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POSITIVITY IN THE THEORY OF SUPERCYCLIC OPERATORS

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Abstract. A bounded linear operator T defined on a Banach space X is said to be supercyclic if there exists a vector $x \in X$ such that the projective orbit $\{\lambda T^n x \ : \ \lambda \in \mathbb{C},\ n \in \mathbb{N}\}$ is dense in X. The aim of this survey is to show the relationship between *positivity* and *supercyclicity.* This relationship comes from the so called Positive Supercyclicity Theorem. Throughout this exposition, interesting new directions and open problems will appear.

1. Introduction. A bounded linear operator T defined on a Banach space X is said to be supercyclic if the projective orbit of some vector x:

$$\{\lambda T^n x : n \in \mathbb{N},\ \lambda \in \mathbb{C}\}$$

is dense in X.

Supercyclicity is an intermediate property between the hypercyclicity ($T \in \mathcal{L}(X)$ is called hypercyclic if the orbit $\{T^n x\}$ is dense for some vector x) and the cyclicity (if the linear span of $\{T^n x\}$ is dense in X). These notions are intimately related to the

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invariant subspace problem. A good reference on hypercyclic operators is the survey of K. G. Große-Erdmann [21] (for a more recent survey see [34]).

Supercyclicity was introduced in the sixties by Hilden and Wallen [24] who proved that weighted backward shifts are all supercyclic. Since 1991 this property has been studied extensively, thanks to the interesting work by Godefroy and Shapiro ([18]). The first example of supercyclic operator in infinite-dimensional Banach spaces (in fact, a hypercyclic operator) was discovered by S. Rolewicz in 1969 ([33]).

This survey paper deals with positive supercyclicity. An operator T is said to be positively supercyclic if the subset

$$\{rT^n x : r \in \mathbb{R}_+,\, n \in \mathbb{N}\},$$

where $\mathbb{R}_+$ denotes the positive real numbers, is dense in X for some x. The Positive Supercyclicity Theorem appeared in [27] where F. León and V. Müller gave a positive answer to a question raised in [6].

Theorem 1.1 (Positive Supercyclicity Theorem). *If T is a supercyclic operator with $\sigma_p(T^\star) = \emptyset$ then T is positive supercyclic. Moreover, if for some vector $x \in X$ the subset*

$$\{\lambda T^n x : n \in \mathbb{N},\, \lambda \in \mathbb{C}\}$$

is dense in X, then the subset $\{rT^n x : n \in \mathbb{N},\, r \in \mathbb{R}_+\}$ is dense in X.

The above result actually simplifies the notion of supercyclicity. Throughout what follows we will see new applications of Theorem 1.1.

The next result is similar to Theorem 1.1 in the context of hypercyclic operators. Let $\mathbb{D}$ be the open unit disc in the complex plane.

Theorem 1.2. *Let T be a hypercyclic operator in $\mathcal{L}(X)$ and suppose that $\alpha \in \partial\mathbb{D}$. Then T has the same hypercyclic vectors as αT.*

Theorem 1.2 solved a question posed independently by several authors at the same time. The same technique used to prove Theorem 1.1 yields the following more general result (see [27]).

Theorem 1.3. *Let $\mathcal{M}$ be a semigroup of operators defined on a Banach space X and for which there exists a vector $x \in X$ such that the subset $\{\lambda Sx : S \in \mathcal{M}, \lambda \in \partial\mathbb{D}\}$ is dense in X. Suppose that there exists a bounded linear operator $T \in \mathcal{M}$ such that $\sigma_p(T^\star) = \emptyset$ and $ST = TS$ for all $S \in \mathcal{M}$. Then the subset $\{Sx : S \in \mathcal{M}\}$ is dense in X.*

In Section 2, we will see the connection between the above result on positive supercyclicity and the invariant subspace problem for positive operators. This connection is very natural. In fact, as an immediate consequence, we will see that each positive operator has many non-supercyclic vectors. In this section we analyze also a recent result of V. Müller ([31],[32]), and we will clarify its relation with the invariant subspace problem for positive operators.

