ +x^{1-\alpha} \left( \frac{n \ln^{n-1} x}{x} + \sum_{k=1}^{n-1} C_n \alpha (n-k) \ln^{n-k-1} x \right) \]
\[ = x^{2-\alpha} \left( \frac{\alpha \ln x}{1-\alpha} + n \ln^{n-1} x \right) \]
\[ + x^{2-\alpha} \left( \sum_{k=1}^{n} \frac{\alpha C_{n-k} (\alpha)}{1-\alpha} \ln^{n-k} x + \sum_{k=2}^{n} C_{n-k+1} (\alpha) (n-k+1) \ln^{n-k} x \right) \]
\[ = x^{2-\alpha} \left[ \frac{\alpha \ln x}{1-\alpha} + \sum_{k=1}^{n} \frac{n!}{(n-k)!} \frac{\alpha^{(k-1)(k-2)/2}(1-\alpha)^{k-1}}{(1-\alpha)\ldots(1-\alpha^k)} \ln^{n-k} x \right], \quad n \in \mathbb{Z}_+. \]

It follows from (2)-(3) that \( \alpha x^{\alpha - 1} f_{n+1} (x^\alpha) = (1-\alpha)\alpha^n f'_{n+1} (x) \). Thus
\[ (V_{\alpha} f_{n+1})(x) = \int_0^x f_{n+1}(t) dt = \int_0^x \alpha^{\alpha - 1} f_{n+1}(t^\alpha) dt = (1-\alpha)\alpha^n \int_0^x f'_{n+1}(t) dt \]
\[ = (1-\alpha)\alpha^n (f_{n+1}(1) - f_{n+1}(0)) = (1-\alpha)\alpha^n f_{n+1}(x), \quad n \in \mathbb{Z}_+. \]

4) The convergence of the series
\[ S := \sum_{k=2}^{\infty} \frac{\alpha^{(k-1)(k-2)/2}}{(1-\alpha)\ldots(1-\alpha^k)} x^{k-1}, \quad x \in [0, 1] \]
follows from d’Alembert rule. Since \( \frac{\alpha^{k-1}}{(\alpha-1)(\alpha^k-1)} = \frac{1}{\alpha} + \ldots + \frac{1}{\alpha^{k-1}} > k-1 \), we obtain that
\( x^{k-1} > x^{\frac{1}{(\alpha-1)\ldots(\alpha^k-1)}} \) for \( x \in [0, 1] \). Now the absolute convergence of \( g_n(x) \) for \( x \in [0, 1] \)
(and hence continuity of \( g_n \)) is implied by the convergence of \( S \).

Let us check that \( g_{n+1}(x) \) is an eigenfunction for the operator \( V_{\alpha}^* \) with the corresponding eigenvalue \( \lambda_{n+1} := (1-\alpha)\alpha^n \):

\[ (V_{\alpha}^* g_{n+1})(x) = \int_{x^{1/\alpha}}^1 g_{n+1}(t) dt \]
\[ = 1 - x^{1/\alpha} + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \alpha^{(k-1)(k-2)/2}}{(1-\alpha)\ldots(1-\alpha^k)} \frac{1}{(1-\alpha)} \left. x^{1-\alpha} \right|_{x^{1/\alpha}} \]
\[ = (1-\alpha)\alpha^n \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \alpha^{(k-1)(k-2)/2}}{(1-\alpha)\ldots(1-\alpha^k)} x^{(1-\alpha)\alpha^k} =: \lambda_{n+1}(S_1 - S_2) \]
\[ = \lambda_{n+1} S_1 - (1 - g_{n+1}(x)) \]
\[ = \lambda_{n+1}(S_1 - (1 - g_{n+1}(x))) = \lambda_{n+1}(S_1 - 1) + \lambda_{n+1} g_{n+1}(x). \]

By Lemma 3.1
\[ S_1 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \alpha^{(k-1)(k-2)/2}}{(1-\alpha)\ldots(1-\alpha^k)} = -\sum_{k=1}^{\infty} \frac{\alpha^{k(k+1)/2}\alpha^{(-n-1)k}}{(\alpha - 1)\ldots(\alpha^k - 1)} = -(F_\alpha(\alpha^{n-1}) - 1) = 1. \]
Combining (4) and (5), we get \( (V_{\alpha}^* g_{n+1})(x) = \lambda_{n+1} g_{n+1}(x) \).
5) It can be proved that $E_{n+1} := \text{span}\{f_1, \ldots, f_{n+1}\} = \text{span}\{x^\frac{n}{\alpha} \ln^k x : k = 0 \ldots n\}$. Hence by Lemma 1

$$E_\infty := \text{span}\{f_k : k \in \mathbb{Z}_+\} = \text{span}\{x^\frac{n}{\alpha} \ln^k x : k \in \mathbb{Z}_+\} = x^\frac{n}{\alpha} L^2(0,1) = L^2(0,1).$$

1), 2) follow from 5) and Lemma 2.

