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ON THE SPECTRUM AND EIGENFUNCTIONS OF THE OPERATOR $(Vf)(x) = \int_0^{x^\alpha} f(t)dt$

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1. Introduction. It is well known that the Volterra operator $V:\ f \to \int_0^x f(t)dt$ defined on $L^p(0,1)$ ($C[0,1]$) is quasinilpotent, that is, $\sigma(V) = \{0\}$. It was pointed out in [5]–[6] that the operator

$$V_\phi:\ f \to \int_0^{\phi(x)} f(t)dt \tag{1}$$

which is a composition of integration and substitution with $\phi \in C[0,1]$ is quasinilpotent on $C[0,1]$ if $\phi(x) \le x$ for all $x \in [0,1]$.

Let $\phi : [0,1] \to [0,1]$ be a measurable function and let $V_\phi : L^p(0,1) \to L^p(0,1)$ ($1 \le p < \infty$) be defined by (1). It was proved in [12]–[13] that V_ϕ is quasinilpotent on $L^p(0,1)$ if and only if $\phi(x) \le x$ for almost all $x \in [0,1]$. It was also noted in [13] and proved in [14] that the spectral radius of V_{x^α} defined on $L^p(0,1)$ or $C[0,1]$ is $1-\alpha$ ($0 < \alpha < 1$).

We note also paper [4], where the hypercyclicity of V_{x^α} was proved on some Fréchet space.

In this note we find the spectrum of V_{x^α} defined on $L^2(0,1)$ and investigate some properties of its eigenfunctions.

Notations. Let X be a Banach space and let T be a bounded operator on X. Then $\ker T := \{x \in X\ : Tx = 0\}$ denotes a kernel of T and $\mathrm{R}(T) := \{Tx\ : x \in X\}$ denotes a range of T. I denotes the identity operator on X; $\mathrm{span}E$ denotes the closed linear span of the set $E \subset X$; $\mathbb{1}$ denotes the function $f \equiv 1$ in $L^2(0,1)$; $\mathbb{Z}_+ := \{0,1,2,\dots\}$. For simplicity we set $\sum_{k=n}^m a_k := 0$ if $n > m$.

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2. Auxiliary results. The following two Lemmas are well known. For the sake of completeness, proofs are given.

Lemma 1. *The system* $\{(\ln x)^n\}_{n=0}^{\infty}$ *is complete in* $L^2(0,1)$.

Proof. Since the Laguerre functions $f_n(x) := e^{-x/2}\frac{1}{n!}e^x\frac{d^n}{dx^n}(x^n e^{-x})$ $(n \in \mathbb{Z}_+)$ form [1] an orthonormal basis in $L^2(0,\infty)$, the system $\{x^n e^{-x/2}\}_{n=0}^{\infty}$ is complete in $L^2(0,\infty)$. Let the operator $T : L^2(0,\infty) \to L^2(0,1)$ be defined by

$$(Tf)(x) := \frac{f(-\ln x)}{x^{1/2}}.$$

It is easily proved that T is a surjective isometry. Thus the system $\{T(x^n e^{-x/2})\}_{n=0}^{\infty} = \{(-\ln x)^n\}_{n=0}^{\infty}$ is complete in $L^2(0,1)$. ■

Remark 1. Consider an operator $C:\ L^2(0,1) \to L^2(0,1)$ defined by $(Cf)(x) = f(x) - \int_x^1 \frac{f(t)}{t}dt$. It is well known [2] that C is a simple unilateral shift. Since $\ker C^* = \{c \cdot \mathbb{1} : c \in \mathbb{C}\}$, it follows [8] that the set $\{C^n \mathbb{1}\}_{n=0}^{\infty}$ forms an orthonormal basis in $L^2(0,1)$. It can easily be checked that $(C^n\mathbb{1})(x) = P_n(\ln x)$, where P_n is a polynomial of degree n. Thus $L^2(0,1) = \operatorname{span}\{(\ln x)^n : n \geqslant 0\}$.

Lemma 2. *Let A be a compact operator defined on a Hilbert space H, $Af_n = \lambda_n f_n$ and* $\operatorname{span}\{f_n : n \geqslant 1\} = H$. *Then*

1) $\sigma_p(A) = \{\lambda_n\}_{n=1}^{\infty}$;

2) if $\lambda_i \neq \lambda_j$ for $i \neq j$ then for every eigenvalue of A the algebraic multiplicity is equal to one.