Section 3 is devoted to the study of the orbits of some integral operators. In Section 3 it is proved that the classical Cesàro operator is hypercyclic on $L^p[0,1]$, $1 < p < \infty$, nevertheless (thanks to the Positive Supercyclicity Theorem), this result is not true on

$C[0,1]$ with the uniform norm. This result is extended in [29], where the authors extensively study the hypercyclic and supercyclic properties of integral operators. Related results on supercyclicity of integral operators appeared in [17].

A bounded linear operator T defined on a Banach space X is said to be weakly supercyclic if there exists a vector $x \in X$ such that the orbit $\{\lambda T^n x : n \in \mathbb{N},\ \lambda \in \mathbb{C}\}$ is weakly dense in X. In [35] the existence of weakly supercyclic operators which are not (norm) supercyclic is shown. In Section 4 we will prove a weak version of the Positive Supercyclicity Theorem. These results are included in [28].

The paper will be concluded with some open questions and further directions.

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2. The invariant subspace problem for positive operators. The invariant subspace problem is one of the most fascinating unsolved problems in operator theory. Even if certain conditions of regularity are imposed on T, then it is not known if the operator T has nontrivial invariant closed subspaces. For instance, let us suppose that $(X, \leq_C)$ is a Banach space with an order $\leq_C$. The order is defined by means of a closed cone C; a cone is a subset C for which $C + C \subset C$ and $rC \subset C$ for all $r \in \mathbb{R}_+$. Let T be a positive operator on X with respect to $\leq_C$; that is, $T(C) \subset C$. Then, in this regular situation it is not known if there exist non-trivial invariant closed subspaces for T.

The above problem is known as the invariant subspace problem for positive operators. The main reference, which collects the general results related to this topic, is the survey of Abramovich, Aliprantis and Burkinshaw (see [1]). In this context, some advances have been made recently by Atzmon-Godefroy and Kalton (see [3],[4]).

In terms of cyclic vectors the above problem can be reformulated in the following way: it is not known if a positive bounded linear operator on $(X, \leq_C)$ has a non-trivial non-cyclic vector.

However as a consequence of the Positive Supercyclicity Theorem, every positive operator has a lot of non-supercyclic vectors. In fact, each vector of the cone C, or $-C$ is non-supercyclic for T.

PROPOSITION 2.1. *Let T be a positive operator defined on the ordered vector space $(X, \leq_C)$ where X is a Banach space. If C is the positive cone which defines the order, then each vector of C is non-supercyclic for T.*

Now we will turn our attention to an "apparently" different problem. Let us consider an operator T defined on a separable infinite-dimensional Hilbert space H. If T is a contraction with spectral radius $r(T)$ equal to 1, then, does a closed non-trivial invariant subspace exist for T? Or equivalently, does a (non-trivial) non-cyclic vector exist for T? In general form:

QUESTION 2.2. Let $T \in \mathcal{L}(H)$ be a power-bounded operator with $r(T) = 1$. Does a (non-trivial) non-cyclic vector exist for T?

The previous problem is related to the articles by Brown-Chevreau and Pearcy. They proved (see [10], [11]) that if the spectrum of a Hilbert space contraction T is rich (namely, the spectrum of T contains the unit circle), then T has a non-trivial non-cyclic vector. In this direction, some recent advances have been made by C. Ambrozie and V. Müller (see [2]) who improved the result of Brown-Chevreau and Pearcy to polynomially bounded operators.

This situation is where the Positive Supercyclicity Theorem appears. Regarding the Question 2.2, V. Müller has recently proved that if T is a power bounded operator with spectral radius 1, then there exists a non-supercyclic vector (non-trivial) for T. In fact, the result in [31] is more general:

THEOREM 2.3 (V. Müller). *Let $T \in \mathcal{L}(H)$ be a power-bounded operator with $1 \in \sigma(T)$ and let us suppose that $T^n x \to 0$ for all $x \in H$. If $\{a_n\}$ is a sequence of positive real numbers converging to zero, then there exists $x \in H \setminus \{0\}$ such that*

$$\operatorname{Re} \langle T^n x, x\rangle \geq a_n$$

for all n.