6) It follows from M"untz-Sz"asz theorem [7], [11] that the system $\{x^{(1-\alpha)n}\}_{n=0}^\infty$ is not complete in $L^2(0,1)$. Since $\text{span}\{g_n : n \geq 1\} \subset \text{span}\{x^{\frac{n}{\alpha}-\alpha n} : n \geq 0\}$, we have that the system $\{g_n\}_{n=1}^\infty$ is not complete in $L^2(0,1)$.

7) Let $E = \text{span}\{g_n : n \geq 1\}$. Then $V_\alpha E \subset E$ and by 5) the operator $V_\alpha|_E$ is quasinilpotent.

**Corollary 1.** Let $0 < \alpha < 1$, $\phi(x) = 1 - (1 - x)^{1/\alpha}$. Then the operators $V_x^\alpha$ and $V_\phi$ are unitarily equivalent and hence $\sigma_p(V_\phi) = \{(1-\alpha)\alpha^{-n-1}\}_{n=1}^\infty$.

**Proof.** Let $U$ be a unitary operator defined by $(Uf)(x) = f(1-x)$. Then simple computations show that $V_x^\alpha = U^{-1}V_\phi U$.

**Remark 2.** Suppose $\phi(x) = (1 - (1 - x)^{1/\alpha})'$, then $\phi'(0) = 1/\alpha$. Thus Corollary 1 states that the condition $\phi'(0) = \infty$ is not necessary for $\text{card}\{\sigma_p(V_\phi)\} = \infty$.

**Remark 3.** It is interesting to note that if $\phi(\phi(x)) = x$ then the operator $V_\phi$ is selfadjoint, and hence eigenfunctions of $V_\phi$ form an orthonormal basis in $L^2(0,1)$. The statements 5) and 6) of Proposition 1 imply that the operator $V_\alpha$ is not similar and even quasisimilar (see definition in [8], [10]) to $V_\alpha^*$. It contrasts to the case $\alpha = 1 : V^* = U^{-1}VU$.

It follows also that $V_\alpha$ is not quasisimilar to any selfadjoint operator.

**Corollary 2.** 1) $f_n(x)$ is a continuous function with $n$ real zeroes which belong to $[0,1]$;

2) zeroes of $f_n(x)$ and $f_{n+1}(x)$ interlace.

**Proof.** 1) The continuity of $f_n(x)$ was proved in Proposition 1. Let us prove that the function $f_{n+1}$ has $n+1$ zeroes which belong to $[0,1]$. By definition, put

$$P_n(x) := \left( \frac{t^{\frac{n}{\alpha}} f_{n+1}(t)}{\ln^n t} \right)_{t=e^{-\frac{1}{\alpha x}}}$$

$$= \left( 1 + \sum_{k=1}^{\infty} \frac{n!}{(n-k)!} \frac{\alpha^{k(1-\alpha)/2}(1-\alpha^k)}{(1-\alpha) \ldots (1-\alpha^k)} \ln^k t \right)_{t=e^{-\frac{1}{\alpha x}}}$$

$$= 1 + \sum_{k=1}^{\infty} \frac{n!}{(n-k)!} \frac{\alpha^{k(1-\alpha)/2}}{(1-\alpha) \ldots (1-\alpha^k)} x^k.$$

It can easily be checked that

$$f_{n+1}(t) = t^{\frac{n}{\alpha}} \ln^n t P_n \left( \frac{-\alpha}{1-\alpha} \ln t \right).$$

It follows from Lemma 3 2) that the polynomial $P_n$ has exactly $n$ positive zeroes. Thus the function $f_{n+1}$ has $n+1$ zeroes which belong $[0,1]$.

2) Let us note that $(x^n P_{n+1}(x^{-1}))' = nx^{n-1} P_n(x^{-1})$. Therefore zeroes of $P_n(x)$ and $P_{n+1}(x)$ interlace. Hence zeroes of $f_n(x)$ and $f_{n+1}(x)$ interlace.
Remark 4. We suppose that eigenfunctions $g_n$ of the operator $V_{x_\alpha}^*$ have the same properties of zeroes as $f_n$. Namely

1) $g_n(x)$ is a continuous function with $n$ real zeroes which belong to $[0,1]$;
2) zeroes of $g_n(x)$ and $g_{n+1}(x)$ interlace.

Remark 5. Proposition 1 as well as Corollary 2 hold also if the operator $V_\alpha$ is defined on $L^p(0,1)$ ($1 \leq p < \infty$). To prove it one can easily check that the operator $V_\alpha$ defined on $L^2(0,1)$ is quasisimilar to the operator $V_\alpha$ defined on $L^p(0,1)$.

Remark 6. It was proved in [4] that $V_\alpha$ is hypercyclic on the Fréchet space $C_0([0,1]) := \{u \in C([0,1]) : u(0) = 0\}$, endowed with the system of seminorms
$$
\|u\|_k = \max_{t \in [0,1-1/(k+1)]} |u(t)|, \quad k = 1, 2, \ldots .
$$
If the operator $V_\alpha$ is defined on $L^p(0,1)$ ($1 \leq p < \infty$) then $\sigma(V_\alpha^*)$ is an infinite set and hence (see [3]) $V_\alpha$ cannot be even supercyclic on $L^p(0,1)$.

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References