Proof. 1) Let $\lambda \in \sigma_p(A)$ and $\lambda \neq \lambda_n$ for all $n = 1, 2, \dots$. Then $\overline{\lambda} \in \sigma_p(A^*)$ and hence

$$H \neq \left(\ker(A^* - \overline{\lambda}I)\right)^{\perp} = \overline{\mathrm{R}(A-\lambda I)} = \operatorname{span}\{(A-\lambda I)f_n : n \geqslant 1\}$$
$$= \operatorname{span}\{(\lambda_n - \lambda)f_n : n \geqslant 1\} = \operatorname{span}\{f_n : n \geqslant 1\} = H.$$

This contradiction proves 1).

2) Let $\lambda_k \in \sigma_p(A)$. Since A is a compact operator and $\operatorname{span}\{f_n : n \geqslant 1\} = H$, we obtain

$$\dim\ker(A - \lambda_k I)^m = \dim\overline{\mathrm{R}(A-\lambda_k I)^m}^{\perp} = \dim\left(\operatorname{span}\{(\lambda_n - \lambda_k)^m f_n : n \geq 0\}\right)^{\perp}$$
$$= \dim\left(\operatorname{span}\{f_n : n \geq 0,\ n \neq k\}\right)^{\perp} = 1, \qquad m = 1, 2, \dots.$$

Hence the algebraic multiplicity of λ_k is equal to one. ■

The following Lemma is a rephrasing of Problems I.50, V.161, V.162 from [9].

Lemma 3. *Let $|q| < 1$ then*

1) $F_q(z) := \prod_{k=1}^{\infty}(1 - q^k z) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}}{(q-1)\cdots(q^k-1)} z^k$ *is an entire function.*

2) The polynomials $P_n(z) := 1 + \sum_{k=1}^{n} \frac{n!}{(n-k)!}\frac{q^{k(k+1)/2}}{(q-1)\cdots(q^k-1)} z^k$ *have only real positive zeroes.*

3. Main results

PROPOSITION 1. *Let $0<\alpha<1$ and $V_\alpha := V_{x^\alpha}$ be defined on $L^2(0,1)$. Then*

1) $\sigma_p(V_\alpha) = \{(1-\alpha)\alpha^{n-1}\}_{n=1}^\infty$;

2) the algebraic multiplicity of every eigenvalue of V_α is equal to one;

3)

$$f_{n+1}(x) = x^{\frac{\alpha}{1-\alpha}}\left(\ln^n x + \sum_{k=1}^n \frac{n!}{(n-k)!}\frac{\alpha^{k(k-1)/2}(1-\alpha)^k}{(1-\alpha)\dots(1-\alpha^k)}\ln^{n-k} x\right), \qquad n\in\mathbb{Z}_+$$

is an eigenfunction for the operator V_α with eigenvalue $\lambda_{n+1} := (1-\alpha)\alpha^n$;

4)

$$g_{n+1}(x) = 1 + \sum_{k=2}^\infty (-1)^{k-1}\frac{\alpha^{(k-1)(k-2-2n)/2}}{(1-\alpha)\dots(1-\alpha^{k-1})}x^{\frac{1-\alpha^{k-1}}{(1-\alpha)\alpha^{k-1}}}, \qquad n\in\mathbb{Z}_+$$

is an eigenfunction for the operator V_α^* with eigenvalue $\lambda_{n+1} := (1-\alpha)\alpha^n$.

5) the system $\{f_n\}_{n=1}^\infty$ is complete in $L^2(0,1)$;

6) the system $\{g_n\}_{n=1}^\infty$ is not complete in $L^2(0,1)$.

7) the operator V_α does not admit a spectral synthesis, i.e. there exists an invariant subspace E such that $V_\alpha|_E$ is quasinilpotent.