Let us observe that by the Positive Supercyclicity Theorem the vector x constructed in Theorem 2.3 is a non-supercyclic vector. It is folklore in the context of the invariant subspace problem (see [5] Chapter XIII) that we can suppose without loss of generality that either $T^n x \to 0$ for all $x \in H$ or $T^{*n} x \to 0$ for all $x \in H$.

Moreover Theorem 2.3 gives a surprising connection between the invariant subspace problem for positive operators and Question 2.2. We believe that the two problems are equivalent. In fact the following conjecture arises.

CONJECTURE 2.4. Let us denote by $(H, \leq_C)$ a separable complex Hilbert space and by $\leq_C$ an order defined by means of a closed cone C. Then each positive operator T on H has a non-trivial invariant closed subspace if and only if for each power-bounded operator T with $r(T) = 1$ there exists a non-trivial invariant closed subspace for T.

In fact, the following result is true.

COROLLARY 2.5. *Let us denote by $(H, \leq_C)$ a separable complex Hilbert space. If each positive operator T on H has a non-trivial invariant closed subspace then for each power-bounded operator T with $r(T) = 1$ there exists a non-trivial invariant closed subspace for T.*

The proof is a direct consequence of Theorem 2.3. We only need to put all the information together. Let us consider a power bounded operator T with $r(T) = 1$. We can suppose without loss of generality that $T^n x \to 0$ for all $x \in H$ or $T^{*n} x \to 0$ for all $x \in H$. In any case, we apply Theorem 2.3. Let us consider the following cone C:

$$C = \{p(T)x : p \text{ polynomial with positive coefficients}\}^-$$

where x is the vector which is guaranteed by Theorem 2.3. Let us observe that T is positive with respect to C, and by hypothesis, there exists a non-trivial invariant subspace for T.

3. Supercyclicity of integral operators. In this section we will see how the Positive Supercyclicity Theorem helps us to prove non-supercyclicity of certain integral operators. In fact, we will prove that the classical Cesàro operator is not supercyclic on $C[0,1]$ with the uniform norm.

In the literature it is a clear fact that few operators of *integral type* (that is, unilateral forward shifts, the Volterra operator, etc.) are supercyclic or hypercyclic. However, the operators of *derivative type* (translation operators, unilateral backward shifts, etc.) are more likely to be supercyclic or hypercyclic. The following question arises.

QUESTION 3.1. For which kernels K is the integral operator

$$(V_K f)(x) = \int_0^x K(x,t)\, f(t)\, dt$$

supercyclic or hypercyclic on $L^p[0,1]$ $(1 \le p < \infty)$?

The results in [29] suggest that if the kernel K is *regular* then the orbit of V_K is also regular. However, if the behaviour of K is irregular then the orbit of some vector under V_K^n can be chaotic. Nevertheless, for different kernels it is possible to obtain different grades of hypercyclicity (see [29]).

For instance, if K is a continuous function on $[0,1]\times[0,1]$ it is well known that V_K is quasinilpotent, so that the orbit of each vector $V_K^n f \to 0$ for all $f \in L^p[0,1]$ $(1 \le p < \infty)$.

However, if we analyze the behaviour of the operator V_K with kernel $K(x,y) = \frac{1}{x}$, we find a surprise.

THEOREM 3.2. *The operator*

$$(Cf)(x) = \frac{1}{x}\int_0^x f(s)\, ds$$

is hypercyclic on $L^p[0,1](1 < p < \infty)$.

Let us observe that in this case the operator C is the classical Hardy operator. The Hardy operator was introduced in [22], and it was intensively studied in the 1980s. For instance, in [9] and [26] the spectrum of C, defined on the space $L^p[0,1]$ $(1 < p < \infty)$, is computed. The Hardy operator is also known as the *Cesàro operator*. However, the name Cesàro operator is also given in the literature (see [12], [37]) to another operator defined on ℓ^p $(1 \le p < \infty)$ by

$$T_C(x_1, x_2, x_3, x_4, \dots) = \left(x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \frac{x_1+\dots+x_4}{4}, \dots\right)$$

which is of a different nature than C.

In fact, T_C is not supercyclic on ℓ^2 because it is subnormal ([25]) and subnormal operators are not supercyclic ([7]). Moreover, T_C is not supercyclic on ℓ^p $(1 \le p < \infty)$; see [29]. Indeed, let $x = (x_1, x_2, \dots, x_n, \dots)$ be a supercyclic vector for T_C, then $x_1 \neq 0$ otherwise x is not cyclic for T_C. Now, an easy check in the first coordinates shows that the positive projective orbit

$$\{rT_C^n x : n \in \mathbb{N},\ r \in \mathbb{R}_+\}$$

cannot be dense in ℓ^p, and using the Positive Supercyclicity Theorem we obtain the desired result.

To prove the hypercyclicity of C we need an extra tool discovered in [18].

LEMMA 3.3 ([18]). *Let T be a bounded linear operator defined on a Banach space X. If the subsets*

$$H_+(T) = \textit{linear span}\{\textit{Ker}(T-\lambda) : |\lambda| > 1\}$$

and

$$H_-(T) = \textit{linear span}\{\textit{Ker}(T-\lambda) : |\lambda| < 1\}$$

are dense in X, then T is hypercyclic.

To prove Theorem 3.2, let us observe that the functions x^α are eigenvectors of C corresponding to the eigenvalues $\frac{1}{\alpha+1}$, that is, if $f_\alpha(x) = x^\alpha$ then

$$(Cf_\alpha)(x) = \frac{1}{\alpha+1}x^\alpha.$$

Using the "full Müntz-Szàsz theorem" (see [14]), the linear span of

$$A = \{x^{n-1} : n \geq 1\}$$

and the linear span

$$B = \{x^{-\frac{1}{n+1}} : n \geq p\}$$

are dense in $L^p[0,1]$ ($1 \leq p < \infty$). Finally, let us observe that the functions in A correspond to the eigenvalues $\frac{1}{n}$ and the functions in B correspond to the eigenvalues $1+\frac{1}{n}$. Therefore the subsets $H_+(C)$ and $H_-(C)$ are dense in $L^p[0,1]$ ($1 < p < \infty$) and Lemma 3.3 yields the desired result. ∎

Nevertheless, this result is not true on $C[0,1]$ with the supremum norm (see [29]). Moreover, the Cesàro operator C is not supercyclic on $C[0,1]$. For this proof, we will need again the Positive Supercyclicity Theorem.

THEOREM 3.4. *The Cesàro operator is not supercyclic on $C[0,1]$.*

Proof. Let us observe that the range of $C-\lambda I$ is dense for all $\lambda \in \mathbb{C}$ (this is due to the Weierstrass Approximation Theorem), that is $\sigma_p(C^\star) = \emptyset$. Thus, we can use the Positive Supercyclicity Theorem. Therefore it is sufficient to show that for every continuous function $f : [0,1] \to \mathbb{R}$ the subset

$$\{rC^n f : n \in \mathbb{N},\, r \in \mathbb{R}_+\}$$

is not dense in the space $C_\mathbb{R}[0,1]$, of all continuous real-valued continuous functions with the supremum norm.

We can assume without loss of generality that $f(0) \geq 0$. By applying L'Hôpital's rule, we obtain

$$(Cf)(0) = \lim_{x\to 0} \frac{\int_0^x f(s)\,ds}{x} = f(0)$$

and, therefore $(C^n f)(0) = f(0)$ for all n. Thus, we have that

$$\sup_{[0,1]} |r(C^n f)(0) - (-1)| > 1,$$

for all $n \in \mathbb{N}$ and $r \in \mathbb{R}_+$, that is the projective orbit $\{rC^n f : n \in \mathbb{N},\, r > 0\}$ is separated from the function (-1) by more than 1, and this finishes the proof. ∎

4. The weak version of the Positive Supercyclicity Theorem. The Positive Supercyclicity Theorem was discovered in [27]. In this section we will prove a weak version of this result.