Proof. 3) Since $x^\varepsilon \ln^m x \in C[0,1]$ for all $\varepsilon>0$ and $m\in\mathbb{Z}_+$, we have that $f_{n+1}\in L^2(0,1)$. Let us check that $f_{n+1}(x)$ is an eigenfunction of V_α corresponding to the eigenvalue $\lambda_{n+1} := (1-\alpha)\alpha^n$. By definition, put

$$C_{n-k}(\alpha) := \frac{n!}{(n-k)!}\frac{\alpha^{k(k-1)/2}(1-\alpha)^k}{(1-\alpha)\dots(1-\alpha^k)}, \qquad k=1\dots n.$$

Then

$$\frac{\alpha}{1-\alpha}C_{n-k}(\alpha)+(n-k+1)C_{n-k+1}(\alpha) = \frac{n!}{(n-k)!}\frac{\alpha^{(k-1)(k-2)/2}(1-\alpha)^{k-1}}{(1-\alpha)\dots(1-\alpha^{k-1})}\left(\frac{\alpha^k}{1-\alpha^k}+1\right)$$
$$= \frac{n!}{(n-k)!}\frac{\alpha^{(k-1)(k-2)/2}(1-\alpha)^{k-1}}{(1-\alpha)\dots(1-\alpha^k)}, \qquad k=1\dots n.$$

Further,

$$\alpha x^{\alpha-1}f_{n+1}(x^\alpha) = \alpha x^{\alpha-1}(x^\alpha)^{\frac{\alpha}{1-\alpha}}\left(\ln^n x^\alpha + \sum_{k=1}^n C_{n-k}(\alpha)\ln^{n-k}x^\alpha\right) \tag{2}$$
$$= \alpha x^{\alpha-1+\frac{\alpha^2}{1-\alpha}}\left(\alpha^n\ln^n x + \sum_{k=1}^n\frac{n!}{(n-k)!}\frac{\alpha^{k(k-1)/2}(1-\alpha)^k}{(1-\alpha)\dots(1-\alpha^k)}\alpha^{n-k}\ln^{n-k}x\right)$$
$$= (1-\alpha)\alpha^n x^{\frac{2\alpha-1}{1-\alpha}}\left(\frac{\alpha\ln^n x}{1-\alpha} + \sum_{k=1}^n \frac{n!}{(n-k)!}\frac{\alpha^{(k-1)(k-2)/2}(1-\alpha)^{k-1}}{(1-\alpha)\dots(1-\alpha^k)}\ln^{n-k}x\right), \quad n\in\mathbb{Z}_+,$$

and

$$f'_{n+1}(x) = \frac{\alpha}{1-\alpha}x^{\frac{\alpha}{1-\alpha}-1}\left(\ln^n x + \sum_{k=1}^n C_{n-k}(\alpha)\ln^{n-k}x\right) \tag{3}$$

$$+x^{\frac{\alpha}{1-\alpha}}\left(\frac{n\ln^{n-1}x}{x}+\sum_{k=1}^{n-1}C_{n-k}(\alpha)\frac{1}{x}(n-k)\ln^{n-k-1}x\right)$$

$$=x^{\frac{2\alpha-1}{1-\alpha}}\left(\frac{\alpha\ln^n x}{1-\alpha}+n\ln^{n-1}x\right)$$

$$+x^{\frac{2\alpha-1}{1-\alpha}}\left(\sum_{k=1}^{n}\frac{\alpha C_{n-k}(\alpha)}{1-\alpha}\ln^{n-k}x+\sum_{k=2}^{n}C_{n-k+1}(\alpha)(n-k+1)\ln^{n-k}x\right)$$

$$=x^{\frac{2\alpha-1}{1-\alpha}}\left[\frac{\alpha\ln^n x}{1-\alpha}+\frac{n}{1-\alpha}\ln^{n-1}x+\sum_{k=2}^{n}\left(\frac{\alpha C_{n-k}(\alpha)}{1-\alpha}+(n-k+1)C_{n-k+1}(\alpha)\right)\ln^{n-k}x\right]$$

$$=x^{\frac{2\alpha-1}{1-\alpha}}\left(\frac{\alpha\ln^n x}{1-\alpha}+\sum_{k=1}^{n}\frac{n!}{(n-k)!}\frac{\alpha^{(k-1)(k-2)/2}(1-\alpha)^{k-1}}{(1-\alpha)\dots(1-\alpha^k)}\ln^{n-k}x\right),\qquad n\in\mathbb{Z}_+.$$

It follows from (2)-(3) that $\alpha x^{\alpha-1}f_{n+1}(x^\alpha)=(1-\alpha)\alpha^n f'_{n+1}(x)$. Thus

$$(V_\alpha f_{n+1})(x)=\int_0^{x^\alpha}f_{n+1}(t)dt=\int_0^x\alpha t^{\alpha-1}f_{n+1}(t^\alpha)dt=(1-\alpha)\alpha^n\int_0^x f'_{n+1}(t)dt$$
$$=(1-\alpha)\alpha^n(f_{n+1}(x)-f_{n+1}(0))=(1-\alpha)\alpha^n f_{n+1}(x),\qquad n\in\mathbb{Z}_+.$$