The proof of this result can be extended without any difficulty to semigroups of operators. We will prove this more general result and as a consequence we deduce the weak version of the Positive Supercyclicity Theorem.

THEOREM 4.1. *Let $\mathcal{M}$ be a semigroup of bounded linear operators acting on a Banach space X, with separable dual. Let us suppose that there is an operator $T \in \mathcal{M}$ with $\sigma_p(T^\star) = \emptyset$ such that $TS = ST$ for all $S \in \mathcal{M}$. If there exists a vector $x \in X$ such that the subset*

$$\{\lambda Sx : |\lambda| = 1,\ S \in \mathcal{M}\}$$

is weakly dense in X, then the subset

$$\{Sx : S \in \mathcal{M}\}$$

is weakly dense in X.

Proof. The proof will be carried out in several steps. If $A \subset X$ we will denote by $A^{-\omega}$ the weak closure of A. The neighborhoods of a point x in the weak topology will be denoted by $V(x, \varepsilon; x_1^\star, \ldots, x_n^\star)$, that is,

$$V(x, \varepsilon; x_1^\star, \ldots, x_n^\star) = \left\{y \in X : |x_j^\star(x-y)| \le \varepsilon : j = 1, \ldots, n\right\}.$$

For each $u \in X$ set $M_u = \{Su : S \in \mathcal{M}\}^{-\omega}$. For $u, v \in X$ set

$$F_{u,v} = \{\mu \in \mathbb{C} : |\mu| = 1,\ \mu v \in M_u\}.$$

STEP (1). *The set $F_{u,v}$ is a closed subset of the unit circle $\mathbb{T} = \{\mu \in \mathbb{C} : |\mu| = 1\}$.*

Proof. If $F_{u,v} = \emptyset$ then (1) is already proved. Let us suppose that $F_{u,v} \neq \emptyset$ and let $\{\mu_n\} \subset F_{u,v}$ be a sequence converging to $\mu \in \mathbb{T}$. Given a neighborhood $V_0 = V(\mu v, \varepsilon; x_1^\star, \ldots, x_n^\star)$, there exists μ_{n_0} such that $|\mu - \mu_{n_0}|\|v\| < \varepsilon/2$. Since $\mu_{n_0} v \in M_u$ there exists $S_0 \in \mathcal{M}$ such that $S_0 u \in V(\mu_0 v, \varepsilon/2; x_1^\star, \ldots, x_n^\star)$. An easy cross-check proves that $S_0 u \in V_0$ and therefore $\mu v \in M_u$ which yields (1). ■

Let X_0 be the set of all vectors $u \in X$ such that $\{\mu Su : \mu \in \mathbb{T},\ S \in \mathcal{M}\}^{-\omega} = X$.

STEP (2). *Let $u \in X_0$. Then $F_{u,v} \neq \emptyset$ for all $v \in X$.*

Proof. Suppose on the contrary that $F_{u,v} = \emptyset$ for some $v \in X$. Let $\{x_n^\star\}_{n\ge1}$ be a dense subset of $X^\star$. Since $u \in X_0$, by induction, there exists a subsequence $\{\mu_n\}_{n\ge1} \subset \mathbb{T}$ such that

$$x_j^\star(v - \mu_n S_n u) < \frac{1}{n}, \qquad j = 1, \ldots, n.$$

Since $\{\mu_n\} \subset \mathbb{T}$ and $\mathbb{T}$ is compact, there exists a subsequence μ_{n_k} converging to some $\mu \in \mathbb{T}$. Therefore, we can suppose without loss of generality that the sequence μ_n converges to μ. Since the sequence $\{\mu_n S_n u - v\}$ converges weakly to 0, it is bounded and we can suppose without loss of generality that it is contained in the unit ball.

We claim that $\mu^{-1} \in F_{u,v}$. Let us fix a weak neighborhood of $\mu^{-1} v$:

$$V_0 = V(\mu^{-1} v, \varepsilon; y_1^\star, \ldots, y_l^\star).$$

Since $\{x_n^\star\}$ is dense there exist $n_1 < n_2 < \cdots < n_l$ and $x_{n_1}^\star, \ldots, x_{n_l}^\star \in X^\star$ such that $\frac{1}{n_1} \leq \frac{\varepsilon}{3}$ and

$$\|y_j^\star - x_{n_j}^\star\| \leq \frac{\varepsilon}{3} \qquad j = 1, \ldots, l.$$

Let $n \geq n_l$ be large enough such that $\|y_j^\star\| \, |\mu_n^{-1} - \mu^{-1}| \|v\| \leq \frac{\varepsilon}{3}$ for all $j = 1, \ldots, l$. Let us observe that

$$\begin{aligned} |y_j^\star(S_n u - \mu^{-1} v)| &= |y_j^\star(S_n u - \mu^{-1} v - \mu_n^{-1} v + \mu_n^{-1} v)| \\ &\leq |y_j^\star(S_n u - \mu_n^{-1} v)| + \|y_j^\star\| \, |\mu_n^{-1} - \mu^{-1}| \, \|v\| \\ &\leq |(y_j^\star - x_{n_j}^\star)(S_n u - \mu_n^{-1} v)| + |x_{n_j}^\star(S_n u - \mu_n^{-1} v)| + \frac{\varepsilon}{3} \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for $j = 1, \ldots, l$. Therefore $\mu^{-1} v \in M_u$, since V_0 is arbitrary. Then $\mu^{-1} \in F_{u,v}$ and the proof is complete. ■

Step (3). *Let $u, v, w \in X$ such that $\mu_1 \in F_{u,v}$ and $\mu_2 \in F_{v,w}$. Then $\mu_1 \mu_2 \in F_{u,w}$.*

Proof. Let $V_0 = V(\mu_1 \mu_2 w, \varepsilon; x_1^\star, \ldots, x_n^\star)$ be an arbitrary neighborhood of $\mu_1 \mu_2 w$. Since $\mu_2 \in F_{v,w}$ there exists $S_1 \in \mathcal{M}$ such that

$$S_1 v \in V(\mu_2 v, \varepsilon/2; x_1^\star, \ldots, x_n^\star).$$

On the other hand, since $\mu_1 \in F_{u,v}$ there exists $S_2 \in \mathcal{M}$ such that

$$S_2 u \in V\left(\mu_1 v, \frac{\varepsilon}{2\|S_1^\star\|}; x_1^\star, \ldots, x_n^\star\right).$$

Let us observe that

$$\begin{aligned} \left|x_j^\star(S_1 S_2 u - \mu_1 \mu_2 w)\right| &\leq \left|x_j^\star(S_1(S_2 u - \mu_1 v))\right| + \left|x_j^\star(\mu_1(S_1 v - \mu_2 w))\right| \\ &\leq \|S_1^\star\| \left|x_j^\star(S_2 u - \mu_1 v)\right| + \left|x_j^\star(S_1 v - \mu_2 w)\right| \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $j = 1, \ldots, n$. That is, $M_u \cap V_0 \neq \emptyset$ for any neighborhood V_0 of $\mu_1 \mu_2 w$. Therefore $\mu_1 \mu_2 \in F_{u,w}$ and the proof is completed. ■

Step (4). $(T - z)x \in X_0$ *for all* $z \in \mathbb{C}$.