4) The convergence of the series

$$S:=\sum_{k=2}^{\infty}\frac{\alpha^{(k-1)(k-2-2n)/2}}{(1-\alpha)\dots(1-\alpha^{k-1})}x^{k-1},\qquad x\in[0,1]$$

follows from d'Alembert rule. Since $\frac{\alpha^{k-1}-1}{(\alpha-1)(\alpha^{k-1})}=\frac{1}{\alpha}+\dots+\frac{1}{\alpha^{k-1}}>k-1$, we obtain that $x^{k-1}>x^{\frac{\alpha^{k-1}-1}{(\alpha-1)(\alpha^{k-1})}}$ for $x\in[0,1]$. Now the absolute convergence of $g_n(x)$ for $x\in[0,1]$ (and hence continuity of g_n) is implied by the convergence of S.

Let us check that $g_{n+1}(x)$ is an eigenfunction for the operator V_α^* with the corresponding eigenvalue $\lambda_{n+1}:=(1-\alpha)\alpha^n$:

$$(V_\alpha^* g_{n+1})(x)=\int_{x^{1/\alpha}}^1 g_{n+1}(t)dt \tag{4}$$

$$=1-x^{1/\alpha}+\sum_{k=2}^{\infty}\frac{(-1)^{k-1}\alpha^{(k-1)(k-2-2n)/2}}{(1-\alpha)\dots(1-\alpha^{k-1})}\frac{(1-\alpha)\alpha^{k-1}}{1-\alpha^k}x^{\frac{1-\alpha^k}{(1-\alpha)\alpha^{k-1}}}\Big|_{x^{1/\alpha}}^1$$

$$=(1-\alpha)\alpha^n\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\alpha^{k(k-1-2n)/2}}{(1-\alpha)\dots(1-\alpha^k)}(1-x^{\frac{1-\alpha^k}{(1-\alpha)\alpha^{k-1}}})=:\lambda_{n+1}(S_1-S_2)$$

$$=\lambda_{n+1}(S_1-(1-g_{n+1}(x)))=\lambda_{n+1}(S_1-1)+\lambda_{n+1}g_{n+1}(x).$$

By Lemma 3 1)

$$S_1=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\alpha^{k(k-1-2n)/2}}{(1-\alpha)\dots(1-\alpha^k)}=-\sum_{k=1}^{\infty}\frac{\alpha^{k(k+1)/2}\alpha^{(-n-1)k}}{(\alpha-1)\dots(\alpha^k-1)}=-(F_\alpha(\alpha^{-n-1})-1)=1. \tag{5}$$

Combining (4) and (5), we get $(V_\alpha^* g_{n+1})(x)=\lambda_{n+1}g_{n+1}(x)$.

5) It can be proved that $E_{n+1} := \operatorname{span}\{f_1, \dots, f_{n+1}\} = \operatorname{span}\{x^{\frac{\alpha}{1-\alpha}} \ln^k x : k = 0 \dots n\}$. Hence by Lemma 1

$$E_\infty := \operatorname{span}\{f_k : k \in \mathbb{Z}_+\} = \operatorname{span}\{x^{\frac{\alpha}{1-\alpha}} \ln^k x : k \in \mathbb{Z}_+\} = \overline{x^{\frac{\alpha}{1-\alpha}} L^2(0,1)} = L^2(0,1).$$

1), 2) follow from 5) and Lemma 2.

6) It follows from Müntz-Szász theorem [7], [11] that the system $\{x^{\frac{1-\alpha^n}{(1-\alpha)\alpha^n}}\}_{n=0}^\infty$ is not complete in $L^2(0,1)$. Since $\operatorname{span}\{g_n : n \geq 1\} \subset \operatorname{span}\{x^{\frac{1-\alpha^n}{(1-\alpha)\alpha^n}} : n \geq 0\}$, we have that the system $\{g_n\}_{n=1}^\infty$ is not complete in $L^2(0,1)$.

7) Let $E = \operatorname{span}\{g_n : n \geq 1\}^\perp$. Then $V_\alpha E \subset E$ and by 5) the operator $V_\alpha|_E$ is quasinilpotent. ∎

Corollary 1. *Let $0 < \alpha < 1$, $\phi(x) = 1 - (1-x)^{1/\alpha}$. Then the operators $V^*_{x^\alpha}$ and V_ϕ are unitarily equivalent and hence $\sigma_p(V_\phi) = \{(1-\alpha)\alpha^{n-1}\}_{n=1}^\infty$.*

Proof. Let U be a unitary operator defined by $(Uf)(x) = f(1-x)$. Then simple computations show that $V^*_{x^\alpha} = U^{-1} V_\phi U$. ∎

Remark 2. Suppose $\phi(x) = (1-(1-x)^{1/\alpha})'$, then $\phi'(0) = 1/\alpha$. Thus Corollary 1 states that the condition $\phi'(0) = \infty$ is not necessary for $\operatorname{card}\{\sigma_p(V_\phi)\} = \infty$.

Remark 3. It is interesting to note that if $\phi(\phi(x)) = x$ then the operator V_ϕ is selfadjoint, and hence eigenfunctions of V_ϕ form an orthonormal basis in $L^2(0,1)$. The statements 5) and 6) of Proposition 1 imply that the operator V_α is not similar and even quasisimilar (see definition in [8], [10]) to V^*_α. It contrasts to the case $\alpha = 1 : V^* = U^{-1} V U$.

It follows also that V_α is not quasisimilar to any selfadjoint operator.

Corollary 2. *1) $f_n(x)$ is a continuous function with n real zeroes which belong to $[0,1]$;*

2) zeroes of $f_n(x)$ and $f_{n+1}(x)$ interlace.

Proof. 1) The continuity of $f_n(x)$ was proved in Proposition 1. Let us prove that the function f_{n+1} has $n+1$ zeroes which belong to $[0,1]$. By definition, put

$$\begin{aligned} P_n(x) &:= \left(\frac{t^{-\frac{\alpha}{1-\alpha}} f_{n+1}(t)}{\ln^n t} \Big|_{t=e^{-\frac{1-\alpha}{\alpha x}}} \right) \\ &= \left(1 + \sum_{k=1}^\infty \frac{n!}{(n-k)!} \frac{\alpha^{k(k-1)/2}(1-\alpha^k)}{(1-\alpha)\dots(1-\alpha^k)} \ln^{-k} t \right) \Big|_{t=e^{-\frac{1-\alpha}{\alpha x}}} \\ &= 1 + \sum_{k=1}^\infty \frac{n!}{(n-k)!} \frac{\alpha^{k(k+1)/2}}{(\alpha-1)\dots(\alpha^k-1)} x^k. \end{aligned}$$

It can easily be checked that

$$f_{n+1}(t) = t^{\frac{\alpha}{1-\alpha}} \ln^n t P_n\left(\frac{-\alpha}{(1-\alpha)\ln t}\right).$$

It follows from Lemma 3 2) that the polynomial P_n has exactly n positive zeroes. Thus the function f_{n+1} has $n+1$ zeroes which belong $[0,1]$.

2) Let us note that $(x^n P_{n+1}(x^{-1}))' = n x^{n-1} P_n(x^{-1})$. Therefore zeroes of $P_n(x)$ and $P_{n+1}(x)$ interlace. Hence zeroes of $f_n(x)$ and $f_{n+1}(x)$ interlace. ∎

REMARK 4. We suppose that eigenfunctions g_n of the operator $V_{x^\alpha}^*$ have the same properties of zeroes as f_n. Namely

1) $g_n(x)$ is a continuous function with n real zeroes which belong to $[0,1]$;
2) zeroes of $g_n(x)$ and $g_{n+1}(x)$ interlace.

REMARK 5. Proposition 1 as well as Corollary 2 hold also if the operator V_α is defined on $L^p(0,1)$ $(1 \le p < \infty)$. To prove it one can easily check that the operator V_α defined on $L^2(0,1)$ is quasisimilar to the operator V_α defined on $L^p(0,1)$.

REMARK 6. It was proved in [4] that V_α is hypercyclic on the Fréchet space $C_0([0,1]) := \{u \in C([0,1]) : u(0) = 0\}$, endowed with the system of seminorms

$$\|u\|_k = \max_{t\in[0,1-1/(k+1)]} |u(t)|, \qquad k = 1, 2, \dots.$$

If the operator V_α is defined on $L^p(0,1)$ $(1 \le p < \infty)$ then $\sigma(V_\alpha^*)$ is an infinite set and hence (see [3]) V_α cannot be even supercyclic on $L^p(0,1)$.

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