Proof. Since $\sigma_p(T^\star) = \emptyset$ the range of $T - z$ is dense for all $z \in \mathbb{C}$. Therefore, the set

$$\begin{aligned} \{\mu S(T - z)x : S \in \mathcal{M}, \, \mu \in \mathbb{T}\}^{-\omega} &\supset (T - z)\{\mu S x : S \in \mathcal{M}, \, \mu \in \mathbb{T}\}^{-\omega} \\ &= (T - z)(X) \end{aligned}$$

is a dense subset of X. ■

Now fix $x \in X_0$. By (2) and (3), $F_{x,x}$ is a non-empty subsemigroup of the unit circle $\mathbb{T}$. Let us suppose that $F_{x,x} = \mathbb{T}$. By (2) and (3) $F_{x,y} = \mathbb{T}$ for all $y \in X$, thus $y \in M_x$ for all $y \in X$, and therefore the set $\{Sx : S \in \mathcal{M}$ is weakly dense in X.

We will suppose from now on that $F_{x,x} \neq \mathbb{T}$, and we will see that this hypothesis leads to a contradiction. The proofs of the steps (4) and (5) are similar to the paragraphs (c) and (d) of [27], page 387, and we omitted them. The remainder of the proof mimics the proof of [27].

STEP (5). *There exists* $k \in \mathbb{N}$ *such that* $F_{x,x} = \{e^{2\pi ij/k} : j = 0, 1, \ldots, k-1\}$.

STEP (6). *Let* $y \in X_0$. *There exists* $\mu_y \in \mathbb{T}$ *such that* $F_{x,y} = \{\mu_y e^{2\pi ij/k} : j = 0, 1, \ldots, k-1\}$.

Now, for any non-zero vector y in the subspace generated by x and Tx, let us define $f(y) = \mu^k$ where μ is any element of $F_{x,y}$. The function f is clearly well defined by Step (6). Moreover, the function f is continuous (see [27] p. 288 for the details).

Proof. The vectors x and Tx are linearly independent. Otherwise, there exists $\alpha \in \mathbb{C}$ such that $Tx = \alpha x$. Since $S \in \mathcal{M}$ commutes with T, $S(\operatorname{Ker}(T-\alpha)) \subset \operatorname{Ker}(T-\alpha)$ for all $S \in \mathcal{M}$, and therefore $\{\mu Sx : \mu \in \mathbb{T}, S \in \mathcal{M}\}^{-\omega} = X \subset \operatorname{Ker}(T-\alpha)$, which is a contradiction because T cannot be a multiple of the identity: $\sigma_p(T^\star) = \emptyset$.

Let us denote by $\mathbb{D}$ the closed unit disk, and let us define the function $g : \mathbb{D} \to \mathbb{T}$ by $g(z) = f(zx + (1-|z|)Tx)$. For all z in the boundary of $\mathbb{D}$, we have that $F_{x,zx} = z^{-1}F_{x,x}$ and therefore $g(z) = f(zx) = z^{-k}f(x) = z^{-k}$. The function g provides a homotopy between the constant path $\gamma_1(t) = 0$ and the path $\gamma_2(t) = g(e^{it}) = e^{-kit}$ which has the winding number $-k$, a contradiction. ∎

As a consequence of Theorem 4.1 we obtain the following corollary.

COROLLARY 4.2. *Let* X *be a separable Banach space with separable dual* $X^\star$, *and let* T *be a bounded linear operator on* X. *Let us suppose that* $\sigma_p(T^\star) = \emptyset$. *Then the vector* x *is weakly supercyclic for* T *if and only if the set*

$$\{rT^n x : n \in \mathbb{N}, r \in \mathbb{R}\}$$

is weakly dense in X.

Proof. The proof follows directly, if we apply Theorem 4.1 to the semigroup $\{rT^n : n \in \mathbb{N}, r \in \mathbb{R}_+\}$. ∎

The weakly hypercyclic version of Theorem 4.1 is the following result.

COROLLARY 4.3. *Let* T *be a weakly hypercyclic operator defined on a Banach space* X *with separable dual* $X^\star$. *Then* $x \in X$ *is weakly hypercyclic for* T *if and only if* x *is weakly hypercyclic for* λT *for all* $\lambda \in \mathbb{T}$.

Proof. Firstly, the operator T is weakly hypercyclic if and only if $\sigma_p(T^\star) = \emptyset$ (see [13]). The results follows by applying Theorem 4.1 to the semigroup $\mathcal{M} = \{T^n : n \in \mathbb{N}\}$. ∎

5. Concluding remarks and open problems. Supercyclicity holds if the space has dimension 1; that is, if $X = \mathbb{C}$ then each non-zero operator on X is supercyclic. However if the dimension of X is finite and greater than 1, then curiously there are no supercyclic operators in $\mathcal{L}(X)$.

The trivial case in dimension 1 has an influence on the structure of the class of all supercyclic operators. Following D. A. Herrero [23] and [19] the class of supercyclic operators is divided into two classes:

1. The class of supercyclic operators T for which the point spectrum of its adjoint is empty: $\sigma_p(T^\star) = \emptyset$.

2. The class of supercyclic operators for which the point spectrum of $T^\star$ is a non-zero complex number α. In that case T is decomposed as the direct sum $T = R \oplus \alpha 1_{\mathbb{C}}$.

Moreover (see [19]), supercyclic operators of the form $T = R \oplus \alpha 1_{\mathbb{C}}$ can be studied in terms of the hypercyclic properties of the operator $\frac{1}{\alpha}R$.

The Positive Supercyclicity Theorem is true only for the first class of supercyclic operators. The second, "pathological" class has been widely studied and it suggests the notion of N-supercyclicity (see [8]).

Let us consider $X = \mathbb{C}$, and the operators defined by $T(1) = re^{2\pi\theta i}$ with $e^{2\pi\theta i}$ a root of unity. It is clear that T is not positive supercyclic. For the same reason, when $T = R \oplus \alpha 1_{\mathbb{C}}$ with $\alpha = re^{2\pi\theta i}$ and $e^{2\pi\theta i}$ is a root of unity, then T is not positive supercyclic, that is, the Positive Supercyclicity Theorem is not true for these operators.

Now we turn our attention to the following example: $X = \mathbb{C}$ and the operators $T(1) = re^{2\pi\theta i}$. Let us suppose that $e^{2\pi\theta i}$ is not a root of unity (that is θ is irrational). In this situation, it is clear that we have positive supercyclicity. To complete the relation between *positivity* and *supercyclicity* we need to answer the following question:

Question 4.1. Let us suppose that θ is an irrational real number, and let us suppose that $T = R \oplus \alpha 1_{\mathbb{C}}$ ($\alpha = re^{2\pi\theta i}$) is supercyclic on the infinite-dimensional Banach space X. Is T positive supercyclic?

We believe that the above question has a positive answer. The Question 4.1 is related to the *bounded gap problem* which is in fact equivalent to the Hypercyclicity Criterion (see [20]).

Now, let us consider the Hardy space $H^p(\mathbb{D})$ ($1 \le p < \infty$), the Banach space of analytic functions f on the open unit disc $\mathbb{D}$ such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\phi})|^p \, d\phi < \infty.$$

If φ is an analytic self-map of the unit disc into itself, then we can consider the composition operator on $H^p(\mathbb{D})$ induced by φ and defined by

$$C_\varphi f = f \circ \varphi.$$

When φ is a Möbius map of parabolic non-automorphism type, it has been proved that C_φ is not supercyclic (see [16], [15]). If fact, more is true: C_φ is decomposable ([36]); let us observe that decomposability is connected with supercyclicity (see [30]). However the proofs in [16] and [36] are very complicated. Both proofs establish decomposability of C_φ to prove that C_φ is not supercyclic. However supercyclicity is a weaker condition, and we believe that it is easier to prove non-supercyclicity of C_φ with the help of the Positive Supercyclicity Theorem. On the other hand, an unexplored field is the study of the supercyclic properties of composition operator C_φ induced by general non-automorphism maps (not necessarily linear fractional maps) of parabolic type.

Addendum. After this paper was accepted, the authors improved Theorem 4.1 to the non-locally convex setting. In fact, Theorem 4.1 is true without the hypothesis on separability of the dual $X^\star$ (see [28]).

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